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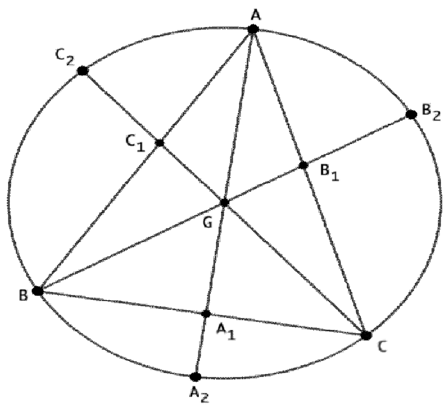
JP.241 Let m'_a, m'_b, m'_c be the circumpedal extensions of cevians of centroid in ΔABC. Prove that:

$$m'_a + m'_b + m'_c \geq 3\sqrt[3]{abc}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1 = m_a; BB_1 = m_b; CC_1 = m_c; AA_1 \cap BB_1 \cap CC_1 = \{G\}$$

$$A_1B = A_1C = \frac{a}{2}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2 \text{ (the power } A_1 \text{ in circumcircle)}$$

$$\frac{a}{2} \cdot \frac{a}{2} = m_a \cdot A_1A_2 \Rightarrow A_1A_2 = \frac{a^2}{4m_a}$$

$$m'_a = AA_2 = AA_1 + A_1A_2 = m_a + \frac{a^2}{4m_a}$$

$$\text{Analogous: } m'_b = m_b + \frac{b^2}{4m_b}; m'_c = m_c + \frac{c^2}{4m_c}$$

$$m'_a + m'_b + m'_c = m_a + m_b + m_c + \frac{a^2}{4m_a} + \frac{b^2}{4m_b} + \frac{c^2}{4m_c} \geq$$

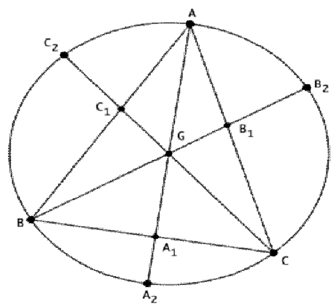
$$\overset{AM-GM}{\geq} 6\sqrt[6]{m_a m_b m_c \cdot \frac{a^2}{4m_a} \cdot \frac{b^2}{4m_b} \cdot \frac{c^2}{4m_c}} = 3\sqrt[3]{abc}$$

JP.242 Let m'_a, m'_b, m'_c be the circumpedal extensions of cevians of centroid in ΔABC. Prove that:

$$m'_a m'_b m'_c \geq abc$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1 = m_a; BB_1 = m_b; CC_1 = m_c; AA_1 \cap BB_1 \cap CC_1 = \{G\}$$

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$$\frac{a}{2} \cdot \frac{a}{2} = m_a \cdot A_1A_2 \Rightarrow A_1A_2 = \frac{a^2}{4m_a}$$

$$m'_a = AA_2 = AA_1 + A_1A_2 = m_a + \frac{a^2}{4m_a} \overset{AM-GM}{\geq} 2\sqrt{m_a \cdot \frac{a^2}{4m_a}} = 2 \cdot \frac{1}{2} \cdot a = a$$

Analogous: $m'_b \geq b; m'_c \geq c$. ***By multiplying:*** $m'_a m'_b m'_c \geq abc$

JP.243. If $x, y, z > 1; xyz = 8$ then:

$$\left(\frac{x}{2}\right)^x + \left(\frac{y}{2}\right)^y + \left(\frac{z}{2}\right)^z \geq 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vișescu-Romania

Let be $f: (1,8) \to \mathbb{R}, f(x) = \left(\frac{x}{2}\right)^x = e^{x(\log x - \log 2)}$

$$f'(x) = (\log x - \log 2 + 1)e^{x(\log x - \log 2)}$$

$$f''(x)=\left((\log x-\log 2+1)^2+\frac{1}{x}\right)e^{x(\log x-\log 2)}>0$$

f –convexe. By Jensen's inequality: $f\left(\frac{x+y+z}{3}\right)\le\frac{1}{3}(f(x)+f(y)+f(z))$

$$3\left(\frac{x+y+z}{6}\right)^{\frac{x+y+z}{3}}\le\left(\frac{x}{2}\right)^x+\left(\frac{y}{2}\right)^y+\left(\frac{z}{2}\right)^z$$

$$\left(\frac{x}{2}\right)^x+\left(\frac{y}{2}\right)^y+\left(\frac{z}{2}\right)^z\ge 3\left(\frac{x+y+z}{6}\right)^{\frac{x+y+z}{3}}\overset{AM-GM}{\ge}$$

$$\ge 3\left(\frac{3\sqrt[3]{xyz}}{6}\right)^{\frac{x+y+z}{3}}=3\left(\frac{3\sqrt[3]{8}}{6}\right)^{\frac{x+y+z}{3}}=3\cdot(1)^{\frac{x+y+z}{3}}=3$$

Equality holds for $x=y=z=2$.

