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ABOUT BĂTINEȚU'S INEQUALITIES

PROBLEMS X.64, X.65, X.74-RMM 24-SPRING EDITION 2020-PAPER VARIANT

By Marin Chirciu – Romania

1) X.64. BĂTINEȚU'S INEQUALITY – 1

If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{y+z}{x}\cdot a^2+\frac{z+x}{y}\cdot b^2+\frac{x+y}{z}\cdot c^2\ge 8\sqrt{3}\cdot S$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Proof.

We prove:

Lemma

2) If $x, y, z > 0$ then in ΔABC

$$\frac{y+z}{x}\cdot a^2+\frac{z+x}{y}\cdot b^2+\frac{x+y}{z}\cdot c^2\ge 2\sum bc$$

Proof.

$$M_s=\sum\frac{y+z}{x}\cdot a^2=\sum\left(\frac{y+z}{x}+1-1\right)\cdot a^2=\sum\frac{x+y+z}{x}a^2-\sum a^2=$$

$$=(x+y+z)\sum\frac{a^2}{x}-\sum a^2\overset{Bergstrom}{\ge}(x+y+z)\frac{(\sum a)^2}{x+y+z}-\sum a^2=$$

$$=\left(\sum a\right)^2-\sum a^2=\sum a^2+2\sum bc-\sum a^2=2\sum bc$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$2\sum bc\ge 8\sqrt{3}\cdot S\Leftrightarrow\sum bc\ge 4\sqrt{3}\cdot S\Leftrightarrow\left(\sum bc\right)^2\ge 48S^2\Leftrightarrow$$

$$\Leftrightarrow(s^2+r^2+4Rr)^2\ge 48r^2s^2\Leftrightarrow s^2(s^2+8Rr-46r^2)+r^2(4R+r)^2\ge 0$$

We've used the known identity in triangle $\sum bc=s^2+r^2+4Rr$

We distinguish the cases:

Case 1). If $(s^2+8Rr-46r^2)\ge 0$, the inequality is obvious.

Case 2). If $(s^2 + 8Rr - 46r^2) < 0$, the inequality can be rewritten:

$r^2(4R + r)^2 \geq s^2(46r^2 - 8Rr - s^2)$, *which follows from Blundon-Gerretsen's inequality*

$16Rr - 5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$r^2(4R + r)^2 \geq \frac{R(4R + r)^2}{2(2R - r)}(46r^2 - 8Rr - 16Rr + 5r^2) \Leftrightarrow 24R^2 - 47Rr - 2r^2 \geq 0$$

$\Leftrightarrow (R - 2r)(24R + r) \geq 0$, *obviously from Euler's inequality $R \geq 2r$.*

Equality holds if and only if the triangle is equilateral.

3) X.65. BĂTINEȚU INEQUALITY – 2

If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{y + z}{x} \cdot a^4 + \frac{z + x}{y} \cdot b^4 + \frac{x + y}{z} \cdot c^4 \geq 32S^2$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution

We prove:

Lemma.

4) If $x, y, z > 0$ then in ΔABC:

$$\frac{y + z}{x} \cdot a^4 + \frac{z + x}{y} \cdot b^4 + \frac{x + y}{z} \cdot c^4 \geq 2\sum b^2c^2$$

Proof.

$$M_s = \sum \frac{y + z}{x} \cdot a^4 = \sum \left(\frac{y + z}{x} + 1 - 1\right) \cdot a^4 = \sum \frac{x + y + z}{x} a^4 - \sum a^4 =$$

$$= (x + y + z)\sum \frac{a^4}{x} - \sum a^4 \overset{Bergstrom}{\geq} (x + y + z)\frac{(\sum a^2)^2}{x + y + z} - \sum a^4 =$$

$$= \left(\sum a^2\right)^2 - \sum a^4 = \sum a^4 + 2\sum b^2c^2 - \sum a^4 = 2\sum b^2c^2$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$2\sum b^2c^2 \geq 32S^2 \Leftrightarrow \sum b^2c^2 \geq 16S^2 \Leftrightarrow s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2 \geq$$

$$\geq 16r^2s^2 \Leftrightarrow s^2(s^2-8Rr-14r^2)+r^2(4R+r)^2 \geq 0$$

We've used the known identity in triangle $\sum b^2c^2 = s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2$

We distinguish the cases:

Case 1). If $(s^2-8Rr-14r^2) \geq 0$*, the inequality is obvious.*

Case 2). If $(s^2-8Rr-14r^2) < 0$*, the inequality can be rewritten:*

$r^2(4R+r)^2 \geq s^2(8Rr+14r^2-s^2)$*, which follows from Blundon-Gerretsen's inequality*

$16Rr-5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$*. It remains to prove that:*

$$r^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(8Rr+14r^2-16Rr+5r^2) \Leftrightarrow 8R^2-17Rr-2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(8R+r) \geq 0$*, obviously from Euler's inequality* $R \geq 2r$*.*

Equality holds if and only if the triangle is equilateral.

