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ABOUT NAGEL AND GERGONNE'S CEVIANS

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ABSTRACT:

In this Math Note we will establish a new geometric identity in triangle:

$$4m_a^2 = n_a^2 + g_a^2 + 2r_br_c \text{ (and the analogs),}$$

where, n_a = Nagel's cevian from A, g_a = Gergonne's cevian from A, $m_a r_b r_c$ being the usual notations.

This identity was proposed as a problem for RMM Magazine (IDENTITY IN TRIANGLE 122).

In this note we will propose the way we've obtained this identity and some of its applications.

$$m_a^2 = \frac{2(b^2+c^2)-a^2}{4} \text{ (and the analogs)}$$

$$m_a^2 = \frac{2(b^2+c^2)-a^2}{4} = \frac{2(b^2+c^2)-a^2+4bc-4bc}{4} =$$

$$= \frac{(b^2+c^2+2bc)}{4} + \frac{(b^2+c^2+2bc)-a^2-4bc}{4}$$

$$= \frac{(b+c)^2}{4} + \frac{(b+c)^2}{4} - \frac{a^2+4bc}{4} = \frac{(a+b+c)(b+c-a)+(b^2+c^2+2bc-4bc)}{4} = \frac{2s\cdot 2(p-a)+(b-c)^2}{4}$$

$$= s(s-a) + \frac{(b-c)^2}{4} = r_br_c + \frac{(b-c)^2}{4} \text{ (and the analogs)} \Rightarrow$$

$$4(m_a^2 - r_br_c) = (b-c)^2 \text{ (and the analogs)}$$

1. **Let be $AE = n_a$ (Nagel's cevian from A);**

$BE = s-c$ and using cosine theorem in ΔABE we have:

$$n_a^2 = c^2 + (s-c)^2 - 2c(s-c)\cos B\,;\ \cos B = \frac{a^2+c^2-b^2}{2ac} \text{ (and the analogs)}$$

$$= s^2 - 2sc + 2c^2 - (s-c)\left(\frac{a^2+c^2-b^2}{a}\right) = s^2 - 2c(s-c) - (s-c)\frac{(a^2+c^2-b^2)}{a}$$

$$= s^2 - (s-c)\left(2c + \frac{a^2+c^2-b^2}{a}\right) = s^2 - (s-c)\left(\frac{a^2+c^2-b^2+2ac}{a}\right) =$$

$$= s^2 + \frac{4s(s-b)(s-c)}{a}$$

$$n_a^2 = s\left(s - a + a - \frac{4(s-b)(s-c)}{a}\right)$$

$$n_a^2 = s(s-a) + s\left(\frac{a^2 - 4(s-b)(s-c)}{a}\right)$$

But $a^2 - 4(s-b)(s-c) = (b-c)^2$ (and the analogs)

$n_a^2 = s(s-a) + \frac{(b-c)^2}{a}s$ (and the analogs)

2. Using Stewart's theorem, we obtain:

$g_a^2 = \frac{c^2(s-c)+b^2(s-b)-a(s-b)(s-c)}{a}$ (and the analogs)

$n_a^2 = \frac{b^2(s-c)+c^2(s-b)-a(s-c)(s-b)}{a}$ (and the analogs)

$$n_a^2 - g_a^2 = \frac{(b-c)}{a}(b^2 - c^2)$$

$$n_a^2 - m_a^2 = \frac{(b-c)^2}{4} + \frac{(b-c)^2(b+c)}{2a}$$

$$n_a^2 - m_a^2 = m_a^2 - r_b r_c + \frac{1}{2}(n_a^2 - g_a^2) \Rightarrow n_a^2 = 4m_a^2 - 2r_b r_c - g_a^2$$

3. Using the inequality between the squared mean and the arithmetic mean:

$\sqrt{\frac{n_a^2+g_a^2+2r_b r_c}{3}} \geq \frac{n_a+g_a+\sqrt{2r_b r_c}}{3}$ (and the analogs)

Which is equivalent with: $2\sqrt{3}m_a \geq n_a + g_a + \sqrt{2r_b r_c}$ (and the analogs), but $r_b r_c \geq w_a$ (and the analogs) $2\sqrt{3}m_a \geq n_a + g_a + w_a\sqrt{2}$ (and the analogs).

