PROPOSED PROBLEM

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If $a, b, c > 0, ab + bc + ca = 3$ then:

$$4 \cdot \tan^{-1} 2 \cdot \tan^{-1}(\sqrt[3]{abc}) \leq \pi \cdot \tan^{-1}(1 + \sqrt[3]{abc})$$

Solution 1 by Khaled Abd Imouti-Damascus - Syria.

If $a, b, c > 0 : ab + bc + ca = 3$. *Then:*

$$4 \tan^{-1} 2 \cdot \tan^{-1}(\sqrt[3]{abc}) \overset{?}{\leq} \pi \cdot \tan^{-1}(1 + \sqrt[3]{abc})$$

$$\tan^{-1}(2) \cdot \tan^{-1}(\sqrt[3]{abc}) \overset{?}{\leq} \frac{\pi}{4} \cdot \tan^{-1}(1 + \sqrt[3]{abc})$$

$$\tan^{-1}(2) \cdot \tan^{-1}(\sqrt[3]{abc}) \overset{?}{\leq} \tan^{-1}(1) \cdot \tan^{-1}(1 + \sqrt[3]{abc})$$

By using AM-GM: $\sqrt[3]{abc} \geq \dfrac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$

$$\sqrt[3]{abc} \geq \frac{3}{\frac{ab+ac+bc}{a \cdot bc}} \Rightarrow \sqrt[3]{abc} \geq \frac{3abc}{3}$$

$$\sqrt[3]{abc} \geq abc. \text{ So: } 0 < abc \leq 1$$

Let be the function: $f(x) = \tan^{-1}(2) \cdot \tan^{-1}(x) - \tan^{-1}(1) \cdot \tan^{-1}(1 + x), \Delta =]0, 1]$

$$f'(x) = \frac{[\tan^{-1}(2) - \tan^{-1}(1)] + [\tan^{-1}(2) - \tan^{-1}(1)]x^2 + \tan^{-1}(2)(2x + 1)}{(1 + x^2)[1 + (x + 1)^2]} > 0$$

x	0	1
$f'(x)$	++++++++++++++++++++++++++++++++++++++	
$f(x)$	$\nearrow$	0

$$x \in]0, 1] : f(x) \leq 0$$

$$\text{So: } \tan^{-1}(2) \cdot \tan^{-1}(x) - \tan^{-1}(1) \cdot \tan^{-1}(1 + x) \leq 0$$

$$\tan^{-1}(2) \cdot \tan^{-1}(x) \leq \tan^{-1}(1) \cdot \tan^{-1}(1 + x)$$

Because: $0 < \sqrt[3]{abc} \leq 1 : \tan^{-1}(2) \cdot \tan^{-1}(\sqrt[3]{abc}) \leq \tan^{-1}(1) \cdot \tan^{-1}(1 + \sqrt[3]{abc})$

□

Solution 2 by Remus Florin Stanca - Romania.

$$4 \tan^{-1}(2) \tan^{-1}(\sqrt[3]{abc}) \leq \pi \tan^{-1}(1 + \sqrt[3]{abc})$$

The inequality can be written as: $\dfrac{\tan^{-1}(\sqrt[3]{abc})}{\tan^{-1}(\sqrt[3]{abc} + 1)} \leq \dfrac{\frac{\pi}{4}}{\tan^{-1}(2)} = \dfrac{\tan^{-1}(1)}{\tan^{-1}(2)}$

$ab+bc+ca \geq 3\sqrt[3]{a^2 + b^2 + c^2} \Rightarrow abc \leq 1$, *let* $f : (0, 1] \to \mathbb{R}$ *such that* $f(x) = \dfrac{\tan^{-1}(x)}{\tan^{-1}(x + 1)}$

$$f'(x) = \frac{\frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1}}{(\tan^{-1}(x+1))^2} \tag{1}$$

$$\frac{1}{x^2} > \frac{1}{x^2+1} \text{ and } \tan^{-1}(x+1) > \tan^{-1}(x) \text{ (as increasing function)} \overset{"\cdot"}{\Rightarrow}$$

$$\Rightarrow \frac{\tan^{-1}(x+1)}{x^2} > \frac{\tan^{-1}(x)}{x^2+1}\Big(\tan^{-1}(x+1) > 0 \text{ and } \tan^{-1}(x) > 0 \text{ because } x > 0 \text{ and}$$

$$-\frac{\pi}{2} < \tan^{-1}(x) < \tan^{-1}(x+1) < \frac{\pi}{2}\Big) \Rightarrow \frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1} > 0$$

$$\Rightarrow \frac{\frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1}}{(\tan^{-1}(x+1)^2} > 0 \overset{(1)}{\Rightarrow} f'(x) > 0 \Rightarrow f \text{ is increasing,}$$

$$\sqrt[3]{abc} \leq 1 \Rightarrow f(\sqrt[3]{abc}) \leq f(1) \Leftrightarrow \frac{\tan^{-1}(\sqrt[3]{abc})}{\tan^{-1}(\sqrt[3]{abc}+1)} \leq \frac{\tan^{-1}(1)}{\tan^{-1}(2)} = \frac{\frac{\pi}{2}}{\tan^{-1}(2)} \Leftrightarrow$$

$$\Leftrightarrow 4\tan^{-1}(2)\tan^{-1}(\sqrt[3]{abc}) \leq \pi\tan^{-1}(1+\sqrt[3]{abc})$$

□

Solution 3 by Tran Hong-Dong Thap-Vietnam.

$$3 = ab + bc + ca \overset{AM-GM}{\geq} 3 \geq 3\sqrt[3]{(abc)^2}$$

$$\leftrightarrow \sqrt[3]{(abc)^2} \leq 1 \leftrightarrow \sqrt[3]{abc} \leq 1$$

We have: $g(x) = x\tan^{-1} x, x > 0 \to g'(x) = \tan^{-1} x + \dfrac{x}{x^2+1} > 0 \to g(x) \uparrow$ *on* $(0; +\infty)$

Because: $\tan^{-1} 1 = \dfrac{\pi}{4}$

Inequality becomes: $\dfrac{\tan^{-1}(\sqrt[3]{abc})}{\tan^{-1} 1} \leq \dfrac{\tan^{-1}(1+\sqrt[3]{abc})}{\tan^{-1} 2} \leftrightarrow \dfrac{\tan^{-1}(\sqrt[3]{abc})}{\tan^{-1}(1+\sqrt[3]{abc})} \leq \dfrac{\tan^{-1} 1}{\tan^{-1} 2}$

$$\leftrightarrow \frac{\tan^{-1} u}{\tan^{-1}(1+u)} \leq \frac{\tan^{-1} 1}{\tan^{-1} 2}; \text{ (with } u = \sqrt[3]{abc}, 0 < u \leq 1)$$

$$\to f'(u) = \frac{((1+u^2)+1)\tan^{-1}(1+u) - (u^2+1)\tan^{-1}(u)}{(u^2+1)((1+u)^2+1)(\tan^{-1}(1+u))^2} > 0$$

(Because: $g(x) \uparrow$ *on* $(0; +\infty) \to 0 < u < 1 + u \to g(u) < g(u+1))$

$$\to f(u) \uparrow \text{ on } (0; 1] \to f(u) \leq f(1) = \frac{\tan^{-1} 1}{\tan^{-1}(1+1)} = \frac{\tan^{-1} 1}{\tan^{-1} 2}$$

Proved. Equality if and only if $a = b = c = 1$.

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