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ABOUT SOME ION IONESCU'S INEQUALITIES

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If , $y, z \in \mathbb{R}_+^* = (0, \infty)$, then the following inequalities hold:

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2} \qquad \text{(N-I)}$$

namely NESBITT-IONESCU inequality, and also:

$$\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq 6 \qquad (*)$$

Let ABC be a triangle, with the usual notations and s is the semi-perimeter, and F its area.

In Mathematical Gazette, 1897, ION IONESCU has established that, in any ABC triangle the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F \qquad \text{(I-W)}$$

and this inequality has been rediscovered by ROLAND WEITZENBOCK, in 1919.

Also, in Mathematical Gazette, vol., XLVIII, (1942-1943) at page 334, it is proved that, in any ABC triangle the following inequalities hold:

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \qquad (**)$$

$$ab + bc + ca = s^2 + r^2 + 4Rr \qquad (***)$$

Next, we will establish other inequalities (some known):

TSINTSIFAS'S INEQUALITY:

If $x, y, z \in \mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{x}{y+z} \cdot a^2 + \frac{y}{z+x} \cdot b^2 + \frac{z}{x+y} \cdot c^2 \geq 2\sqrt{3}F \qquad \text{(T)}$$

Proof. We have:

$$\sum_{cyc} \frac{x}{y+z} a^2 = \sum_{cyc} \frac{x}{y+z} a^2 + 4s^2 - 4s^2 = \sum_{cyc} \frac{x}{y+z} a^2 + (a+b+c)^2 - 4s^2 =$$

$$= \sum_{cyc} \frac{x}{y+z} a^2 + a^2 + b^2 + c^2 + 2(ab+bc+ca) - 4s^2 =$$

$$= \sum_{cyc} \left(\frac{x}{y+z} + 1\right) a^2 + 2(ab+bc+ca) - 4s^2 = (x+y+z) \cdot \sum_{cyc} \frac{a^2}{y+z} +$$

$$+2(ab+bc+ca)-4s^2 \overset{Bergstrom}{\geq} (x+y+z)\cdot\frac{(a+b+c)^2}{\sum_{cyc}(y+z)}+2(ab+bc+ca)-4s^2$$

$$=(x+y+z)\cdot\frac{4s^2}{2(x+y+z)}+2(ab+bc+ca)-4s^2=2(ab+bc+ca)-2s^2\overset{(***)}{=}$$

$$=2(s^2+r^2+4Rr)-2s^2=2r(4R+r)\overset{Doucet}{\geq}2r(s\sqrt{3})=2\sqrt{3}sr=2\sqrt{3}F$$

BĂTINEȚU-GIURGIU'S INEQUALITY:

If $x,y,z\in\mathbb{R}_+^*$, then in ABC triangle the following inequality holds:

$$\frac{y+z}{x}a^2+\frac{z+x}{y}b^2+\frac{x+y}{z}c^2\geq 8\sqrt{3}F \quad \textbf{(B-G)}$$

Proof 1. We have:

$$\sum_{cyc}\frac{y+z}{x}a^2\overset{Means}{\geq}2\sum_{cyc}\frac{\sqrt{yz}}{x}a^2\overset{Means}{=}2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}}{x}a^2}=$$

$$=6\cdot\sqrt[3]{(abc)^2}\overset{Carliz}{\geq}6\cdot\frac{4\sqrt{3}}{3}F=8\sqrt{3}F$$

Proof 2. We have:

$$\sum_{cyc}\frac{y+z}{x}a^2=\sum_{cyc}\frac{(y+z)^2}{xy+xz}a^2\overset{Means}{\geq}4\sum_{cyc}\frac{yz}{xy+xz}a^2\overset{(T)}{\geq}4\cdot 2\sqrt{3}F=8\sqrt{3}F$$

Proof 3. We have:

$$\sum_{cyc}\frac{y+z}{x}a^2=\sum_{cyc}\frac{y+z}{x}a^2+4s^2-4s^2=\sum_{cyc}\frac{y+z}{x}a^2+(a+b+c)^2-4s^2=$$

$$=\sum_{cyc}\left(\frac{y+z}{x}+1\right)a^2+2(ab+bc+ca)-4s^2=$$

$$=(x+y+z)\cdot\sum_{cyc}\frac{a^2}{x}+2(ab+bc+ca)-4s^2\overset{Bergstrom}{\geq}$$

$$\geq(x+y+z)\cdot\frac{(a+b+c)^2}{x+y+z}+2(ab+bc+ca)-4s^2=$$

$$=4s^2+2(ab+bc+ca)-4s^2=2(ab+bc+ca)\overset{V.O.Gordon}{\geq}2\cdot 4\sqrt{3}F=8\sqrt{3}F$$

THE SECOND INEQUALITY OF G. TSINTSIFAS

If $x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{x}{y+z}a^4+\frac{y}{z+x}b^4+\frac{z}{x+y}c^4\ge 8F^2 \quad \text{(G.T.)}$$