Solution 2 by proposer

$$\left(\frac{x}{2}\right)^x=\left(1+\left(\frac{x}{2}-1\right)\right)^x\overset{BERNOULLI}{\ge}1+x\left(\frac{x}{2}-1\right)\ge x-1$$

$$\Leftrightarrow 1+\frac{x^2}{2}-x\ge x-1$$

$$\frac{x^2}{2}-2x+2\ge 0\Leftrightarrow x^2-4x+4\ge 0\Leftrightarrow (x-2)^2\ge 0$$

$$\left(\frac{x}{2}\right)^x\ge x-1;\ \left(\frac{y}{2}\right)^y\ge y-1;\ \left(\frac{z}{2}\right)^z\ge z-1$$

By adding: $\left(\frac{x}{2}\right)^x+\left(\frac{y}{2}\right)^y+\left(\frac{z}{2}\right)^z\ge x+y+z-3\ge$

$$\overset{AM-GM}{\ge}3\sqrt[3]{xyz}-3=3\sqrt[3]{8}-3=3\cdot 2-3=3$$

Equality holds for $x=y=z=2$.

JP.244 If $x,y,z\in\mathbb{R}$ then:

$$\max\{x,y,z\}\ge\min\{x,y,z\}+\sqrt{x^2+y^2+z^2-xy-yz-zx}$$

When equality holds?

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution by Ravi Prakash-New Delhi-India

WLOG: $x\ge y\ge z$. ***Hence:*** $\max\{x,y,z\}=x,\min\{x,y,z\}=z$

$$(y-x)(y-z)\le 0\to y^2-xy-yz+xz\le 0$$

$$(x-z)^2\ge (x-z)^2+(y-x)(y-z)$$

$$(x-z)^2\ge x^2-2xz+z^2+y^2-xy-yz+xz$$

$$(x-z)^2\ge x^2+y^2+z^2-xy-yz-zx$$

$$x-z\ge \sqrt{x^2+y^2+z^2-xy-yz-zx}$$

$$\max\{x,y,z\}-\min\{x,y,z\}\ge \sqrt{x^2+y^2+z^2-xy-yz-zx}$$

JP.245. Let x,y,z be non-negative real numbers such that $x^2+y^2+z^2=1$. Find the minimum and maximum value of Ω.

$$\Omega=\sqrt{1+x}+\sqrt{1+y}+\sqrt{1+z}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Michael Sterghiou-Greece

$$\Omega=\sum_{cyc}\sqrt{1+x}\quad \textit{(1)}$$

Let $(p,q,r)=\left(\sum_{cyc}x,\sum_{cyc}xy,xyz\right)$. The function $f(t)=\sqrt{t}$ is convex therefore by Jensen $\Omega\le 3\sqrt{1+\frac{p}{3}}$ (2) But $1=\sum_{cyc}x^2\ge\frac{p^2}{3}\to p\le\sqrt{3}$ so $\sqrt{2}$ becomes $\Omega\le 3\sqrt{1+\frac{\sqrt{3}}{3}}$ which is the required maximum. Equality for $x=y=z=\frac{\sqrt{3}}{3}$. We will show that

$$\Omega\ge 2+\sqrt{2}\quad \textit{(3)}$$

(2) becomes: $\left(\sum_{cyc}\sqrt{17x}\right)^2\ge\left(2+\sqrt{2}\right)^2\leftrightarrow$

$$\leftrightarrow p+3+2\sum_{cyc}\sqrt{(1+x)(1+y)}\ge\left(2+\sqrt{2}\right)^2$$

or $\sum_{cyc}\sqrt{(1+x)(1+y)}\ge\frac{\left(2+\sqrt{2}\right)^2-(p+3)}{2}=\theta(p)\to\sum_{cyc}(1+x)(1+y)+$

$+2\sum_{cyc}\sqrt{1+x}\prod_{cyc}\sqrt{1+x}\ge\theta(p)^2\to(2p+q+3)+2\Omega\sqrt{p+q+r+1}\ge\Omega^2(p)$

which becomes $\Omega\ge g(q)=\frac{\theta^2(p)-(2p+q+3)}{2\sqrt{p+q+r+1}}$ with equivalence through as all squared amounts are positive. As $g(q)$ is a decreasing function of q according to V. Cîrtoaje *theorem with p, r fixed q becomes maximal when $y = z$ assuming WLOG $x \le y \le z$.*

It is enough therefore to show (3) for $y = z$ in which case $x^2 + 2y^2 = 1$ or $y^2 \le \frac{1}{2}$.

$$(3) \rightarrow 2\sqrt{1+y} + \sqrt{1+\sqrt{1-2y^2}} \ge 2+\sqrt{2} \text{ with } y \in \left[0, \frac{1}{\sqrt{2}}\right]. \text{ Let}$$

$$\theta(y) = 2\sqrt{1+y} + \sqrt{1+\sqrt{1-2y^2}} - (2+\sqrt{2}).\ \theta'(y) = -\frac{A}{T} \text{ where}$$

$$A = y\sqrt{y+1} - \sqrt{1-2y^2} \cdot \sqrt{(1-2y^2)+1} \text{ and } T > 0. \text{ It is easy to show that}$$