5) X.74. If $x,y,z > 0$ then in ΔABC the following relationship holds:

$$\left(\frac{y^2+z^2}{x^2}+\frac{z^2+x^2}{y^2}+\frac{x^2+y^2}{z^2}\right)(a^4+b^4+c^4) \geq 96S^2$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

Solution

We prove:

Lemma.

6) In ΔABC:

$$a^4+b^4+c^4 \geq 16S^2$$

F. Goldner, 1949

Proof.

Using $\sum a^4 = 2[s^4-s^2(8Rr+6r^2)+r^2(4R+r)^2]$*, we have* $a^4+b^4+c^4 \geq 16S^2 \Leftrightarrow$

$$\Leftrightarrow 2[s^4-s^2(8Rr+6r^2)+r^2(4R+r)^2] \geq 16r^2s^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2-8Rr-14r^2)+r^2(4R+r)^2 \geq 0.$$

We distinguish the cases:

Case 1). If $(s^2-8Rr-14r^2) \geq 0$*, the inequality is obvious.*

Case 2). If $(s^2 - 8Rr - 14r^2) < 0$, the inequality can be rewritten:

$r^2(4R + r)^2 \geq s^2(8Rr + 14r^2 - s^2)$*, which follows from Blundon-Gerretsen inequality*

$16Rr - 5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$*. It remains to prove that:*

$$r^2(4R + r)^2 \geq \frac{R(4R + r)^2}{2(2R - r)}(8Rr + 14r^2 - 16Rr + 5r^2) \Leftrightarrow 8R^2 - 17Rr - 2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(8R + r) \geq 0$*, obviously from Euler's inequality $R \geq 2r$.*

Equality holds if and only if the triangle is equilateral.

Let's get back to the main problem:

We have $\frac{y^2+z^2}{x^2} + \frac{z^2+x^2}{y^2} + \frac{x^2+y^2}{z^2} \geq 6$, which follows from $\frac{x^2}{y^2} + \frac{y^2}{x^2} \geq 2$, with equality for $x = y$ and the analogs.

Using the Lemma and the above inequality we obtain the conclusion.

Equality holds if and only if the triangle is equilateral and $x = y = z$.

Remark.

In the same way we can propose:

7) If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{y + z}{x} \cdot (b + c)^2 + \frac{z + x}{y} \cdot (c + a)^2 + \frac{x + y}{z} \cdot (a + b)^2 \geq 32\sqrt{3} \cdot S$$

Proposed by Marin Chirciu – Romania

Solution.

We prove:

Lemma.

8) If $x, y, z > 0$ the in ΔABC:

$$\frac{y + z}{x} \cdot (b + c)^2 + \frac{z + x}{y} \cdot (c + a)^2 + \frac{x + y}{z} \cdot (a + b)^2 \geq 8s^2 + 2\sum bc$$

Proof.

$$M_s = \sum \frac{y + z}{x} \cdot (b + c)^2 = \sum \left(\frac{y + z}{x} + 1 - 1\right) \cdot (b + c)^2 =$$

$$=\sum\frac{x+y+z}{x}(b+c)^2-\sum(b+c)^2=(x+y+z)\sum\frac{(b+c)^2}{x}-\sum(b+c)^2\ge$$

$$\overset{Bergstrom}{\ge}(x+y+z)\frac{(2\sum a)^2}{x+y+z}-\sum(b+c)^2=4\left(\sum a\right)^2-\sum(b^2+c^2+2bc)=$$

$$=4\sum a^2+8\sum bc-2\sum a^2-2\sum bc=2\sum a^2+6\sum bc=8s^2+2\sum bc$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$8s^2+2\sum bc\ge 32\sqrt{3}\cdot S\Leftrightarrow 5s^2+4Rr+r^2\ge 16\sqrt{3}\cdot S\Leftrightarrow$$

$$\Leftrightarrow(5s^2+4Rr+r^2)^2\ge 768r^2s^2\Leftrightarrow$$

$$\Leftrightarrow s^2(25s^2+40Rr-758r^2)+r^2(4R+r)^2\ge 0$$

We've used the known identity in triangle $\sum bc=s^2+r^2+4Rr$.

