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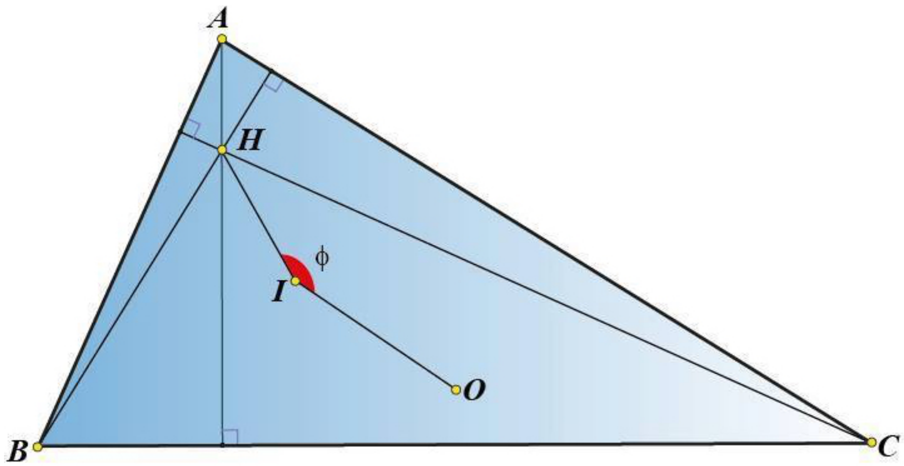

ABC – acute angled triangle, H – orthocenter, O – circumcircle center, I – incenter. Prove: $\angle OIH > 135°$

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In ΔABC – acute. We have: $OI = \sqrt{R^2 - 2Rr} \to OI^2 = R^2 - 2Rr$

$$OH = \sqrt{9R^2 - (a^2 + b^2 + c^2)} \to OH^2 = 9R^2 - (a^2 + b^2 + c^2) =$$
$$= 9R^2 + 8rR + 2r^2 - 2s^2$$

$$HI = \sqrt{2r^2 - 4R^2 \cos A \cos B \cos C} = \sqrt{2r^2 - (s^2 - (2R + r)^2)} =$$
$$= \sqrt{4R^2 + 4Rr + 3r^2 - s^2} \to HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$$

Applying the Cosine rule we have: $\cos \widehat{OHI} = \frac{HI^2 + OI^2 - OH^2}{2 \cdot HI \cdot OI}$

$$= \frac{4R^2 + 4Rr + 3r^2 - s^2 + R^2 - 2Rr - (9R^2 + 8rR + 2r^2 - 2s^2)}{2\sqrt{4R^2 + 4Rr + 3r^2 - s^2} \cdot \sqrt{R^2 - 2Rr}}$$
$$= \frac{r^2 - 4R^2 - 6Rr + s^2}{2\sqrt{4R^2 + 4Rr + 3r^2 - s^2} \cdot \sqrt{R^2 - 2Rr}}$$

We must show that: $\cos \widehat{OHI} < -\frac{\sqrt{2}}{2} \leftrightarrow \frac{r^2 - 4R^2 - 6Rr + s^2}{2\sqrt{4R^2 + 4Rr + 3r^2 - s^2} \cdot \sqrt{R^2 - 2Rr}} < -\frac{\sqrt{2}}{2}$

$$\leftrightarrow r^2 - 4R^2 - 6Rr + s^2 < -\sqrt{2}\sqrt{4R^2 + 4Rr + 3r^2 - s^2}\sqrt{R^2 - 2Rr}$$
$$\leftrightarrow 4R^2 + 6Rr - r^2 - s^2 > \sqrt{2}\sqrt{4R^2 + 4Rr + 3r^2 - s^2}\sqrt{R^2 - 2Rr} \quad (*)$$

Because: $s^2 \le 4R^2 + 4Rr + 3r^2 \to 4R^2 + 6Rr - r^2 - s^2 \ge$

$$\geq 4R^2+6Rr-r^2-(4R^2+4Rr+3r^2)=2Rr-2r^2=$$

$$=2r(R-r)\overset{Euler}{\geq}2r(2r-r)=2r^2>0$$

(*) $\leftrightarrow (4R^2+6Rr-r^2-s^2)^2>2(4R^2+4Rr+3r^2-s^2)(R^2-2Rr)$

$$\leftrightarrow (4R^2+6Rr-r^2-s^2)(4R^2+6Rr-r^2-s^2)>$$

$$>2(4R^2+4Rr+3r^2-s^2)(R^2-2Rr)$$

Which is true because:

$$4R^2+6Rr-r^2-s^2>2(4R^2+4Rr+3r^2-s^2)\leftrightarrow s^2\overset{(1)}{>}4R^2+2Rr+7r^2$$

But: ΔABC ***– acute*** $\rightarrow s^2>(2R+r)^2\rightarrow$ ***(1) true because:***

$$(2R+r)^2>4R^2+2Rr+7r^2\leftrightarrow 4Rr>6r^2\leftrightarrow R>\frac{3}{2}r\left(\because R\geq 2r>\frac{3}{2}r\right)$$

$$4R^2+6Rr-r^2-s^2>(R^2-2Rr)\leftrightarrow 3R^2+8Rr-r^2\overset{(2)}{>}s^2$$

But: ΔABC ***– acute*** $\rightarrow \cos 2A+\cos 2B+\cos 2C=\frac{3R^2+4Rr+r^2-s^2}{R^2}>0$

$\rightarrow 3R^2+4Rr+r^2-s^2>0\rightarrow 3R^2+4Rr+r^2>s^2\rightarrow$ ***(2) true because:***

$3R^2+8Rr-r^2\geq 3R^2+4Rr+r^2\leftrightarrow 4Rr\geq 2r^2\leftrightarrow R\geq 2r$ ***(Euler)***

From (1) and (2) we have (*) is true: $\rightarrow \cos\widehat{OHI}<-\frac{\sqrt{2}}{2}\rightarrow\widehat{OHI}>135^\circ$