$A'(y) > 0$ [sum of positive factors] so $A(y)$ has only one root equal to $\frac{\sqrt{3}}{3}$. This root is unique for $\theta'(y)$ and the minimum value of $\theta(y)$ on $\left[0, \frac{1}{\sqrt{2}}\right]$ is $\theta(0) = 0$, hence $\theta(y) \ge 0$. Equality for $x = y = 0, z = 1$ and permutations $2 + \sqrt{2} \le \Omega \le 3\sqrt{1+\frac{\sqrt{3}}{3}}$

JP.246 If $a, b, c, d > 0$; $a^2cd + b^2da + c^2ab + d^2bc = 4abcd$ then:

$$\frac{a^2}{b^2}\left(\frac{a}{b}-1\right) + \frac{b^2}{c^2}\left(\frac{b}{c}-1\right) + \frac{c^2}{d^2}\left(\frac{c}{d}-1\right) + \frac{d^2}{a^2}\left(\frac{d}{a}-1\right) = 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Florentin Vișescu-Romania

$$a^2cd + b^2da + c^2ab + d^2bc = 4abcd \leftrightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{d}{a} = 4$$

$$AM\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}\right) = \frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{d}{a}}{4} = \frac{4}{4} = 1$$

$$GM\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}\right) = \sqrt[4]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{d}{a}} = 1$$

$$AM\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}\right) = GM\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}\right) \rightarrow \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{a} = 1 \rightarrow LHS = 0$$

Solution 2 by proposer

$$\text{If } u > 0 \Rightarrow (u-1)^2(u+1) \ge 0$$

$$(u^2 - 2u + 1)(u + 1) \ge 0$$

$$u^3 + u^2 - 2u^2 - 2u + u + 1 \geq 0$$

$$u^3 - u^2 - u + 1 \geq 0 \Rightarrow u^3 \geq u^2 + u - 1$$

For $u = \frac{a}{b}$ and successively $u = \frac{b}{c}; u = \frac{c}{d}; u = \frac{a}{b}$ then:

$$\left(\frac{a}{b}\right)^3 \geq \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 1 \quad (1)$$

$$\left(\frac{b}{c}\right)^3 \geq \left(\frac{b}{c}\right)^2 + \left(\frac{b}{c}\right) - 1 \quad (2)$$

$$\left(\frac{c}{d}\right)^3 \geq \left(\frac{c}{d}\right)^2 + \left(\frac{c}{d}\right) - 1 \quad (3)$$

$$\left(\frac{d}{a}\right)^3 \geq \left(\frac{d}{a}\right)^2 + \left(\frac{d}{a}\right) - 1 \quad (4)$$

By adding (1); (2); (3); (4):

$$\sum_{cyc}\left(\frac{a}{b}\right)^3 \geq \sum_{cyc}\left(\frac{c}{d}\right)^2 + \sum_{cyc}\frac{a}{b} - 4 = \sum_{cyc}\left(\frac{c}{d}\right)^2 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} - 4 =$$

$$= \sum_{cyc}\left(\frac{a}{b}\right)^2 + \frac{a^2cd + b^2da + c^2ab + d^2bc - 4abcd}{abcd} =$$

$$= \sum_{cyc}\left(\frac{a}{b}\right)^2 + \frac{4abcd - 4abcd}{abcd} = \sum_{cyc}\left(\frac{a}{b}\right)^2$$

For $a = b = c = d = 1, LHS = 0$ which results by condition from enunciation.

JP.247. Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that:

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 + a^2 + b^2 + c^2 \geq 10(ab + bc + ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

Let $\sqrt{a} = \alpha, \sqrt{b} = \beta, \sqrt{c} = \gamma$, $\alpha, \beta, \gamma \geq 0, \alpha^2 + \beta^2 + \gamma^2 = 1$. We must prove:

$$(\alpha + \beta + \gamma)^2 + \alpha^4 + \beta^4 + \gamma^4 \geq 10(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2), (1)$$

Let $\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}, \beta = \frac{y}{\sqrt{x^2+y^2+z^2}}, \gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$$(1) \Leftrightarrow \frac{(x+y+z)^2}{x^2+y^2+z^2} + \frac{x^4+y^4+z^4}{(x^2+y^2+z^2)^2} \ge \frac{10(x^2y^2+y^2z^2+z^2x^2)}{(x^2+y^2+z^2)^2}$$

$$(x+y+z)^2(x^2+y^2+z^2) + x^4+y^4+z^4 - 10(x^2y^2+y^2z^2+z^2x^2) \ge 0 \ (2)$$

$$f_4 - symmetric\ polynomial, \qquad f_4(x,y,z) \ge 0 \Leftrightarrow f_4(x,1,1) \ge 0$$

$$f_4(x,y,z) = (x+y+z)^2(x^2+y^2+z^2) + x^4+y^4+z^4 - 10(x^2y^2+y^2z^2+z^2x^2), \forall x,y,z \ge 0$$

(Cîrtoaje's theorem)

$$f_4(x,1,1) = (x+1+1)^2(x^2+1+1^2) + x^4+1^4+1^4 - 10(x^21^2+1^21^2+1^2x^2)$$

$$f_4(x,1,1) = (x+2)^2(x^2+2) + x^4 + 2 - 10(2x^2+1) =$$

$$= 2x^4 + 4x^3 - 14x^2 + 8x = 2x(x-1)^2(x+4) \ge 0$$

Solution 2 by proposer

We rewrite the inequality in homogenous form as follows:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2(a+b+c) + a^2+b^2+c^2 \ge 10(ab+bc+ca)$$

