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UP.238. If $m \geq 0$ then find:

$$\Omega = \lim_{x\to\infty}\left(\left((x+1)^m \cdot \Gamma(x+2)\right)^{\frac{1}{x+1}} - \left(x^m \cdot \Gamma(x+1)\right)^{\frac{1}{x}}\right)$$

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Solution 1 by Marian Ursărescu-Romania, Solution 2 by Remus Florin Stanca-Romania, Solution 3 by Rohan Shinde-India

Solution 1 by Marian Ursărescu-Romania

(another approach by sequence)

Because $\Gamma(n+1) = n!$ we must calculate (from Heine):

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^m(n+1)!} - \sqrt[n]{n^m n!}\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{n^m n!}}{n}\cdot n\left(\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}} - 1\right) \quad \textit{(1)}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n^m \cdot n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^m n!}{n^m}} \overset{C.S.}{=} \lim_{n\to\infty}\frac{(n+1)^m(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n^m - n!}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^m\cdot\left(\frac{n}{n+1}\right)^n = \frac{1}{e} \quad \textit{(2)}$$

$$\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}\right) =$$

$$= \lim_{n\to\infty} \frac{n\left(e^{\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m\cdot n!}}} - 1\right)}{\ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}} \cdot \ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}}{\sqrt[n]{n^m n!}}$$

$$= \lim_{n\to\infty} \ln\frac{\sqrt[n+1]{(n+1)^m(n+1)!}^{\,n}}{n^m n!} = \ln\left(\lim_{n\to\infty}\frac{(n+1)^m(n+1)!}{n^m n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)^m(n+1)!}}\right) =$$

$$= \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^m\cdot\frac{n+1}{\sqrt[n+1]{(n+1)^m(n+1)!}}\right) \overset{(2)}{=} \ln e = 1$$ ***(3) From (1)+(2)+(3)*** $\Rightarrow \Omega = \frac{1}{e}$

Solution 2 by Remus Florin Stanca-Romania

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{x\Gamma(x)} \overset{n\in\mathbb{N}}{=} \lim_{n\to\infty}\frac{n!}{n\cdot(n-1)!} = 1 \quad \textbf{(1)}$$

$\Gamma(x)$ ***is continuous and differentiable so the function*** $f(x) = \ln(\Gamma(x)) - x\ln x$ ***is differentiable and continuous*** $\Rightarrow$

$$\Rightarrow \frac{\ln(\Gamma(x+1)) - (x+1)\ln(x+1) - \ln(\Gamma(x)) + x\ln(x)}{x+1-x} = \frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln c_x,$$

$c_x \in (x; x+1)$ ***(Lagrange)*** $\Rightarrow \ln(\Gamma(x+1)) - \ln(\Gamma(x)) - x\ln\left(\frac{x+1}{x}\right) - \ln(x+1) =$

$$= \frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln(c_x) \Rightarrow \lim_{x\to\infty}\left(\ln\left(\frac{\Gamma(x+1)}{\Gamma(x)}\right) - \ln(x+1)\right) - 1 =$$

$$= \lim_{x\to\infty}\frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln(c_x) \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\left(\ln\left(\frac{\Gamma(x+1)}{\Gamma(x)}\right) - \ln(x)\right) - 1 = \lim_{x\to\infty}\left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1\right) \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\ln\left(\frac{\Gamma(x+1)}{x\Gamma(x)}\right) - 1 = \lim_{x\to\infty}\left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1\right)$$

$$\overset{(1)}{\Rightarrow} \lim_{x\to\infty}\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1 = -1 \quad \textbf{(2)}$$

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{x\to\infty}\left(\frac{\Gamma(x+1)}{x^x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln(\Gamma(x+1)) - x\ln(x)}{x}} =$$

$$\overset{\frac{L'H}{\frac{\infty}{\infty}}}{=} \lim_{x\to\infty} e^{\frac{\Gamma'(x)}{\Gamma(x)} - 1 - \ln(x)} \overset{(2)}{=} e^{-1} = \frac{1}{e} \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty}\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}=\frac{1}{e}\quad (3)$$

$$\Gamma=\lim_{x\to\infty}\left(x^m\Gamma(x+1)\right)^{\frac{1}{x}}\left(\frac{\left((x+1)^m\Gamma(x+2)\right)^{\frac{1}{x+1}}}{\left(x^m\Gamma(x+1)\right)^{\frac{1}{x}}}-1\right)=$$

$$=\lim_{x\to\infty}\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x}x\ln\left(\frac{\left((x+1)^m\Gamma(x+2)\right)^{\frac{1}{x+1}}}{\left(x^m\Gamma(x+1)\right)^{\frac{1}{x}}}\right)=$$

$$=\frac{1}{e}\lim_{x\to\infty}\ln\left(\frac{(x+1)^m}{x^m}\cdot\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{\left((x+1)^m\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)=$$

$$=\frac{1}{e}\lim_{x\to\infty}\ln\left(\frac{\Gamma(x+2)}{(x+1)\Gamma(x+1)}\cdot\frac{x+1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)=\frac{1}{e}\ln(e)=\frac{1}{e}\Rightarrow\Omega=\frac{1}{e}$$

Solution 3 by Rohan Shinde-India

Using Lemma 1 of Stolz Cesaro theorem that for two sequences $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *of real numbers, if* $0<b_1<b_2<\cdots<b_n<\cdots$ *and* $\lim_{n\to\infty}b_n=\infty$ *then if* $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\Omega$ *then also* $\lim_{n\to\infty}\frac{a_n}{b_n}=\Omega$

Here if $a_n=\left(n^m\Gamma(n+1)\right)^{\frac{1}{n}}$ *and* $b_n=n$ *then*

$$\lim_{n\to\infty}\left\{\left((n+1)^m\Gamma(n+2)\right)^{\frac{1}{n+1}}-\left(n^m\Gamma(n+1)\right)^{\frac{1}{n}}\right\}=\Omega$$

$$\Rightarrow\lim_{n\to\infty}\frac{\left(n^m\Gamma(n+1)\right)^{\frac{1}{n}}}{n}=\Omega\Rightarrow\lim_{n\to\infty}\left(n^m\cdot\frac{\Gamma(n+1)}{n^n}\right)^{\frac{1}{n}}$$

Using Stirling's approximation for Gamma function,

$$\Omega=\lim_{n\to\infty}\left(n^m\times\frac{\sqrt{2\pi n}}{n^n}\times\left(\frac{n}{e}\right)^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(n^{\frac{m}{n}}\times\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}\times\frac{1}{e}\right)$$

But

$$\lim_{n\to\infty}n^{\frac{m}{n}}=\lim_{n\to\infty}e^{\frac{m\ln n}{n}}=e^{\lim_{n\to\infty}\frac{m\ln n}{n}}=e^0=1$$

and $\lim_{n\to\infty}\left(\sqrt{2\pi n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{\ln(2\pi n)}{2n}} = e^{\lim_{n\to\infty}\left(\frac{\ln(2\pi n)}{2n}\right)} = e^{0} = 1$

$$\Rightarrow \Omega = \frac{1}{e}$$