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If $a,b,c,d>0, ac=bd$ then:

$$\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)} \ge 4\sqrt{ac}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Abdul Aziz-Semarang-Indonesia, Solution 3 by Ruangkhaw Chaoka-Chiangrai-Thailand, Solution 4 by Tran Hong-Dong Thap-Vietnam, Solution 5 by Boris Colakovic-Belgrade-Serbie, Solution 6 by Soumava Chakraborty-Kolkata-India

Solution 1 by Marian Ursărescu-Romania

Because $d=\frac{ac}{b} \Rightarrow$ *we must show:* $\frac{(a+b)(b+c)\left(c+\frac{ac}{b}\right)\left(\frac{ac}{b}+a\right)}{ab\left(c+\frac{ac}{b}\right)+c\cdot\frac{ac}{b}(a+b)} \ge 4\sqrt{ac} \Leftrightarrow$

$$\Leftrightarrow \frac{\frac{(a+b)(b+c)c(b+a)\cdot a(c+b)}{b^2}}{abc\frac{(a+b)}{b}+\frac{ac^2(a+b)}{b}} \ge 4\sqrt{ac} \Leftrightarrow \frac{(a+b)^2(b+c)^2ac}{ac(a+b)(b+c)\cdot b} \ge 4\sqrt{ac} \Leftrightarrow$$

$$(a+b)(b+c) \ge 4b\sqrt{ac} \quad (1)$$

(1) it is true because: $\left.\begin{matrix} a+b \ge 2\sqrt{ab} \\ b+c \ge 2\sqrt{bc} \end{matrix}\right\} \Rightarrow (a+b)(b+c) \ge 4b\sqrt{ac}$

Solution 2 by Abdul Aziz-Semarang-Indonesia

Since $ac=bd$ *then:* $\frac{a}{d}=\frac{b}{c}=\frac{a+b}{d+c} \Rightarrow \frac{a}{a+b}=\frac{d}{d+c}$ *and* $\frac{ac}{d}=b$

$$\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)}=\frac{(a+b)(b+c)(c+d)(d+a)}{(c+d)(a+b)\left(\frac{ab}{a+b}+\frac{cd}{c+d}\right)}$$

$$=\frac{(b+c)(d+a)}{\left(\frac{db}{d+c}+\frac{cd}{c+d}\right)}=\frac{(b+c)(d+a)}{d\left(\frac{b+c}{c+d}\right)}=\frac{(c+d)(d+a)}{d}$$

$$=a+b+c+d\ge 4\sqrt[4]{abcd}=4\sqrt{ac}$$

Solution 3 by Ruangkhaw Chaoka-Chiangrai-Thailand

$a,b,c,>0;ac=bd$. **Prove that** $K=\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)}\ge 4\sqrt{ac}$

$$ac=bd\Leftrightarrow\frac{a}{b}=\frac{d}{c}\Leftrightarrow\frac{a+b}{b}=\frac{d+c}{c}$$

$$K=\frac{(a+b)(b+c)(c+d)(d+a)}{ac(a+b)+cd(a+b)}=\frac{(b+c)(c+d)}{c}$$

$$\overset{AM-GM}{\ge}\frac{2\sqrt{bc}\cdot 2\sqrt{cd}}{c}=4\sqrt{bd}=4\sqrt{ac}.$$ **Equality holds at** $b=c=d=a$.

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$ab(c+d)+cd(a+b)=abc+abd+cda+cdb\overset{ac=bd}{=}abc+aac+$$
$$+cda+cac=ac(b+a+d+c)=ac(a+b+c+d)$$

$$(a+b)(b+c)(c+d)(d+a)=(ab+b^2+bc+ac)(ca+cd+d^2+da)\overset{ac=bd}{=}$$
$$=(ab+b^2+bc+bd)(bd+cd+d^2+da)=b(a+b+c+d)\cdot d(b+c+d+a)=$$
$$=bd(a+b+c+d)^2\overset{ac=bd}{=}ac(a+b+c+d)^2$$
$$\to\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)}=\frac{ac(a+b+c+d)^2}{ac(a+b+c+d)}$$
$$=a+b+c+d\overset{AM-GM}{\ge}4\sqrt[4]{abcd}\overset{ac=bd}{=}4\sqrt[4]{(ac)^2}=4\sqrt{ac}$$

Proved. Equality if and only if $a=b=c=d$

Solution 5 by Boris Colakovic-Belgrade-Serbie

$$ab(c+d)+cd(a+b)=ab\left(c+\frac{ac}{b}\right)+cd(a+b)=$$
$$=ac(a+b)+cd(a+b)=(a+b)(a+d)c$$

$$LHS=\frac{(b+c)(c+d)}{c}\overset{AM-GM}{\ge}\frac{4c\sqrt{bd}}{c}=4\sqrt{ac}$$

Solution 6 by Soumava Chakraborty-Kolkata-India

$$4\sqrt{ac} = 2\sqrt{ac} + 2\sqrt{ac} \overset{ac=bd}{=} 2\sqrt{ac} + 2\sqrt{bd} \overset{G\le A}{\le} a + c + b + d$$

$$\overset{?}{\le} \frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d) + cd(a+b)}$$

$$\Leftrightarrow (a+b)(b+c)(c+d)(d+a) \overset{?}{\ge} \{ab(c+d) + cd(a+b)\}(a+b+c+d)$$

$\Leftrightarrow (ac-bd)^2 \overset{?}{\ge} 0 \to$ ***true*** $\Rightarrow$ ***proposed inequality is true, equality when***

$ac = bd, a = c, b = d \Rightarrow$ ***when*** $a = b = c = d$ ***(Proved)***