This is equivalent to:

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)(a+b+c) + a^2+b^2+c^2 \ge 4(ab+bc+ca)$$

After replacing (a,b,c) **by** (a^2,b^2,c^2) ***the inequality becomes:***

$$a^4+b^4+c^4 + (ab+bc+ca)(a^2+b^2+c^2) \ge 4(a^2b^2+b^2c^2+c^2a^2)$$

Or equivalently

$$a^4+b^4+c^4 + abc(a+b+c) + \sum_{cyc} ab(a^2+b^2) \ge 4(a^2b^2+b^2c^2+c^2a^2)$$

According to Schur's inequality of fourth degree we have:

$$a^4+b^4+c^4 + abc(a+b+c) \ge \sum_{cyc} ab(a^2+b^2)$$

Hence it suffices to prove:

$$\sum_{cyc} ab(a^2+b^2) \ge 2(a^2b^2+b^2c^2+c^2a^2)$$

But this is true because:

$$\sum_{cyc} ab(a^2+b^2) - 2(a^2b^2+b^2c^2+c^2a^2) = \sum_{cyc} ab(a-b)^2 \ge 0$$

JP.248. If $a, b, c > 0$ then:

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)\ge 2(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

By Bergstrom's inequality: $\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\ge\frac{9}{4(a+b+c)}$

Remains to prove:

$$(a^2+b^2+c^2)(a+b+c)+9abc-2(ab+bc+ca)(a+b+c)\ge 0, (1)$$

Reminder Cîrtoaje's theorem:

If $f_3(a,b,c)$ is a symmetric and homogenous polynomial og degree 3 then:

$$f_3(a,b,c)\ge 0\Leftrightarrow f_3(1,1,1)\ge 0, f_3(a,1,0)\ge 0, \forall a\ge 0$$

In our case $f_3(1,1,1)=9+9-18=0$

$$f_3(a,1,0)=(a^2+1)(a+1)-2a(a+1)=(a-1)^2(a+1)\ge 0\Rightarrow(1)$$

Solution 2 by Do Chinh-Vietnam

$$4abc\sum_{cyc}\frac{1}{2a+b+c}\overset{BERGSTROM}{\geq}4abc\cdot\frac{9}{4(a+b+c)}=\frac{9}{a+b+c}$$

Remains to prove:

$$(a^2+b^2+c^2)(a+b+c)+9abc-2(ab+bc+ca)(a+b+c)\ge 0$$

$$\sum_{cyc}a^3+3abc\ge\sum_{cyc}ab(a+b)$$

$$\sum_{cyc}ab(a+b)-\sum_{cyc}a^3-2abc\le abc$$

$$(a+b-c)(b+c-a)(c+a-b)\le abc$$

$$(a+b-c)(b+c-a)(c+a-b)\le|a+b-c|\cdot|b+c-a|\cdot|c+a-b|=$$

$$=\prod_{cyc}\sqrt{|a+b-c|\cdot|b+c-a|}\overset{(2)}{\leq}|abc|=abc$$

Observation: (2) ***is true if*** a, b, c ***are sides in triangle.***

Solution 3 by Michael Sterghiou-Greece

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)\ge 2(ab+bc+ca)(1)$$

Inequality is homogenous so WLOG we can assume:

$$a+b+c=3,(p,q,r)=(a+b+c,ab+bc+ca,abc),p=3$$

$$LHS\text{ of }(1)\overset{CBS}{\geq} p^2-2q+4r\cdot\frac{9}{4p}\ge 2q\ \textit{(to prove)}$$

This becomes the stronger inequality: $p^2-4q+\frac{9r}{p}\ge 0\Leftrightarrow 9-4q+3r\ge 0$

Which is 3***-rd degree Schur's inequality-usually written:*** $q\le\frac{p^3+9r}{4p}$

Solution 4 and generalizations by Marin Chirciu-Romania

1) If $a,b,c>0$ then:

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{a+2b+c}+\frac{1}{a+b+2c}\right)\ge 2(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution

Schur's lemma:

2) If $a,b,c>0$ then:

$$a^2+b^2+c^2+\frac{9abc}{a+b+c}\ge 2(ab+bc+ca)$$

Proof.

Inequality can be written:

$a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$***, which follows by Schur's inequality:***

$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b)\ge 0, a,b,c>0, r>0$***, for*** $r=1$***. Equality holds for*** $a=b=c$***.***

Back to the main problem: By Bergström's inequality:

$$\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\ge\frac{(1+1+1)^2}{\sum(2a+b+c)}=\frac{9}{4(a+b+c)}\quad (1)$$

By (1) and Schur's lemma:

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\right)\geq$$

$$\geq a^2+b^2+c^2+\frac{9abc}{a+b+c}\geq 2(ab+bc+ca)$$

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\right)\geq 2(ab+bc+ca)$$

Equality holds for $a=b=c$.