But we know that $\frac{b^2+c^2}{2bc} \geq \frac{m_a}{w_a}$ (and the analogs) $\Rightarrow \frac{2m_a}{w_a} \leq \frac{b}{c} + \frac{c}{b}$

$\frac{2m_a}{w_a} \geq \frac{n_a+g_a+\sqrt{r_b r_c}}{w_a\sqrt{3}}$ (and the analogs)

So, we'll obtain the inequality: $\frac{b}{c} + \frac{c}{b} \geq \frac{n_a+g_a+\sqrt{2r_b r_c}}{w_a\sqrt{3}}$ (and the analogs)

Summing we will obtain a new inequality, namely:

$$\frac{1}{\sqrt{3}}\sum \frac{n_a + g_a + \sqrt{2r_b r_c}}{w_a} \leq \sum \frac{b+c}{a}$$

But $\sum\frac{b+c}{a}=\sum\frac{h_b+h_c}{h_a}\Rightarrow\frac{1}{\sqrt{3}}\sum\frac{n_a+g_a+\sqrt{r_br_c}}{w_a}\le\sum\frac{h_b+h_c}{h_a}$

But $\frac{r_a+r}{r_a-r}=\frac{b+c}{a}$ and the analogs $\Rightarrow\frac{1}{\sqrt{3}}\sum\frac{n_a+g_a+\sqrt{2r_br_c}}{w_a}\le\sum\frac{r_a+r}{r_{a-r}}$

$1+\frac{h_a}{r_r}=\frac{b+c}{a}$ (and the analogs) $\Rightarrow$

$$\Rightarrow\sum\frac{h_a}{r_a}\ge\frac{1}{\sqrt{3}}\sum\frac{n_a+g_a+\sqrt{2r_br_c}}{w_a}-3$$

4. We know that $3(a^2+b^2+c^2)=4\sum m_a^2$

$$4(m_a^2+m_b^2+m_c^2)=\sum n_a^2+\sum g_a^2+2\sum r_br_c$$

$$\sum r_br_c=s^2\Rightarrow 3(a^2+b^2+c^2)=\sum n_a^2+\sum g_a^2+2s^2$$

But $a^2=2R\frac{h_bh_c}{h_a}$ (and the analogs), so we will obtain a new identity, namely:

$$\sum\frac{h_bh_c}{h_a}=\frac{\sum n_a^2+\sum g_a^2+2s^2}{6R}$$

5. $m_a\ge\frac{b^2+c^2}{4R}$ (Tereshin Inequality) $\Rightarrow 4Rm_a\ge b^2+c^2$

$16R^2m_a^2\ge(b^2+c^2)^2$ (and the analogs) $\Leftrightarrow 16R^2\frac{3}{4}(a^2+b^2+c^2)\ge\sum(b^2+c^2)^2$

$$4R^2\left(\sum n_a^2+\sum g_a^2+2s^2\right)\ge\sum(b^2+c^2)^2$$

$$\Rightarrow 2R\ge\sqrt{\frac{\sum(b^2+c^2)^2}{\sum n_a^2+\sum g_a^2+2s^2}}$$

$h_a=\frac{2r_br_c}{r_b+r_c}$ (and the analogs); $2r_br_c=4m_a^2-n_a^2-g_a^2\Rightarrow h_a=\frac{4m_a^2-n_a^2-g_a^2}{r_b+r_c}$

6. From the above expressions of n_a^2 and m_a^2 we can easily observe that

$n_a\ge m_a$ (and the analogs)

So, we can write that:

$h_a\le\frac{3m_a^2-g_a^2}{r_b+r_c}\Rightarrow h_a+h_b+h_c\le\sum\frac{3m_a^2-g_a^2}{r_b+r_c}$;

7. We know that $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$

So, $\frac{1}{r}=\sum\frac{r_b+r_c}{4m_a^2-n_a^2-g_a^2}\ \geq\sum\frac{r_b+r_c}{3m_a^2-g_a^2}$

Taking into account the above we can write that:

$$r_b+r_c=\frac{4m_a^2-n_a^2-g_a^2}{h_a}$$

Summing we will obtain the identity:

$$2(r_a+r_b+r_c)=\sum\frac{4m_a^2-n_a^2-g_a^2}{h_a}$$

$$2(r_a+r_b+r_c)\leq\sum\frac{3m_a^2-g_a^2}{h_a}$$

8. We know that $m_a\geq\frac{(b+c)}{2}\cos\frac{A}{2}$; $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$; $\cos\frac{A}{2}=\sqrt{\frac{r_b+r_c}{4R}}$

