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A GENERALIZATION OF FINSLER-HADWIGER'S INEQUALITY

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ABSTRACT: Finsler-Hadwiger's inequality:

In any ABC triangle having the area F, the following inequality holds:

$$a^2+b^2+c^2 \geq 4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2 \qquad \textbf{(F-H)}$$

Let's suppose and generalize this inequality, but first, we state some important results.

IMPORTANT RESULTS:

In Mathematical Gazette vol. XLVIII (1942-1943) at page 334 it is proved that:

- in any ABC triangle the following inequality holds:

$$a^2+b^2+c^2=2(s^2-r^2-4Rr) \qquad \textbf{(i)}$$

and $$ab+bc+ca=s^2+r^2+4Rr \qquad \textbf{(ii)}$$

where s is the semiperimeter of the triangle.

Also, we will take into account Doucet's inequality:

$$4R+r \geq s\sqrt{3} \qquad \textbf{(D)}$$

and D.S. Mitrinovic's inequality:

$$s \geq 3\sqrt{3}r \qquad \textbf{(M)}$$

MAIN RESULT:

If $u,v \in \mathbb{R}_+^* = (0,\infty), v \in \left[\frac{u}{3},u\right]$, then in any ABC triangle having the area F has the inequality:

$$u(a^2+b^2+c^2) \geq 4u\sqrt{3}F+v((a-b)^2+(b-c)^2+(c-a)^2) \qquad \textbf{(*)}$$

Proof.

We have:

$$u\sum_{cyc}a^2-v\sum_{cyc}(a-b)^2=(u-2v)\sum_{cyc}a^2+2v\sum_{cyc}ab=$$

$$\overset{(i),(ii)}{=}(u-2v)2(s^2-r^2-4Rr)+2v(s^2+r^2+4Rr)=$$

$$=(2u-4v+2u)s^2+(2v+2u+4v)r(4R+r)=$$

$$= 2(u-v)s^2 + 2(3v-u)r(4R+r) \overset{(H)}{\geq}$$

$$\geq 2(u-v)s\left(3\sqrt{3}r\right) + 2(3v-u)r(4R+r) \overset{(D)}{\geq}$$

$\geq 6(u-v)\sqrt{3}F + 2(3u-u)rs\sqrt{3} = (6u-6v+6v-2u)\sqrt{3}F = 4u\sqrt{3}F$ **Q.E.D.**

If $u = v$ then inequality (*) becomes $u(a^2+b^2+c^2) \geq 4uF\sqrt{3} + u\sum_{cyc}(a-b)^2 \Leftrightarrow$

$\Leftrightarrow a^2+b^2+c^2 \geq 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2$**, namely inequality (F-H).**

If $v = \frac{u}{2}$ then inequality (*) becomes:

$$u(a^2+b^2+c^2) \geq 4u\sqrt{3}F + \frac{u}{2}\sum_{cyc}(a-b)^2 \Leftrightarrow$$

$$\Leftrightarrow a^2+b^2+c^2 \geq 4\sqrt{3}F + \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$$

REFFERENCES:

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**