Remark:

3) If $a,b,c>0$ ***and*** $n\geq 0$ ***then:***

$$a^2+b^2+c^2+(n+2)abc\left(\frac{1}{na+b+c}+\frac{1}{nb+c+a}+\frac{1}{nc+a+b}\right)\geq 2(ab+bc+ca)$$

Marin Chirciu

Solution:

By Bergström's inequality: $\frac{1}{na+b+c}+\frac{1}{nb+c+a}+\frac{1}{nc+a+b}\geq\frac{(1+1+1)^2}{\sum(na+b+c)}=\frac{9}{(n+2)(a+b+c)}$ ***(1)***

By (1) and Schur's lemma: $a^2+b^2+c^2+(n+2)abc\left(\frac{1}{na+b+c}+\frac{1}{nb+c+a}+\frac{1}{nc+a+b}\right)\geq$

$$\geq a^2+b^2+c^2+\frac{9abc}{a+b+c}\geq 2(ab+bc+ca)$$

$$a^2+b^2+c^2+(n+2)abc\left(\frac{1}{na+b+c}+\frac{1}{nb+c+a}+\frac{1}{nc+a+b}\right)\geq 2(ab+bc+ca)$$

Equality holds for $a=b=c$.

4) If $a,b,c>0$ ***then:***

$$a^2+b^2+c^2+2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 2(ab+bc+ca)$$

Marin Chirciu

Solution

By Bergström's inequality: $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{(1+1+1)^2}{\sum(b+c)}=\frac{9}{2(a+b+c)}$ ***(1)***

By (1) and Schur's lemma:

$$a^2+b^2+c^2+2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq a^2+b^2+c^2+\frac{9abc}{a+b+c}\geq$$

$$\geq 2(ab+bc+ca)$$

$$a^2+b^2+c^2+2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 2(ab+bc+ca)$$

Equality holds for $a = b = c$.

5) In ΔABC ***the following relationship holds:***

$$a^2+b^2+c^2+abc\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)\ge 2(ab+bc+ca)$$

Marin Chirciu

Solution

By Bergström's inequality: $\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\ge\frac{(1+1+1)^2}{\sum(b+c-a)}=\frac{9}{a+b+c}$ ***(1)***

By (1) and Schur's lemma:

$$a^2+b^2+c^2+abc\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)\ge a^2+b^2+c^2+\frac{9abc}{a+b+c}\ge$$
$$\ge 2(ab+bc+ca)$$

$$a^2+b^2+c^2+abc\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)\ge 2(ab+bc+ca)$$

Equality holds for $a = b = c$.

6) In $\Delta ABC, n\le 1$ ***the following relationship holds:***

$$a^2+b^2+c^2+(2-n)abc\left(\frac{1}{b+c-na}+\frac{1}{c+a-nb}+\frac{1}{a+b-nc}\right)\ge$$
$$\ge 2(ab+bc+ca)$$

Marin Chirciu

Solution

By Begström's inequality: $\frac{1}{b+c-na}+\frac{1}{c+a-nb}+\frac{1}{a+b-nc}\ge\frac{(1+1+1)^2}{\sum(b+c-na)}=\frac{9}{(2-n)(a+b+c)}$ ***(1)***

By (1) and Schur's lemma:

$$a^2+b^2+c^2+(2-n)abc\left(\frac{1}{b+c-na}+\frac{1}{c+a-nb}+\frac{1}{a+b-nc}\right)\ge$$
$$\ge a^2+b^2+c^2+\frac{9abc}{a+b+c}\ge 2(ab+bc+ca)$$

$$a^2+b^2+c^2+(2-n)abc\left(\frac{1}{b+c-na}+\frac{1}{c+a-nb}+\frac{1}{a+b-nc}\right)\ge$$
$$\ge 2(ab+bc+ca)$$

Equality holds for $a = b = c$.

7) In ΔABC the following relationship holds:

$$\sum \tan^2\frac{A}{2}+\frac{2r}{s}\sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}\geq 2$$

Marin Chirciu

Solution

By inequality 4) above, we replace (a,b,c) with $\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ and we use the known relationships: $\prod\tan\frac{A}{2}=\frac{r}{s},\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$. Equality holds for $a=b=c$.

Solution (alternative):

By the known relationships:

$$\sum \tan^2\frac{A}{2}=\left(\frac{4R+r}{s}\right)^2-2,\sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right],\prod\tan\frac{A}{2}=\frac{r}{s}$$

The inequality can be written:

$\left(\frac{4R+r}{s}\right)^2-2+\frac{2r}{s}\cdot\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\geq 2\Leftrightarrow\left(\frac{4R+r}{s}\right)^2\left(1+\frac{r}{2R}\right)\geq 4-\frac{r}{2R}$*, which follows by Blundon-Gerretsen's inequality* $s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$*. It remains to prove:*

$$\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\left(1+\frac{r}{2R}\right)\geq 4-\frac{r}{2R}\Leftrightarrow R\geq 2r$$ *(Euler). Equality holds for $a=b=c$.*

8) In ΔABC the following relationship holds:

$$3+\sum \tan^2\frac{A}{2}\geq\frac{8r}{s}\sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}$$

Marin Chirciu

Solution

Taking $n=-8$ in the above inequality. Equality holds for $a=b=c$.

