R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT PROBLEM JP.235

16 RMM SPRING EDITION 2020

By Marin Chirciu – Romania

1) In $\Delta ABC; I$ – incenter; A', B', C' - lies on circumcircle such that:

$(A, I, A'), (B, I, B'), (C, I, C')$ are collinear. Prove that:

$$\frac{a}{IA'}+\frac{b}{IB'}+\frac{c}{IC'}\ge\frac{a+b+c}{R}$$

Proposed by Marian Ursărescu – Romania

Solution

We prove the following lemma:

Lemma

2) In $\Delta ABC; I$ – incenter; A', B', C' - lies on circumcircle such that:

$(A, I, A'), (B, I, B'), (C, I, C')$ are collinear. Prove that:

$$\frac{a}{IA'}+\frac{b}{IB'}+\frac{c}{IC'}=\frac{1}{2R}\sum\frac{a}{\sin\frac{A}{2}}$$

Proof.

Using the power of point I towards the circumcircle, we obtain:

$IA\cdot IA'=(R-OI)(R+OI)=R^2-OI^2=R^2-(R^2-2Rr)=2Rr$, so $IA\cdot IA'=2Rr$.

Using $IA=\frac{r}{\sin\frac{A}{2}}$, *it follows* $IA'=\frac{2Rr}{IA}=\frac{2Rr}{\frac{r}{\sin\frac{A}{2}}}=2R\sin\frac{A}{2}$, so $IA'=2R\sin\frac{A}{2}$

We obtain $\sum\frac{a}{IA'}=\sum\frac{a}{2R\sin\frac{A}{2}}=\frac{1}{2R}\sum\frac{a}{\sin\frac{A}{2}}$

Inequality from enunciation can be written:

$\frac{1}{2R}\sum\frac{a}{\sin\frac{A}{2}}\ge\frac{2s}{R}\Leftrightarrow\sum\frac{a}{\sin\frac{A}{2}}\ge 4s$, *which follows from:*

3) In ΔABC the following relationship holds:

$$\sum\frac{a}{\sin\frac{A}{2}}\ge 4s$$

Proof.

With means inequality we obtain:

$$\sum\frac{a}{\sin\frac{A}{2}}\geq 3\sqrt[3]{\prod\frac{a}{\sin\frac{A}{2}}}=3\sqrt[3]{\frac{abc}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}=3\sqrt[3]{\frac{4Rrs}{\frac{r}{4R}}}=3\sqrt[3]{16R^2s}=6\sqrt[3]{2R^2s}$$

It remains to prove that:

$6\sqrt[3]{2R^2s}\geq 4s\Leftrightarrow 3\sqrt[3]{2R^2s}\geq 2s\Leftrightarrow 27\cdot 2R^2s\geq 8s^3\Leftrightarrow 27R^2\geq 4s^2$, ***obviously from***

Mitrinovic's inequality $s^2\leq\frac{27R^2}{4}$

Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

4) In ΔABC; I ***– incenter;*** A', B', C' ***- lies on circumcircle such that:***

$(A,I,A'),(B,I,B'),(C,I,C')$ ***are collinear. Prove that:***

$$\frac{a}{IA'}+\frac{b}{IB'}+\frac{c}{IC'}\leq\frac{a+b+c}{2r}$$

Marin Chirciu

Solution

Using Lemma, the inequality can be written:

$$\frac{1}{2R}\sum\frac{a}{\sin\frac{A}{2}}\leq\frac{a+b+c}{2r}\Leftrightarrow\frac{1}{2R}\sum\frac{a}{\sin\frac{A}{2}}\leq\frac{2s}{2r}\Leftrightarrow\sum\frac{a}{\sin\frac{A}{2}}\leq\frac{2Rs}{r}$$

The triplets (a,b,c) ***and*** $\left(\frac{1}{\sin\frac{A}{2}},\frac{1}{\sin\frac{B}{2}},\frac{1}{\sin\frac{C}{2}}\right)$ ***are reverse ordered. With Cebyshev's***

inequality we obtain:

$$\sum\frac{a}{\sin\frac{A}{2}}\leq\frac{1}{3}\sum a\sum\frac{1}{\sin\frac{A}{2}}=\frac{1}{3}\cdot 2s\sum\frac{1}{\sin\frac{A}{2}}\leq\frac{2s}{3}\cdot\frac{2(R+r)}{r}=\frac{4s(R+r)}{3r}\leq\frac{2Rs}{r}$$

which follows from:

5) In ΔABC the following relationship holds:

$$\sum \frac{1}{\sin\frac{A}{2}} \le \frac{2(R+r)}{r}$$

Proof.

Using CBS inequality, we obtain:

$$\left(\sum \frac{1}{\sin\frac{A}{2}}\right)^2 = \left(\sum \frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}}\right)^2 =$$

$$= \left(\sum \frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}\right)^2 \le \sum bc \sum \frac{1}{(s-b)(s-c)} =$$

$$= (s^2+r^2+4Rr)\cdot\frac{1}{r^2} = \frac{s^2+r^2+4Rr}{r^2} \le \frac{4R^2+4Rr+3r^2+r^2+4Rr}{r^2} = \frac{4(R+r)^2}{r^2}$$

We've used:

$\sum bc = s^2+r^2+4Rr, \sum \frac{1}{(s-b)(s-c)} = \frac{1}{r^2}$ ***and*** $s^2 \le 4R^2+4Rr+3r^2$ ***(Gerretsen)***

It follows that $\sum \frac{1}{\sin\frac{A}{2}} \le \frac{2(R+r)}{r}$

Equality holds if and only if the triangle is equilateral.

Remark.

The double inequality can be written:

6) In ΔABC the following relationship holds:

$$\frac{a+b+c}{R} \le \frac{a}{IA'}+\frac{b}{IB'}+\frac{c}{IC'} \le \frac{a+b+c}{2r}$$

Solution

See inequalities 1) and 4).

Equality holds if and only if the triangle is equilateral.

Refferences:

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**