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ABOUT PROBLEM X.58.

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By Marin Chirciu – Romania

1) Let $\Delta A'B'C'$ be the circumcevian triangle of orthocenter in acute ΔABC.

Prove that:

$$\sum \frac{AA'}{BA' \cdot CA'} \geq 2\left(\frac{1}{r} + \frac{1}{R}\right)$$

Proposed by Marian Ursărescu – Romania

Solution

We prove the following lemma:

Lemma.

2) Let $\Delta A'B'C'$ be the circumcevian triangle of orthocenter in acute ΔABC.

Prove that:

$$\sum \frac{AA'}{BA' \cdot CA'} = \frac{2}{R} \cdot \frac{s^2 - (3R^2 + 4Rr + r^2)}{s^2 - (2R + r)^2}$$

Proof.

Let D be the leg of the height from A on the side BC.

Using the power of the point D we obtain $AD \cdot DA' = BD \cdot DC$, wherefrom $DA' = \frac{BD \cdot DC}{AD}$

Then, $AA' = AD + DA' = h_a + \frac{BD \cdot DC}{h_a} = \frac{h_a^2 + BD \cdot DC}{h_a}$

Expressing the area of $\Delta A'BC$ in two ways we obtain:

$[A'BC] = \frac{BA' \cdot CA' \cdot \sin A}{2}$ *and* $[A'BC] = \frac{BC \cdot DA'}{2} = \frac{a \cdot \frac{BD \cdot DC}{h_a}}{2} = \frac{a \cdot BD \cdot DC}{2h_a}$*, wherefrom*

$$BA' \cdot CA' = \frac{a \cdot BD \cdot DC}{h_a \sin A}$$

We calculate $\sum \frac{AA'}{BA' \cdot CA'} = \sum \frac{\frac{h_a^2 + BD \cdot DC}{h_a}}{\frac{a \cdot BD \cdot DC}{h_a \sin A}} = \sum \frac{\sin A}{a} \cdot \frac{h_a^2 + BD \cdot DC}{BD \cdot DC} = \frac{1}{2R} \sum \frac{h_a^2 + BD \cdot DC}{BD \cdot DC} =$

$$=\frac{1}{2R}\left(\sum\frac{h_a^2}{BD\cdot DC}+3\right)=\frac{1}{2R}\left(\sum\frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}+3\right)=\frac{2}{R}\cdot\frac{s^2-(3R^2+4Rr+r^2)}{s^2-(2R+r)^2}$$

Which follows from

$$\sum\frac{h_a^2}{BD\cdot DC}=\sum\frac{h_a}{BD}\cdot\frac{h_a}{DC}=\sum\tan B\tan C=\frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}$$

Let's get back to the main problem.

Using the Lemma, the problem returns to proving the inequality:

$$\frac{2}{R}\cdot\frac{s^2-(3R^2+4Rr+r^2)}{s^2-(2R+r)^2}\geq 2\left(\frac{1}{r}+\frac{1}{R}\right)\Leftrightarrow\frac{s^2-(3R^2+4Rr+r^2)}{s^2-(2R+r)^2}\geq\frac{R+r}{r}\Leftrightarrow$$

$s^2\leq 4R^2+5Rr+r^2$, *right away from Gerretsen's inequality* $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that: $4R^2+4Rr+3r^2\leq 4R^2+5Rr+r^2\Leftrightarrow R\geq 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark: The inequality can be strengthened.

3) Let $\Delta A'B'C'$ be the circumcevian triangle of orthocenter in acute ΔABC. Prove that:

$$\sum\frac{AA'}{BA'\cdot CA'}\geq\frac{3}{r}$$

Proposed by Marin Chirciu – Romania

Solution

Using Lemma, the problem returns to proving the inequality:

$$\frac{2}{R}\cdot\frac{s^2-(3R^2+4Rr+r^2)}{s^2-(2R+r)^2}\geq\frac{3}{r}\Leftrightarrow(3R-2r)s^2\leq 3R(2R+r)^2-2r(3R^2+4Rr+r^2)$$

$\Leftrightarrow s^2(2R-3r)\leq 12R^3+6R^2r-5Rr^2-2r^3$, *which follows from Gerretsen's inequality* $s^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(2R-3r)\leq 12R^3+6R^2r-5Rr^2-2r^3\Leftrightarrow R^2-3Rr+2r^2\geq 0$$

$\Leftrightarrow (R-2r)(R-r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Refferences:

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**