Solution (alternative):

By the known relationships:

$$\sum \tan^2\frac{A}{2}=\left(\frac{4R+r}{s}\right)^2-2,\sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right],\prod\tan\frac{A}{2}=\frac{r}{s}$$

Inequality can be written:

$3+\left(\frac{4R+r}{s}\right)^2-2\ge\frac{8r}{s}\cdot\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\Leftrightarrow\left[1+\left(\frac{4R+r}{s}\right)^2\right]\left(1-\frac{2r}{R}\right)\ge 0$, *obviously by Euler's inequality* $R\ge 2r$. *Equality holds for* $a=b=c$.

9) If in ΔABC, $-8\le n\le 2$ ***the following relationship holds:***

$$\sum\tan^2\frac{A}{2}+\frac{nr}{s}\sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}\ge 1+\frac{n}{2}$$

Marin Chirciu

Solution

By the known relationships:

$$\sum\tan^2\frac{A}{2}=\left(\frac{4R+r}{s}\right)^2-2,\ \sum\frac{1}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right],\ \prod\tan\frac{A}{2}=\frac{r}{s}$$

The inequality can be written:

$\left(\frac{4R+r}{s}\right)^2-2+\frac{nr}{s}\cdot\frac{s}{4R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\ge 1+\frac{n}{2}\Leftrightarrow\left(\frac{4R+r}{s}\right)^2\left(1+\frac{nr}{4R}\right)\ge 3+\frac{n}{2}-\frac{nr}{4R}$, *which follows by Blundon – Gerretsen's inequality* $s^2\le\frac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove:*

$$\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\left(1+\frac{nr}{4R}\right)\ge 3+\frac{n}{2}-\frac{nr}{4R}\Leftrightarrow(4-2n)R^2+(5n-8)Rr-2nr^2\ge 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)[(4-2n)R+nr]\ge 0$, *obviously by Euler's inequality* $R\ge 2r$ *and* $-8\le n\le 2$, *which assure us* $[(4-2n)R+nr]\ge 0$

Equality holds for $a=b=c$.

JP.249. **ABOUT PROBLEM JP.249**

17 RMM SUMMER EDITION 2020

By Marin Chirciu – Romania

1) If $a,b,c>0$ **then:**

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}\ge\sqrt{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+12}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution

$$\text{By squaring: } \sum \frac{b+c}{a} + 2\sum \sqrt{\frac{c+a}{b}}\sqrt{\frac{a+b}{c}} \ge \sum \frac{b+c}{a} + 12 \Leftrightarrow \sum \sqrt{\frac{c+a}{b}}\sqrt{\frac{a+b}{c}} \ge 6$$

$$\sum \sqrt{\frac{c+a}{b}}\sqrt{\frac{a+b}{c}} \ge 3\sqrt[3]{\sqrt{\prod\left(\frac{b+c}{a}\right)^2}} = 3\sqrt[3]{\prod\left(\frac{b+c}{a}\right)} =$$

$$= 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}} \overset{Cesaro}{\ge} 3\sqrt[3]{8} = 3\cdot 2 = 6$$

Equality holds for $a = b = c$.

Remark.

We propose and solve a few problems of the same kind:

2) In ΔABC the following relationship holds:

$$\sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}} \ge \sqrt{\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} + 12}$$

Marin Chirciu

Solution

$$\text{By squaring: } \sum \frac{a}{s-a} + 2\sum \sqrt{\frac{b}{s-b}}\sqrt{\frac{c}{s-c}} \ge \sum \frac{a}{s-a} + 12 \Leftrightarrow \sum \sqrt{\frac{s}{s-b}}\sqrt{\frac{c}{s-c}} \ge 6$$

$$\sum \sqrt{\frac{s}{s-b}}\sqrt{\frac{c}{s-c}} \ge 3\sqrt[3]{\sqrt{\prod\left(\frac{a}{s-a}\right)^2}} = 3\sqrt[3]{\prod\left(\frac{a}{s-a}\right)} = 3\sqrt[3]{\frac{abc}{\prod(s-a)}} =$$

$$= 3\sqrt[3]{\frac{4Rrs}{r^2 s}} = 3\sqrt[3]{\frac{4R}{r}} \overset{Euler}{\ge} 3\sqrt[3]{8} = 3\cdot 2 = 6$$

Equality holds for an equilateral triangle.

3) In ΔABC the following relationship holds:

$$\sqrt{\frac{s-a}{a}} + \sqrt{\frac{s-b}{b}} + \sqrt{\frac{s-c}{c}} \ge \sqrt{\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} + \frac{6r}{R}}$$