Proof 1. We have:

$$\sum_{cyc}\frac{x}{y+z}a^4=\sum_{cyc}\frac{x^2}{xy+xz}a^4=\sum_{cyc}\frac{(xa^2)^2}{xy+xz}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(xa^2+yb^2+zc^2)^2}{\sum_{cyc}(xy+xz)}=\frac{(xa^2+yb^2+zc^2)^2}{2(xy+yz+zx)}\overset{Oppenheimer}{\ge}\frac{16(xy+yz+zx)F^2}{2(xy+yz+zx)}=8F^2$$

Proof 2. We have:

$$\sum_{cyc}\frac{x}{y+z}a^4=\sum_{cyc}\left(\frac{x}{y+z}+1\right)a^4-(a^4+b^4+c^4)=(x+y+z)\cdot\sum_{cyc}\frac{a^4}{y+z}-$$

$$-(a^4+b^4+c^4)\overset{Bergstrom}{\ge}(x+y+z)\cdot\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}(y+z)}-(a^4+b^4+c^4)=$$

$$=(x+y+z)\cdot\frac{a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)}{2(x+y+z)}-(a^4+b^4+c^4)=$$

$$=a^2b^2+b^2c^2+c^2a^2-\frac{1}{2}(a^4+b^4+c^4)=$$

$$=\frac{1}{2}(a+b+c)(-a+b+c)(a-b+c)(a+b-c)=$$

$$=\frac{1}{2}\cdot 16\cdot s(s-a)(s-b)(s-c)=8F^2$$

Theorem 1. If $m,n,x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{x}{my+nz}a^4+\frac{y}{mz+nx}b^4+\frac{z}{mx+ny}c^4\ge\frac{16}{m+n}F^2 \quad \text{(i)}$$

Proof. We have:

$$\sum_{cyc}\frac{x}{my+nz}a^4=\sum_{cyc}\frac{x^2a^4}{mxy+nxz}=\sum_{cyc}\frac{(xa^2)^2}{mxy+nxz}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(xa^2+yb^2+zc^2)^2}{\sum_{cyc}(mxy+nxz)}=\frac{(xa^2+yb^2+zc^2)^2}{(m+n)(xy+yz+zx)}\overset{Klamkin}{\ge}$$

$$\ge\frac{16(xy+yz+zx)F^2}{(m+n)(xy+yz+zx)}=\frac{16}{m+n}F^2$$

If in (i) we take $m = n$ we obtain inequality (G.T.)

BĂTINEȚU-GIURGIU-SITARU INEQUALITY

If $x, y, z \in \mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{y+z}{x}a^4 + \frac{z+x}{y}b^4 + \frac{x+y}{z}c^4 \ge 32F^2 \quad \text{(B-G-S)}$$

Proof 1. We have:

$$\sum_{cyc}\frac{y+z}{x}a^4 = \sum_{cyc}\frac{(y+z)^2a^4}{xy+xz} \overset{Means}{\ge} 4\sum_{cyc}\frac{yz}{xy+xz}a^4 \overset{(G.T.)}{\ge} 4\cdot 8\cdot F^2 = 32F^2$$

Proof 2. We have:

$$\sum_{cyc}\frac{y+z}{x}a^4 \overset{Means}{\ge} 2\sum_{cyc}\frac{\sqrt{yz}}{x}a^4 \overset{Means}{\ge} 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}}{x}a^4} =$$

$$= 6\sqrt[3]{(abc)^4} = 6\left(\sqrt[3]{(abc)^2}\right)^2 \overset{Cavliz}{\ge} 6\left(\frac{4\sqrt{3}F}{3}\right)^2 = 6\cdot 16\cdot\frac{3}{9}F = 2\cdot 16\cdot F^2 = 32F^2$$

Proof 3. We have:

$$\sum_{cyc}\frac{y+z}{x}a^4 = \sum_{cyc}\left(\frac{y+z}{x}+1\right)a^4 - (a^4+b^4+c^4) =$$

$$= (x+y+z)\sum_{cyc}\frac{a^4}{x} - (a^4+b^4+c^4) \overset{Bergstrom}{\ge} (x+y+z)\cdot\frac{(a^2+b^2+c^2)^2}{x+y+z} -$$

$$-(a^4+b^4+c^4) = (a^2+b^2+c^2)^2 - (a^4+b^4+c^4) = 2(a^2b^2+b^2c^2+c^2a^2) \ge$$

$$\overset{Bergstrom}{\ge} \frac{2}{3}(ab+bc+ca)^2 \overset{V.O.Gordon}{\ge} \frac{2}{3}\left(4\sqrt{3}F\right)^2 = 32F^2$$

Theorem 2. If $m, n, x, y, z \in \mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{my+nz}{x}a^4 + \frac{mz+nx}{y}b^4 + \frac{mx+ny}{z}c^4 \ge \frac{64mn}{m+n}F^2 \quad \text{(ii)}$$

