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ABOUT PROBLEM JP.206

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By Marin Chirciu – Romania

1) In ΔABC:

$$\frac{4r}{R^2}\le\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}\le\frac{R}{2r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution

We prove the following Lemma:

Lemma

2) In ΔABC:

$$\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}=\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof.

Using $h_a=\frac{2S}{a}$ and $r_a=\frac{S}{s-a}$ we obtain:

$$\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}=\sum\frac{\frac{2S}{a}}{\frac{S}{s-b}\cdot\frac{S}{s-c}}=\frac{2}{S}\sum\frac{(s-b)(s-c)}{a}$$

$$=\frac{2}{rs}\cdot\frac{r}{4Rs}\left[1+\left(\frac{4R+r}{s}\right)^2\right]=$$

$=\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$***, which follows from*** $\sum\frac{(s-b)(s-c)}{a}=\frac{r}{4Rs}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$

Let's get back to the main problem:

Using Lemma, the enunciation can be written: $\frac{4r}{R^2}\le\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{R}{2r^2}$

The right-hand inequality:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{R}{2r^2}\Leftrightarrow s^2(R^2-r^2)\ge r^2(4R+r)^2$***, which follows from Gerretsen's inequality*** $s^2\ge 16Rr-5r^2$***. It remains to prove that:***

$(16Rr-5r^2)(R^2-r^2)\geq r^2(4R+r)^2 \Leftrightarrow 16R^3-21R^2r-24Rr^2+4r^3\geq 0 \Leftrightarrow$

$(R-2r)(16R^2+11Rr-2r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

The left-hand inequality:

$$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\geq\frac{4r}{R^2}\Leftrightarrow R(4R+r)^2\geq s^2(8r-R)$$

We distinguish the following cases:

Case 1). If $(8r-R)<0$, *the inequality is obvious.*

Case 2). If $(8r-R)\geq 0$, *the inequality follows from Gerretsen's inequality:*

$s^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$R(4R+r)^2\geq(4R^2+4Rr+3r^2)(8r-R)\Leftrightarrow 5R^3-5R^2r-7Rr^2-6r^3\geq 0\Leftrightarrow$

$\Leftrightarrow(R-2r)(5R^2+5Rr+3r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

The double inequality 1) can be strengthened:

3) In ΔABC:

$$\frac{2}{R}\leq\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}\leq\frac{1}{R}+\frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

Solution

Using Lemma, the enunciation can be written: $\frac{2}{R}\leq\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\leq\frac{1}{R}+\frac{1}{2r}$.

The right-hand inequality:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\leq\frac{1}{R}+\frac{1}{2r}\Leftrightarrow s^2\geq\frac{r(4R+r)^2}{R+r}$, *which follows from Gerretsen's inequality*

$$s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$$

Equality holds if and only if the triangle is equilateral.

The left-hand inequality:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \geq \frac{2}{R} \Leftrightarrow (4R+r)^2 \geq 3s^2$, *(Doucet's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

A sequence of inequalities can be written:

4) In ΔABC:

$$\frac{4r}{R^2} \leq \frac{2}{R} \leq \frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} \leq \frac{1}{R} + \frac{1}{2r} \leq \frac{R}{2r^2}$$

Solution

See the double inequality 3) and Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

Interchanging r_a with h_a we can propose:

5) In ΔABC:

$$\frac{1}{r} \leq \frac{r_a}{h_b h_c} + \frac{r_b}{h_c h_a} + \frac{r_c}{h_a h_b} \leq \frac{1}{2r}\left(1 + \frac{R}{2r}\right)$$

Proposed by Marin Chirciu – Romania

Solution

We prove the following lemma:

Lemma

6) In ΔABC:

$$\frac{r_a}{h_b h_c} + \frac{r_b}{h_c h_a} + \frac{r_c}{h_a h_b} = \frac{1}{4r}\left[1 + \left(\frac{4R+r}{s}\right)^2\right]$$

Proof.

Using $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$$\frac{r_a}{h_b h_c} + \frac{r_b}{h_c h_a} + \frac{r_c}{h_a h_b} = \sum \frac{\frac{S}{s-a}}{\frac{2S}{a} \cdot \frac{2S}{c}} = \frac{1}{4S}\sum \frac{bc}{s-a} = \frac{1}{4rs} \cdot \frac{s^2 + (4R+r)^2}{s} =$$

$= \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$, *which follows from* $\sum \frac{bc}{s-a} = \frac{s^2+(4R+r)^2}{s}$

Let's get back to the main problem:

Using Lemma, the enunciation can be written: $\frac{1}{r} \le \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \le \frac{1}{2r}\left(1+\frac{R}{2r}\right)$

The right-hand inequality:

$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \le \frac{1}{2r}\left(1+\frac{R}{2r}\right) \Leftrightarrow s^2 \ge \frac{r(4R+r)^2}{R+r}$, *which follows from Gerretsen's inequality*

$$s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$$

Equality holds if and only if the triangle is equilateral.

The left-hand inequality:

$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \ge \frac{1}{r} \Leftrightarrow (4R+r)^2 \ge 3s^2$, *(Doucet's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\frac{h_a}{r_b r_c} + \frac{h_b}{r_c h_a} + \frac{h_c}{r_a r_b}$ *and* $\frac{r_a}{h_b h_c} + \frac{r_b}{h_c h_a} + \frac{r_c}{h_a h_b}$ *the following relationship holds:*

7) In ΔABC:

$$\frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} \le \frac{r_a}{h_b h_c} + \frac{r_b}{h_c h_a} + \frac{r_c}{h_a h_b}$$

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Solution

Using the above lemmas: $\sum \frac{h_a}{r_b r_c} = \frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$ *and* $\sum \frac{r_a}{h_b h_c} = \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$

the inequality can be written:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \le \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Refferences:
Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**