Marin Chirciu

Solution

$$\textit{By squaring: } \sum\frac{s-a}{a}+2\sum\sqrt{\frac{s-b}{b}}\sqrt{\frac{s-c}{c}}\ge\sum\frac{s-a}{a}+\frac{6r}{R}\Leftrightarrow\sum\sqrt{\frac{s-b}{b}}\sqrt{\frac{s-c}{c}}\ge\frac{3r}{R}$$

$$\sum\sqrt{\frac{s-b}{b}}\sqrt{\frac{s-c}{c}}\ge 3\sqrt[3]{\sqrt{\prod\left(\frac{s-a}{a}\right)^2}}=3\sqrt[3]{\prod\left(\frac{s-a}{a}\right)}=3\sqrt[3]{\frac{\prod(s-a)}{abc}}=$$

$$=3\sqrt[3]{\frac{r^2s}{4Rrs}}=3\sqrt[3]{\frac{r}{4R}}\overset{Euler}{\ge}\frac{3r}{R}$$

$$3\sqrt[3]{\frac{r}{4R}}\overset{Euler}{\ge}\frac{3r}{R}\Leftrightarrow\sqrt[3]{\frac{r}{4R}}\ge\frac{r}{R}\Leftrightarrow\frac{r}{4R}\ge\frac{r^3}{R^3}\Leftrightarrow R^2\ge 4r^2\Leftrightarrow R\ge 2r$$

Equality holds for an equilateral triangle.

4) In ΔABC the following relationship holds:

$$\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}\ge\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}+\frac{6r}{R}}$$

Marin Chirciu

Solution

$$\textit{By squaring: } \sum\sin\frac{A}{2}+2\sum\sqrt{\sin\frac{B}{2}}\sqrt{\sin\frac{C}{2}}\ge\sum\sin\frac{A}{2}+\frac{6r}{R}\Leftrightarrow\sum\sqrt{\sin\frac{B}{2}}\sqrt{\sin\frac{C}{2}}\ge\frac{3r}{R}$$

$$\sum\sqrt{\sin\frac{B}{2}}\sqrt{\sin\frac{C}{2}}\ge 3\sqrt[3]{\sqrt{\prod\sin^2\frac{A}{2}}}=3\sqrt[3]{\prod\sin\frac{A}{2}}=3\sqrt[3]{\frac{r}{4R}}\overset{Euler}{\ge}\frac{3r}{R}$$

$$3\sqrt[3]{\frac{r}{4R}}\overset{Euler}{\ge}\frac{3r}{R}\Leftrightarrow\sqrt[3]{\frac{r}{4R}}\ge\frac{r}{R}\Leftrightarrow\frac{r}{4R}\ge\frac{r^3}{R^3}\Leftrightarrow R^2\ge 4r^2\Leftrightarrow R\ge 2r$$

Equality holds for an equilateral triangle.

5) In ΔABC the following relationship holds:

$$\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\ge\sqrt{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}+\frac{2s}{R}}$$

Marin Chirciu

Solution

By squaring: $\sum\cos\frac{A}{2}+2\sum\sqrt{\cos\frac{B}{2}}\sqrt{\cos\frac{C}{2}}\ge\sum\cos\frac{A}{2}+\frac{2s}{R}\Leftrightarrow\sum\sqrt{\cos\frac{B}{2}}\sqrt{\cos\frac{C}{2}}\ge\frac{s}{R}$

$$\sum\sqrt{\cos\frac{B}{2}}\sqrt{\cos\frac{C}{2}}\ge 3\sqrt[3]{\sqrt{\prod\cos^2\frac{A}{2}}}=3\sqrt[3]{\prod\cos\frac{A}{2}}=3\sqrt[3]{\frac{s}{4R}}\overset{Mitrinovic}{\ge}\frac{s}{R}$$

$$3\sqrt[3]{\frac{s}{4R}}\overset{Mitrinovic}{\ge}\frac{s}{R}\Leftrightarrow 3\sqrt[3]{\frac{s}{4R}}\ge\frac{s}{R}\Leftrightarrow\frac{27s}{4R}\ge\frac{s^3}{R^3}\Leftrightarrow s^2\le\frac{27R^2}{4}\Leftrightarrow s\le\frac{3R\sqrt{3}}{2}$$

Equality holds for an equilateral triangle.

6) In ΔABC the following relationship holds:

$$\sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\ge\sqrt{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\frac{18r}{s}}$$

Marin Chirciu

Solution

By squaring: $\sum\tan\frac{A}{2}+2\sum\sqrt{\tan\frac{B}{2}}\sqrt{\tan\frac{C}{2}}\ge\sum\tan\frac{A}{2}+\frac{18r}{s}\Leftrightarrow\sum\sqrt{\tan\frac{B}{2}}\sqrt{\tan\frac{C}{2}}\ge\frac{9r}{s}$

$$\sum\sqrt{\tan\frac{B}{2}}\sqrt{\tan\frac{C}{2}}\ge 3\sqrt[3]{\sqrt{\prod\tan^2\frac{A}{2}}}=3\sqrt[3]{\prod\tan\frac{A}{2}}=3\sqrt[3]{\frac{r}{s}}\overset{Mitrinovic}{\ge}\frac{9r}{s}$$

$$3\sqrt[3]{\frac{r}{s}}\overset{Mitrinovic}{\ge}\frac{9r}{s}\Leftrightarrow\sqrt[3]{\frac{r}{s}}\ge\frac{3r}{s}\Leftrightarrow\frac{r}{s}\ge\frac{27r^3}{s^3}\Leftrightarrow s^2\ge 27r^2\Leftrightarrow s\ge 3r\sqrt{3}.$$

Equality holds for an equilateral triangle.

