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UP.239. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{1}{2}\int_a^b \int_a^b (1+\tan x)(1+\tan y)(1+\tan x \tan y)dxdy \geq (\tan b - \tan a)^2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\frac{2}{\cos^2 x \cos^2 y} = 2 \cdot \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 y} = 2 \cdot \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \cdot \frac{\sin^2 y + \cos^2 y}{\cos^2 y} =$$

$$= 2(\tan^2 x + 1)(\tan^2 y + 1) =$$

$$= (\tan^2 x + 1)(\tan^2 y + 1) + (\tan^2 x + 1)(\tan^2 y + 1) \geq$$

$$\overset{QM-AM}{\geq} 2\left(\frac{\tan x + 1}{2}\right)^2 (\tan^2 y + 1) + (\tan^2 x + 1) \cdot 2\left(\frac{\tan y + 1}{2}\right)^2 =$$

$$= \frac{1}{2}[(\tan x + 1)^2(\tan^2 y + 1) + (\tan^2 x + 1)(\tan y + 1)^2] =$$

$$= \frac{1}{2}[(\tan^2 x + 2\tan x + 1)(\tan^2 y + 1) + (\tan^2 x + 1)(\tan^2 y + 2\tan y + 1)] =$$

$$= \frac{1}{2}\left(\begin{matrix}\tan^2 x \tan^2 y + \tan^2 x + 2\tan x \tan^2 y + 2\tan x + \\ + \tan^2 y + 1 + \tan^2 x \tan^2 y + 2\tan^2 x \tan y + \tan^2 x + \tan^2 y + 2\tan y + 1\end{matrix}\right) =$$

$$= \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x + \tan y + 1 + \tan^2 x + \tan^2 y \geq$$

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$$\overset{AM-GM}{\geq} \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x + \tan y + 1 + 2\tan x \tan y =$$

$$= \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x \tan y + \tan x \tan y + \tan x + \tan y + 1 =$$

$$= \tan x \tan y (\tan x \tan y + \tan x + \tan y + 1) + (\tan x \tan y + \tan x + \tan y + 1) =$$

$$= (\tan x \tan y + \tan x + \tan y + 1)(\tan x \tan y + 1) =$$

$$= [\tan x (\tan y + 1) + (\tan y + 1)](\tan x \tan y + 1) =$$

$$= (\tan x + 1)(\tan y + 1)(\tan x \tan y + 1)$$

$$\int_a^b \int_a^b (1 + \tan x)(1 + \tan y)(1 + \tan x \tan y) dx dy \geq$$

$$\geq \int_a^b \int_a^b \frac{2}{\cos^2 x \cos^2 y} dx dy = 2\left(\int_a^b \frac{1}{\cos^2 x} dx\right)\left(\int_a^b \frac{1}{\cos^2 y} dy\right)$$

$$\frac{1}{2}\int_a^b \int_a^b (1 + \tan x)(1 + \tan y)(1 + \tan x \tan y) dx dy \geq$$

$$\geq \left(\tan x \Big|_a^b\right) \cdot \left(\tan y \Big|_a^b\right) = (\tan b - \tan a)^2$$

Equality holds for $a = b$.