$bc=2Rh_a$ (and the analogs)

$$m_aw_a\ \geq\frac{(b+c)}{2}\cos\frac{A}{2}\ \frac{2bc}{b+c}\cos\frac{A}{2}=bc\ \frac{r_b+r_c}{4R}\ =h_a\ \frac{(r_b+r_c)}{2}$$

$$\frac{2m_aw_a}{h_a}\geq r_b+r_c=\frac{4m_a^2-n_a^2-g_a^2}{h_a}\Rightarrow n_a^2+g_a^2\geq 2m_a(2m_a-w_a)\Rightarrow$$

$$\Rightarrow\frac{n_a^2+g_a^2}{2m_a}\geq 2m_a-w_a$$

$$\frac{1}{2}\sum\frac{n_a^2+g_a^2}{m_a}\geq 2\sum m_a-\sum w_a$$

From the above we can write that: $\sum(n_a^2+g_a^2)=2\sum r_br_b+\sum(b-c)^2$

9. We know that $\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$ (and the analogs); $\cos\frac{B-C}{2}=\frac{h_a}{w_a}\Rightarrow m_a\sqrt{\frac{R}{2r}}\geq m_a\frac{w_a}{h_a}$

From the above we will have: $m_a\sqrt{\frac{R}{2r}}\geq\frac{r_b+r_c}{2}$ (and the analogs) $\Rightarrow m_a\sqrt{\frac{R}{2r}}\geq\frac{4m_a^2-n_a^2-g_a^2}{2h_a}$

From the above we will have: $m_a\sqrt{\frac{R}{2r}}\geq\frac{r_b+r_c}{2}$ (and the analogs) $\Rightarrow m_a\sqrt{\frac{R}{2r}}\geq\frac{4m_a^2-n_a^2-g_a^2}{2h_a}$

$$m_a\sqrt{\frac{2R}{r}}\geq\frac{4m_a^2-n_a^2-g_a^2}{h_a}$$

Summing we will obtain: $(m_a + m_b + m_c)\sqrt{\frac{2R}{r}} \ge \sum \frac{4m_a^2 - n_a^2 - g_a^2}{h_a}$

$$h_a\sqrt{\frac{2R}{r}} \ge \frac{4m_a^2 - n_a^2 - g_a^2}{m_a} = 4m_a - \frac{n_a^2 + g_a^2}{m_a}$$

Summing we will have the following inequality:

$$(h_a + h_b + h_c)\sqrt{\frac{2R}{r}} \ge 4(m_a + m_b + m_c) - \sum \frac{n_a^2 + g_a^2}{m_a}$$

10. From inequality $\frac{n_a^2+g_a^2}{2m_a} \ge 2m_a - w_a \Rightarrow$

$$\frac{n_a^2 + g_a^2}{2m_a - w_a} \ge 2m_a \Rightarrow \sum \frac{n_a^2 + g_a^2}{2m_a - w_a} \ge 2(m_a + m_b + m_c)$$

$m_a \ge \frac{b^2+c^2}{4R}$ **(Tereshin's inequality)** $\Rightarrow \frac{m_a}{h_a} \ge \frac{b^2+c^2}{2bc} = \frac{1}{2}\left(\frac{b}{c} + \frac{c}{b}\right) \Rightarrow \sum \frac{m_a}{h_a} \ge \frac{1}{2}\sum \frac{b+c}{a}$

$$\frac{1}{\sqrt{3}}\sum \frac{n_a + g_a + \sqrt{2r_br_c}}{w_a} \le \sum \frac{b+c}{a};$$

$$\frac{1}{\sqrt{3}}\sum \frac{n_a + g_a + \sqrt{2r_br_c}}{l_a} \le \sum \frac{b+c}{a} \le 2\sum \frac{m_a}{h_a}$$

11. $\frac{R}{2r} \ge \frac{m_a}{h_a}$ **(Laurențiu Panaitopol – Romanian Mathematical Gazzete)**

$$\Rightarrow \frac{3R}{2r} \ge \sum \frac{m_a}{h_a} \ge \frac{1}{2\sqrt{3}}\sum \frac{n_a+g_a+\sqrt{2r_br_c}}{w_a};$$

So, finally, we have as being true a new inequality, namely:

$$\frac{R}{r} \ge \sum \frac{m_a}{h_a} \ge \frac{1}{3\sqrt{3}}\sum \frac{n_a+g_a+\sqrt{2r_br_c}}{w_a};$$

So, $m_a^2 = r_br_c + \frac{(b-c)^2}{4}$ **(and analogs);** $4m_a^2 = n_a^2 + g_a^2 + 2r_br_c$ **(and analogs)**

