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SOLVING SOME PROBLEMS WITH DETERMINANTS

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Abstract: In this article, we will solve some problems with determinants. A lot of these problems had appeared in math magazines or were proposed to various mathematic contests.

For the start we will remember the next lemma:

Lemma: Let be $A, B \in M_n(\mathbb{C})$. Then, $f(x) = \det(A + xB)$ is a polynomial function having the grade n, which has the form: $f(x) = \det A + a_1 x + \cdots + a_{n-1} + x^{n-1} + \det Bx^n$,

$$a_1, a_2, \ldots, a_n \in \mathbb{C}$$

Remarks:

1. If $A, B \in M_n(\mathbb{R})$, then $f \in \mathbb{R}[x]$; if $A, B \in M_n(\mathbb{Q})$ then $f \in \mathbb{Q}[x]$, and if $A, B \in M_n(\mathbb{Z})$ then $f \in \mathbb{Z}[x]$
2. If $A, B \in M_2(\mathbb{C})$ then:

$f(x) = \det A + a_1 x + \det Bx^2$, where $a_1 = Tr(AB^*)$ or $a_1 = Tr\,A \cdot Tr\,B - Tr(AB)$ or

$$a_1 = \det(A + B) - \det A - \det B$$

Applications:

1.Let be $A, B \in M_2(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + B^2) = 0$.

Prove that $\det A = \det B$.

Proof:

$$\det(A^2 + B^2) = \det(A^2 - iB^2) = \det(A + iB) \cdot \det(A - iB) = 0$$

(we have used the fact that $AB = BA$) $\Rightarrow \det(A + iB) = 0$ or $\det(A - iB) = 0$ (1)

Let be $f(x) = \det(A + xB) = \det A + a_1 x + \det Bx^2$, $a_1 \in \mathbb{R}$. From (1)$\Rightarrow f(i) = 0$ or $f(-i) = 0 \Rightarrow f(\pm i) = 0 \Rightarrow \det A \pm ai - \det B = 0 \Rightarrow \det A = \det B$

2. Let $A, B \in M_2(\mathbb{R})$ such that $\det(AB + BA) \leq 0$. Prove that $\det(A^2 + B^2) \geq 0$

(NMO-Romania)

Proof:

Let be $f(x) = \det\left(A^2 + B^2 + x(AB + BA)\right), f \in R[x]$

$$f(x) = \det(AB + B)\, x^2 + a_1 x + \det(A^2 + B^2)$$

We have $f(1) = \det(A^2 + B^2 + AB + BA) = (\det(A + B))^2 \geq 0$

$$f(-1) = \det(A^2 + B^2 - AB - BA) = (\det(A - B))^2 \geq 0$$

But $grade\ f\ = 2$ and from hypothesis $\det(AB + BA) \leq 0$ and because

$$0 \in (-1,1) \Rightarrow f(0) \geq 0 \Rightarrow \det(A^2 + B^2) \geq 0$$

3. Let be $A, B \in M_3(\mathbb{Z})$ such that $\det A = \det B = 1$. Prove that the matrix $A + \sqrt{2}B$ is invertible.

(Mathematical Gazette)

Proof:

We must prove that $\det\left(A + \sqrt{2}B\right) \neq 0$. By absurdum suppose that $\det\left(A + \sqrt{2}B\right) = 0$. Let $f(x) = \det(A + xB) = \det A + a_1 x + a_2 x^2 + \det B x^3 = 1 + a_1 x + a_2 x^2 + x^3$, with $a_1, a_2 \in \mathbb{Z} \Rightarrow f\left(\sqrt{2}\right) = 0 \Rightarrow 1 + a_1\sqrt{2} + 2a_2 + 2\sqrt{2} = 0 \Rightarrow \sqrt{2}(a_1 + 2) = -1 - 2a_2 \Rightarrow$
$\Rightarrow \sqrt{2} = \frac{-1-2a_2}{a_1+2} \in \mathbb{Q}$, false, $\sqrt{2} \notin \mathbb{Q}$

4. Let $A, B \in M_n(\mathbb{Z})$ such that $\det A$ and $\det(A + B)$ are odds. Prove that the matrix $A + kB$ is invertible $\forall k \in \mathbb{Z}$.

(Mathematical Gazette)

Proof:

We must prove that $\det(A + kB) \neq 0, \forall k \in \mathbb{Z}$. By absurdum suppose that $\exists k \in \mathbb{Z}$ such that $\det(A + kB) = 0$. Let be $f = \det(A + xB) = \det A + a_1 x + \cdots + a_{n-1}x^{n-1} + \det B x^n$, $f \in \mathbb{Z}[x]$. We have: $f(0) = \det A =$ odd. $f(1) = \det(A + B) =$ odd.