We distinguish the cases:

Case 1). If $(25s^2+40Rr-758r^2)\ge 0$*, the inequality is obvious.*

Case 2). If $(25s^2+40Rr-758r^2)<0$*, the inequality can be rewritten:*

$r^2(4R+r)^2\ge s^2(46r^2-8Rr-s^2)$*, which follows from Blundon-Gerretsen inequality*

$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$*. It remains to prove that:*

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}\left(758r^2-40Rr-25(16Rr-5r^2)\right)\Leftrightarrow$$

$\Leftrightarrow 440R^2-879Rr-2r^2\ge 0\Leftrightarrow(R-2r)(440R+r)\ge 0$*, obvious from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

9) If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{y+z}{x}\cdot(b+c)^4+\frac{z+x}{y}\cdot(c+a)^4+\frac{x+y}{z}\cdot(a+b)^4\ge 512S^2$$

Proposed by Marin Chirciu – Romania

Solution

We prove:

Lemma

10) If $x, y, z > 0$ then in ΔABC:

$$\frac{y+z}{x}\cdot(b+c)^4+\frac{z+x}{y}\cdot(c+a)^4+\frac{x+y}{z}\cdot(a+b)^4\ge 2\sum(a+b)^2(a+c)^2$$

Proof.

$$M_s=\sum\frac{y+z}{x}\cdot(b+c)^4=\sum\left(\frac{y+z}{x}+1-1\right)\cdot(b+c)^4=$$

$$=\sum\frac{x+y+z}{x}(b+c)^4-\sum(b+c)^4=(x+y+z)\sum\frac{(b+c)^4}{x}-\sum(b+c)^4\ge$$

$$\overset{Bergstrom}{\ge}(x+y+z)\frac{(\sum(b+c)^2)^2}{x+y+z}-\sum(b+c)^4=\left(\sum(b+c)^2\right)^2-\sum(b+c)^4=$$

$$=\sum(b+c)^4+2\sum(a+b)^2(a+c)^2-\sum(b+c)^4=2\sum(a+b)^2(a+c)^2$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$2\sum(a+b)^2(a+c)^2\ge 512S^2\Leftrightarrow\sum(a+b)^2(a+c)^2\ge 256S^2\Leftrightarrow$$

$$\Leftrightarrow 9s^4+s^2(8Rr-6r^2)+r^2(4R+r)^2\ge 256r^2s^2\Leftrightarrow$$

$$\Leftrightarrow s^2(9s^2+8Rr-262r^2)+r^2(4R+r)^2\ge 0$$

We've used the known identity in triangle

$$\sum(a+b)^2(a+c)^2=9s^4+s^2(8Rr-6r^2)+r^2(4R+r)^2$$

We distinguish the cases:

Case 1). If $(9s^2+8Rr-262r^2)\ge 0$, the inequality is obvious.

Case 2). If $(9s^2+8Rr-262r^2)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\ge s^2(262r^2-8Rr-9s^2)$***, which follows from Blundon-Gerretsen***

$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$***. It remains to prove that:***

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}\left(262r^2-8Rr-9(16Rr-5r^2)\right)\Leftrightarrow$$

$\Leftrightarrow 152R^2-303Rr-2r^2\ge 0\Leftrightarrow(R-2r)(152R+r)\ge 0$***, obvious from Euler's inequality $R\ge 2r$.***

Equality if and only if the triangle is equilateral.

11) If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{y+z}{x}\cdot b^2c^2+\frac{z+x}{y}\cdot c^2a^2+\frac{x+y}{z}\cdot a^2b^2\geq 32S^2$$

Proposed by Marin Chirciu – Romania

Solution

We prove:

Lemma.