$bc = r_br_c + rr_a$ **(and the analogs);** $n_a^2 + g_a^2 + 2r_br_c - 4r_br_c = (b-c)^2;$

$$n_a^2 + g_a^2 - 2r_br_c = b^2 + c^2 - 2bc$$

$b^2 + c^2 - n_a^2 - g_a^2 = 2rr_a = 2(s-b)(s-c)$ **(and the analogs)**

12. We know that: $\sin\frac{A}{2} = \sqrt{\frac{r_ar}{bc}} = \sqrt{\frac{r_a+r}{4R}}; bc = 2Rh_a \Rightarrow$

$2rr_a = h_a(r_a - r)$ **(and the analogs)**

$$b^2+c^2-n_a^2-g_a^2=h_a(r_a-r);\sum\frac{r_a}{h_a}=\frac{2R}{r}-1;\ \frac{b^2+c^2-n_a^2-g_a^2}{h_a^2}=\frac{h_a(r_a-r)}{h_a^2}\Rightarrow$$

$\Rightarrow\sum\frac{b^2+c^2-n_a^2-g_a^2}{h_a^2}=\frac{2R}{r}-1-\sum\frac{r}{h_a};\sum\frac{1}{h_a}=\frac{1}{r}$, so, we have the following:

$$\sum\frac{b^2+c^2-n_a^2-g_a^2}{h_a^2}=2\left(\frac{R}{r}-1\right);$$

From the above we have: $\frac{b^2+c^2-n_a^2-g_a^2}{h_a}=r_a-r$ (and the analogs),

but $r_a+r_b+r_c=4R+r$

So, we have the identity: $\sum\frac{b^2+c^2-n_a^2-g_a^2}{h_a}=2(2R-r)$;

13. We proved that: $b^2+c^2-n_a^2-g_a^2=h_a(r_a-r)$ (and the analogs)

But $n_a^2+g_a^2\ge 2n_ag_a$ (and the analogs) $\Rightarrow$

$b^2+c^2-2n_ag_a\ge h_a(r_a-r)$ (and the analogs)

$2rr_a=h_a(r_a-r)\Rightarrow b^2+c^2-2n_ag_a\ge 2rr_a$ (and the analogs)

$\frac{b^2+c^2}{2}\ge n_ag_a+rr_a$ (and the analogs)

Taking into account the above inequality we have: $\frac{1}{8}\prod(b^2+c^2)\ge\prod(n_ag_a+rr_a)$

From the above we have: $8r_ar_br_cr^3=\prod(b^2+c^2-n_a^2-g_a^2)$

But $r_ar_br_c=Ss=s^2r$ we will obtain the following:

$$8S^2r^2=\prod(b^2+c^2-n_a^2-g_a^2);$$

$$8s^2r^4=\prod(b^2+c^2-n_a^2-g_a^2);$$

$h_ah_bh_c=\frac{2r}{R}r_ar_br_c$ we will obtain the following: $h_ah_bh_c=\frac{1}{4Rr^2}\prod(b^2+c^2-n_a^2-g_a^2)$

14. From $m_a\ge\frac{b^2+c^2}{4R}$ and $a^2=2R\frac{h_bh_c}{h_a}$ (and the analogs), we will obtain:

$$m_a\ge\frac{h_a}{2}\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right);2r_br_c=h_a(r_b+r_c)$$

$\frac{4m_a^2}{h_a^2}\ge\left(\frac{h_b}{h_c}\right)^2+\left(\frac{h_c}{h_b}\right)^2+2$ (and the analogs)

$\frac{n_a^2+g_a^2}{h_a^2}+\frac{2r_br_c}{h_a^2}\ge 2+\left(\frac{h_b}{h_c}\right)^2+\left(\frac{h_c}{h_b}\right)^2\Rightarrow\frac{n_a^2+g_a^2}{h_a^2}+\frac{r_b+r_c}{h_a}\ge 2+\left(\frac{h_b}{h_c}\right)^2+\left(\frac{h_c}{h_b}\right)^2$ (and the analogs)

$$(r_a+r_b+r_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\sum\frac{r_a}{h_a}+\sum\frac{r_b+r_c}{h_a}\Rightarrow\frac{4R+r}{r}=\frac{2R}{r}-1+\sum\frac{r_b+r_c}{h_a}$$

So, finally we have the identity: $\sum\frac{r_b+r_c}{h_a}=2\left(1+\frac{R}{r}\right)$

Taking into account the above we have the following inequality:

$$2\left(1+\frac{R}{r}\right)+\sum\frac{n_a^2+g_a^2}{h_a^2}\geq 6+\sum\frac{h_b^2+h_c^2}{h_a^2}$$

$$2\left(1+\frac{R}{r}\right)+\sum\frac{n_a^2+g_a^2-h_b^2-h_c^2}{h_a^2}\geq 6$$

$m_a\geq\frac{b+c}{2}\cos\frac{A}{2}$ (and the analogs)