Proof. We have:

$$\sum_{cyc}\frac{my+nz}{x}a^4 = \sum_{cyc}\frac{(my+nz)^2}{mxy+nxz} \overset{Means}{\ge} 4mn\sum_{cyc}\frac{yz}{mxy+nxz}a^4 \overset{(i)}{\ge} 4mn\cdot\frac{16}{m+n}F^2 =$$

$$= \frac{64mn}{m+n}F^2$$

Theorem 3. If $u,v,x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{uy+vz}{x}a^4+\frac{uz+vx}{y}b^4+\frac{ux+vy}{z}c^4\geq 32\sqrt{uv}F^2 \quad \text{(iii)}$$

Proof. We have:

$$\sum_{cyc}\frac{uy+vz}{x}a^4\overset{Means}{\geq}2\sqrt{uv}\cdot\sum_{cyc}\frac{\sqrt{yz}}{x}a^4\overset{Means}{\geq}$$

$$\geq 2\sqrt{uv}\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}}{x}a^4}=6\sqrt{uv}\cdot\sqrt[3]{(abc)^4}=$$

$$=6\sqrt{uv}\cdot\left(\sqrt[3]{(abc)^2}\right)^2\overset{Carliz}{\geq}6\sqrt{uv}\cdot\left(\frac{4\sqrt{3}}{3}F\right)^2=$$

$$=6\sqrt{uv}\cdot\frac{16}{3}F^2=32\sqrt{uv}F^2$$

If $m=n$ relationship (ii) is transformed in relationship (B-G-S), and if $u=v=1$ relationship (iii), becomes relationship (B-G-S).

If in (B-G-S) inequality $x=y=z$ we obtain inequality:

$$a^4+b^4+c^4\geq 16F^2 \quad \text{(F.G)}$$

namely inequality F. GOLDNER.

Theorem 4. If $x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$\frac{x^2+y^2+z^2+2yz}{xy+xz}a^2+\frac{x^2+y^2+z^2+2zx}{yz+yx}b^2+\frac{x^2+y^2+z^2+2xy}{zx+zy}c^2\geq 10\sqrt{3}F \quad \text{(j)}$$

Proof 1. We have:

$$\sum_{cyc}\frac{x^2+y^2+z^2+2yz}{xy+xz}a^2=\sum_{cyc}\frac{(x+y+z)^2-2(xy+xz)}{xy+xz}a^2=$$

$$=(x+y+z)^2\cdot\sum_{cyc}\frac{a^2}{xy+xz}-2(a^2+b^2+c^2)\geq 3(xy+yz+zx)\sum_{cyc}\frac{a^2}{xy+xz}-$$

$$-2(a^2+b^2+c^2)\overset{Bergstrom}{\geq}3(xy+yz+zx)\frac{(a+b+c)^2}{\sum_{cyc}(xy+xz)}-2(a^2+b^2+c^2)=$$

$$=3(xy+yz+zx)\cdot\frac{4s^2}{2(xy+yz+zx)}-2(a^2+b^2+c^2)=6s^2-2(a^2+b^2+c^2)=$$

$$\stackrel{(**)}{=} 6s^2 - 2 \cdot 2(s^2 - r^2 - 4Rr) = 2s^2 + 4r(4R + r) \geq$$

$$\stackrel{Mitrinovic}{\geq} 2s(3\sqrt{3}r) + 4r(4R + r) \stackrel{Doucet}{\geq} 6\sqrt{3}sr + 4r(s\sqrt{3}) =$$

$$= 10\sqrt{3}sr = 10\sqrt{3}F$$

Proof 2. According to inequalities (T) and (B-G) we have:

$$\sum_{cyc} \frac{x}{y+z} a^2 + \sum_{cyc} \frac{y+z}{x} a^2 = \sum_{cyc} \frac{x^2 + (y+z)^2}{xy + xz} a^2 = \sum_{cyc} \frac{x^2 + y^2 + z^2 + 2yz}{xy + xz} a^2 \geq$$

$$\geq 2\sqrt{3}F + 8\sqrt{3}F = 10\sqrt{3}F$$

We conclude with some remarkable observations

1. If in inequalities (T), (B-G), (j) we take $x = y = z$, we obtain Ion Ionescu's inequality, namely inequality (I-W).

2. If ABC triangle is equilateral then, from inequalities (T), (G.T) we obtain inequality (N-I), and from inequalities (B-G), (D-G-S), we obtain inequality (*).

3. If ABC triangle is equivalent then from inequality (j) we obtain inequality:

$$\frac{x^2+y^2+z^2+2yz}{xy+xz} + \frac{x^2+y^2+z^2+2zx}{yz+yx} + \frac{x^2+y^2+z^2+2xy}{zx+zy} \geq \frac{15}{2} \text{ namely}$$

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq \frac{3}{2} + 6 = \frac{15}{2}$$

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