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A SIMPLE PROOF OF J.RADON'S INEQUALITY(1913)

If $a_1, a_2, b_1, b_2 > 0; k \in \mathbb{N} \setminus \{0\}$ then:

$$\frac{a_1^k}{b_1^{k-1}} + \frac{a_2^k}{b_2^{k-1}} \geq \frac{(a_1 + a_2)^k}{(b_1 + b_2)^{k-1}}$$

By Daniel Sitaru-Romania

Proof:

Let be $f_1, f_2: (0, \infty) \to \mathbb{R}; f_1(x) = a_1 x^k - b_1 x^{k-1}; f_2(x) = a_2 x^k - b_2 x^{k-1}$

$$f'_1(x) = a_1 k x^{k-1} - b_1(k-1)x^{k-2}$$

$$f'_1(x) = 0 \Rightarrow x^{k-2}\big(a_1 k x - b_1(k-1)\big) = 0 \Rightarrow x = \frac{b_1(k-1)}{a_1 k}$$

$$\min_{x>0} f_1(x) = f_1\left(\frac{b_1(k-1)}{a_1 k}\right) = a_1 \cdot \left(\frac{b_1(k-1)}{a_1 k}\right)^k - b_1 \cdot \left(\frac{b_1(k-1)}{a_1 k}\right)^{k-1} =$$

$$= \frac{b_1^k (k-1)^k}{a_1^{k-1} \cdot k^k} - \frac{b_1^k (k-1)^{k-1}}{a_1^{k-1}(k)^{k-1}} = \frac{b_1^k}{a_1^{k-1}} \cdot \left(\frac{k-1}{k}\right)^{k-1} \left(\frac{k-1}{k} - 1\right) =$$

$$= \frac{b_1^k}{a_1^{k-1}}\left(\frac{k-1}{k}\right)^{k-1}\left(-\frac{1}{k}\right)$$

Analogous:

$$\min_{x>0} f_2(x) = \frac{b_2^k}{a_2^{k-1}}\left(\frac{k-1}{k}\right)^{k-1}\left(-\frac{1}{k}\right)$$

$$\min_{x>0}(f_1 + f_2)(x) = \frac{(b_1 + b_2)^k}{(a_1 + a_2)^{k-1}} \cdot \left(\frac{k-1}{k}\right)^{k-1}\left(-\frac{1}{k}\right)$$

$$\min_{x>0} f_1(x) + \min_{x>0} f_2(x) \le \min_{x>0}(f_1 + f_2)(x)$$

$$\frac{b_1^k}{a_1^{k-1}}\left(\frac{k-1}{k}\right)^{k-1}\left(-\frac{1}{k}\right) + \frac{b_2^k}{a_2^{k-1}}\left(\frac{k-1}{k}\right)^{k-1}\left(-\frac{1}{k}\right) \le$$

$$\le \frac{(b_1 + b_2)^k}{(a_1 + a_2)^{k-1}} \cdot \left(\frac{k-1}{k}\right)^{k-1} \cdot \left(-\frac{1}{k}\right)$$

$$\frac{b_1^k}{a_1^{k-1}} + \frac{b_2^k}{a_2^{k-1}} \ge \frac{(b_1 + b_2)^k}{(a_1 + a_2)^{k-1}}$$

Analogous for $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0$***:***

$n, k \in \mathbb{N} \setminus \{0\}$ ***then:***

$$\frac{b_1^k}{a_1^{k-1}} + \frac{b_2^k}{a_2^{k-1}} + \frac{b_3^k}{a_2^{k-1}} \ge \frac{(b_1 + b_2 + b_3)^k}{(a_1 + a_2 + a_3)^{k-1}}$$

$$\frac{b_1^k}{a_1^{k-1}} + \frac{b_2^k}{a_2^{k-1}} + \dots + \frac{b_n^k}{a_n^{k-1}} \ge \frac{(b_1 + b_2 + \dots + b_n)^k}{(a_1 + a_2 + \dots + a_n)^{k-1}}$$

Equality holds for:

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \dots = \frac{b_n}{a_n}$$

Refferences:

https://www.cut-the-knot.org/m/Algebra/RadonInequality.shtml

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**

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Radon's Inequality and Applications

▸ Radon's Inequality

▸ Proof of Radon's Inequality

▾ Reverse Radon's Inequality

Dan Sitaru has kindly alerted me to the validity of what's known as the reverse Radon's inequality:

If $x_k, a_k > 0,\ \ k \in \{1, 2, \ldots, n\},\ \ 0 \le p \le 1,$ *then*

$$\frac{x_1^p}{a_1^{p-1}} + \frac{x_2^p}{a_2^{p-1}} + \cdots + \frac{x_n^p}{a_n^{p-1}} \le \frac{(x_1 + x_2 + \cdots + x_n)^p}{(a_1 + a_2 + \cdots + a_n)^{p-1}}.$$