$\cos\frac{A}{2}=\sqrt{\frac{r_br_c}{bc}}$ (and the analogs)

Squaring and taking into account:

$4m_a^2=n_a^2+g_a^2+2r_br_c$ (and the analogs)

$$n_a^2+g_a^2+2r_br_c\geq(b+c)^2\frac{r_br_c}{bc};$$

$2+\frac{n_a^2+g_a^2}{r_br_c}\geq 2+\frac{b^2+c^2}{bc}\Rightarrow\frac{n_a^2+g_a^2}{r_br_c}\geq\frac{b^2+c^2}{bc}$ (and the analgos)

So, finally we have:

$\frac{n_a^2+g_a^2}{r_br_c}\geq\frac{b}{c}+\frac{b}{c}$ (and the analogs). Summing we will obtain:

$$\sum\frac{n_a^2+g_a^2}{r_br_c}\geq\sum\frac{b+c}{a}$$

15. But $\sum\frac{b+c}{a}=\sum\frac{h_b+h_c}{h_a}=\sum\frac{r_a+r}{r_a-r}$, so we will obtain the following:

$$\sum\frac{n_a^2+g_a^2}{r_br_c}\geq\sum\frac{h_b+h_c}{h_a};$$

$$\sum\frac{n_a^2+g_a^2}{r_br_c}\geq\sum\frac{r_a+r}{r_a-r};$$

$\frac{n_a^2+g_a^2}{b^2+c^2}\geq\left(\cos\frac{A}{2}\right)^2$ (and the analogs); $\cos\frac{A}{2}=\sqrt{\frac{r_b+r_c}{4R}}$ (and the analogs)

$$\Rightarrow\sum\left(\cos\frac{A}{2}\right)^2=\frac{r_a+r_b+r_c}{2R}=2+\frac{r}{2R}$$

$$\sum \frac{n_a^2 + g_a^2}{b^2 + c^2} \geq 2 + \frac{r}{2R}$$

But $\prod \cos\frac{A}{2} = \frac{s}{4R}$ **so, we will have the inequality:** $\prod \frac{n_a^2+g_a^2}{b^2+c^2} \geq \frac{p^2}{16R^2}$

16. From Tereshin's inequality presented above and the new identity we will obtain the following:

$$n_a^2 + g_a^2 + 2r_b r_c \geq \left(\frac{b^2+c^2}{2R}\right)^2 \text{ (and the analogs)}$$

But $\frac{b^2+c^2}{2bc} \geq \frac{m_a}{l_a}$ **(and the analogs) and from the above we can write the following:**

$$\frac{n_a^2+g_a^2}{r_b r_c} \geq \frac{b}{c} + \frac{b}{c} \geq \frac{2m_a}{w_a} \text{ (and the following)}$$

$$\frac{n_a^2+g_a^2}{r_b r_c} \geq \frac{2m_a}{w_a} \text{ we have the following: } \frac{n_a^2+g_a^2}{m_a} \geq \frac{2r_b r_c}{w_a} \text{ (and the following)}$$

Summing we will obtain a new inequality, namely: $\sum \frac{n_a^2+g_a^2}{m_a} \geq 2\sum \frac{r_b r_c}{l_a}$

$$\frac{n_a^2+g_a^2}{b^2+c^2} \geq \left(\cos\frac{A}{2}\right)^2 \text{ (and the analogs)} \Rightarrow \frac{n_a^2+g_a^2}{b^2+c^2} \geq \frac{r_b+r_c}{4R} \text{ (and the analogs)}$$

$$\cos\frac{A}{2} = \sqrt{\frac{r_b+r_c}{4R}} \text{ (and the analogs)}$$

From the above we have: $\frac{n_a^2+g_a^2}{r_b+r_c} \geq \frac{b^2+c^2}{4R}$ **(and the analogs)**

Summing we will obtain: $\sum \frac{n_a^2+g_a^2}{r_b+r_c} \geq \frac{a^2+b^2+c^2}{2R}$

17. From $a^2 = 2R\frac{h_b h_c}{h_a}$ **(and the analogs) we have** $\sum \frac{n_a^2+g_a^2}{r_b+r_c} \geq \sum \frac{h_b h_c}{h_a}$

$$3(a^2 + b^2 + c^2) = \sum n_a^2 + \sum g_a^2 + 2p^2 \Rightarrow \sum \frac{n_a^2 + g_a^2}{r_b + r_c} \geq \frac{\sum n_a^2 + \sum g_a^2 + 2p^2}{6R}$$

These are just a little part of the relationships that can be obtained using the established relationship in this note. Any other new obtained relationships will be published in the form of problems for RMM.

Refferences:

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**