12) If $x, y, z > 0$ then in ΔABC:

$$\frac{y+z}{x}\cdot b^2c^2+\frac{z+x}{y}\cdot c^2a^2+\frac{x+y}{z}\cdot a^2b^2\geq 2abc\sum a$$

Proof.

$$M_s=\sum\frac{y+z}{x}\cdot b^2c^2=\sum\left(\frac{y+z}{x}+1-1\right)\cdot b^2c^2=\sum\frac{x+y+z}{x}b^2c^2-\sum b^2c^2=$$

$$=(x+y+z)\sum\frac{b^2c^2}{x}-\sum b^2c^2\overset{Bergstrom}{\geq}(x+y+z)\frac{(\sum bc)^2}{x+y+z}-\sum b^2c^2=$$

$$=\left(\sum bc\right)^2-\sum b^2c^2=\sum b^2c^2+2abc\sum a-\sum b^2c^2=2abc\sum a$$

Let's get to the main problem:

Using the Lemma, it suffices to prove that:

$2abc\sum a\geq 32S^2\Leftrightarrow 8Rrs^2\geq 16r^2s^2\Leftrightarrow R\geq 2r$ ***(Euler's inequality)***

Equality holds if and only if the triangle is equilateral.

13) If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{y+z}{x}\cdot r_b^2r_c^2+\frac{z+x}{y}\cdot r_c^2r_a^2+\frac{x+y}{z}\cdot r_a^2r_b^2\geq 18S^2$$

Proposed by Marin Chirciu – Romania

Solution

We prove:

Lemma.

14) If $x,y,z>0$ then in ΔABC:

$$\frac{y+z}{x}\cdot r_b^2r_c^2+\frac{z+x}{y}\cdot r_c^2r_a^2+\frac{x+y}{z}\cdot r_a^2r_b^2\ge 2r_ar_br_c\sum r_a$$

Proof.

$$M_s=\sum\frac{y+z}{x}\cdot r_b^2r_c^2=\sum\left(\frac{y+z}{x}+1-1\right)\cdot r_b^2r_c^2=\sum\frac{x+y+z}{x}r_b^2r_c^2=$$

$$=(x+y+z)\sum\frac{r_b^2r_c^2}{x}-\sum r_b^2r_c^2\overset{Bergstrom}{\ge}(x+y+z)\frac{(\sum r_br_c)^2}{x+y+z}-\sum r_b^2r_c^2=$$

$$=\left(\sum r_br_c\right)^2-\sum r_b^2r_c^2=\sum r_b^2r_c^2+2r_ar_br_c\sum r_a-\sum r_b^2r_c^2=2r_ar_br_c\sum r_a$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$2r_ar_br_c\sum r_a\ge 18S^2\Leftrightarrow r_ar_br_c\sum r_a\ge 9r^2s^2\Leftrightarrow 2rs^2(4R+r)\ge 9r^2s^2\Leftrightarrow R\ge 2r$$

(Euler)

Equality holds if and only if the triangle is equilateral.

15) If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{y+z}{x}\cdot r_a^2+\frac{z+x}{y}\cdot r_b^2+\frac{x+y}{z}\cdot r_c^2\ge 6\sqrt{3}\cdot S$$

Proposed by Marin Chirciu – Romania

Solution

We prove:

Lemma:

16) If $x,y,z>0$ then in ΔABC:

$$\frac{y+z}{x}\cdot r_a^2+\frac{z+x}{y}\cdot r_b^2+\frac{x+y}{z}\cdot r_c^2\ge 2\sum r_br_c$$

Proof.

$$M_s=\sum\frac{y+z}{x}\cdot r_a^2=\sum\left(\frac{y+z}{x}+1-1\right)\cdot r_a^2=\sum\frac{x+y+z}{x}r_a^2-\sum r_a^2=$$

$$=(x+y+z)\sum\frac{r_a^2}{x}-\sum r_a^2\overset{Bergstrom}{\ge}(x+y+z)\frac{(\sum r_a)^2}{x+y+z}-\sum r_a^2=$$

$$= \left(\sum r_b r_c\right)^2 - \sum r_a^2 = \sum r_a^2 + 2\sum r_b r_c - \sum r_a^2 = 2\sum r_b r_c$$

Let's get back to the main problem:

Using the Lemma, it suffices to prove that:

$$2\sum r_b r_c \ge 6\sqrt{3}\cdot S \Leftrightarrow \sum r_b r_c \ge 3\sqrt{3}\cdot rs \Leftrightarrow s^2 \ge 3\sqrt{3}\cdot rs \Leftrightarrow s \ge 3r\sqrt{3}$$ ***(Mitrinovic)***

Equality holds if and only if the triangle is equilateral.

Refferences:
Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**