But $f(x) = 0$ (from Bézout) $\Rightarrow f(x) = (x - k)q(x), q \in k[x]$

$\left.\begin{array}{r} f(0) = -kq(0) = odd \\ f(1) = (1-k)q(1) = odd \end{array}\right\} \Rightarrow -k, 1 - k$ are odd numbers, false, because $-k, 1 - k$ are consecutives.

5. Let $A, B \in M_n(\mathbb{C})$ and $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}; n \in \mathbb{N}^*, n \geq 2$.

Prove that:

$$\det(A+B) + \det(A+\varepsilon B) + \cdots + \det(A+\varepsilon^{n-1}B) = n(\det A + \det B)$$

(RMO-Romania)

Proof:

Let be $f(x) = \det(A + xB) = \det A + a_1 x + \cdots + a_{n-1}x^{n-1} + \det B\, x^n$

$$\det(A+B) = \det A + a_1 + \cdots + a_{n-1} + \det B$$

$$\det(A+\varepsilon B) = \det A + a_1\varepsilon + \cdots + a_{n-1}\varepsilon^{n-1} + \det B\, \varepsilon^n$$

$$\vdots$$

$$\det(A+\varepsilon^{n-1}B) = \det A + a_1\varepsilon^{n-1} + \cdots + a_{n-1}(\varepsilon^{n-1})^{n-1} + \det B\, (\varepsilon^{n-1})^n$$

By summing $\Rightarrow$

$$\det(A+B) + \det(A+\varepsilon B) + \cdots + \det(A+\varepsilon^{n-1}B) =$$

$$n\det A + a_1(1 + \varepsilon + \cdots \varepsilon^{n-1}) + \cdots + a_{n-1}(\varepsilon^{n-1} + \cdots + (\varepsilon^{n-1})^{n-1}) +$$

$$+ \det B\, (\varepsilon^n + \varepsilon^n + \cdots + (\varepsilon^n)^{n-1}) \quad (1)$$

But ε is the root having the order n of the unit $\Rightarrow \varepsilon^n = 1$ (2)

But (1)+(2)$\Rightarrow \det(A+B) + \det(A+\varepsilon B) + \cdots + \det(A+\varepsilon^{n-1}B) =$

$$= n\det A + a_1\frac{\varepsilon^n - 1}{\varepsilon - 1} + \cdots + a_{n-1}\frac{\varepsilon^{n-1}((\varepsilon^n)^{n-1} - 1)}{\varepsilon - 1} + n\det B = n(\det A + \det B)$$

In the ending, some proposed problems on R.M.M.:

1. Let $A \in M_2(\mathbb{R})$ with $\det A = d \neq 0$ such that $\det(A + dA^*) = 0$. Prove that: $\det(A - dA^*) = 4$

(RMO-Romania)

2. Let be $A, B \in M_2(\mathbb{R})$ such that $\det(A-B)\cdot\det(A+B) \geq 0$. Prove that:

$$\det(A^2 - B^2) + \det(AB - BA) \geq 0$$

(R.M.M.)

3. Let be $A, B \in M_2(\mathbb{C})$ such that $\det(A+B) = 1$. Prove that:

$$\det(\det BA + \det AB) = \det(AB)$$

(R.M.M.)

4. Let be $A \in M_2(\mathbb{Z})$. Prove that:

$$\det(A + A^T + A^*) + \det(-A + A^T + A^*) + \det(-A^T + A + A^*) + \det(-A^* + A + A^T) \vdots 12$$

(R.M.M.)

5. Let be $A, B \in M_2(\mathbb{R})$ such that $Tr(A^2B^2) = Tr\,A^2 + Tr\,B^2$. Prove that:

$$\det(A^2 + \alpha^2 B^2) + \det\left(A^1 + \frac{1}{\alpha^2}B^2\right) \geq (\det A + \det B)^2, \forall \alpha \in \mathbb{R}^*$$

(R.M.M.)

6. Let be $A, B \in M_2(\mathbb{R})$ such that $\det A = \det B$. Prove that:

$$\det(xAB + yBA) \geq xy\det(AB + BA)$$

(R.M.M.)

7. Let $A, B \in M_3(\mathbb{C})$. Prove that:

$$\det(A + B) + \det(A - B) = 2(\det A + Tr\,(AB^*))$$

8. Let be $A \in M_n(\mathbb{Z})$. If $\det A =$ odd, then the matrix $A - 2kI_n$ is invertible, $\forall k \in \mathbb{Z}$

(R.M.M.)

REFFERENCES:

-Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**