7) In ΔABC the following relationship holds:

$$\sqrt{\cot\frac{A}{2}}+\sqrt{\cot\frac{B}{2}}+\sqrt{\cot\frac{C}{2}}\ge\sqrt{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}+\frac{54r}{s}}$$

Marin Chirciu

Solution

$$\sum\cot\frac{A}{2}+2\sum\sqrt{\cot\frac{B}{2}}\sqrt{\cot\frac{C}{2}}\ge\sum\cot\frac{A}{2}+\frac{54r}{s}\Leftrightarrow\sum\sqrt{\cot\frac{B}{2}}\sqrt{\cot\frac{C}{2}}\ge\frac{27r}{s}$$

$$\sum\sqrt{\cot\frac{B}{2}}\sqrt{\cot\frac{C}{2}}\geq 3\sqrt[3]{\sqrt{\prod\cot^2\frac{A}{2}}}=3\sqrt[3]{\prod\cot\frac{A}{2}}=3\sqrt[3]{\frac{s}{r}}\overset{Mitrinovic}{\geq}\frac{27r}{s}$$

$$3\sqrt[3]{\frac{s}{r}}\overset{Mitrinovic}{\geq}\frac{27r}{s}\Leftrightarrow\sqrt[3]{\frac{s}{r}}\geq\frac{9r}{s}\Leftrightarrow\frac{s}{r}\geq\frac{729r^3}{s^3}\Leftrightarrow s^4\geq 729r^4\Leftrightarrow s^2\geq 27r^2\Leftrightarrow s\geq 3r\sqrt{3}$$

Equality holds for an equilateral triangle.

8) In ΔABC the following relationship holds:

$$\sqrt{\sec\frac{A}{2}}+\sqrt{\sec\frac{B}{2}}+\sqrt{\sec\frac{C}{2}}\geq\sqrt{\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}+\frac{36r}{s}}$$

Marin Chirciu

Solution

$$\sum\sec\frac{A}{2}+2\sum\sqrt{\sec\frac{B}{2}}\sqrt{\sec\frac{C}{2}}\geq\sum\sec\frac{A}{2}+\frac{36r}{s}\Leftrightarrow\sum\sqrt{\sec\frac{B}{2}}\sqrt{\sec\frac{C}{2}}\geq\frac{18r}{s}$$

$$\sum\sqrt{\sec\frac{B}{2}}\sqrt{\sec\frac{C}{2}}\geq 3\sqrt[3]{\sqrt{\prod\sec^2\frac{A}{2}}}=3\sqrt[3]{\prod\sec\frac{A}{2}}=3\sqrt[3]{\frac{4R}{s}}\overset{Mitrinovic}{\geq}\frac{18r}{s}$$

$$3\sqrt[3]{\frac{4R}{s}}\overset{Mitrinovic}{\geq}\frac{18r}{s}\Leftrightarrow\sqrt[3]{\frac{4R}{s}}\geq\frac{6r}{s}\Leftrightarrow\frac{4R}{s}\geq\frac{216r^3}{s^3}\Leftrightarrow s^2R\geq 54r^3\overset{Euler}{\Leftrightarrow}s^2\geq 27r^2$$

$$\Leftrightarrow s\geq 3r\sqrt{3}$$

Equality holds for an equilateral triangle.

9) In ΔABC the following relationship holds:

$$\sqrt{\csc\frac{A}{2}}+\sqrt{\csc\frac{B}{2}}+\sqrt{\csc\frac{C}{2}}\geq\sqrt{\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}+12}$$

Marin Chirciu

Solution

By squaring:

$$\sum\csc\frac{A}{2}+2\sum\sqrt{\csc\frac{B}{2}}\sqrt{\csc\frac{C}{2}}\geq\sum\csc\frac{A}{2}+12\Leftrightarrow\sum\sqrt{\csc\frac{B}{2}}\sqrt{\csc\frac{C}{2}}\geq 6$$

$$\sum\sqrt{\csc\frac{B}{2}}\sqrt{\csc\frac{C}{2}}\geq 3\sqrt[3]{\sqrt{\prod\csc^2\frac{A}{2}}}=3\sqrt[3]{\prod\csc\frac{A}{2}}=3\sqrt[3]{\frac{4R}{r}}\overset{Euler}{\geq}3\sqrt[3]{8}=3\cdot 2=6$$

Equality holds for an equilateral triangle.

10) In ΔABC the following relationship holds:

$$\sqrt{r_a}+\sqrt{r_b}+\sqrt{r_c}\geq\sqrt{r_a+r_b+r_c+18r}$$

Marin Chirciu

Solution

By squaring: $\sum r_a+2\sum\sqrt{r_br_c}\geq\sum r_a+18r\Leftrightarrow\sum\sqrt{r_br_c}\geq 9r$

$$\sum\sqrt{r_br_c}\geq 3\sqrt[3]{\sqrt{\prod r_a^2}}=3\sqrt[3]{\prod r_a}=3\sqrt[3]{rs^2}\overset{Mitrinovic}{\geq}3\sqrt[3]{r\cdot 27r^2}=9r$$

Equality holds for an equilateral triangle.

11) In ΔABC the following relationship hold:

$$\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c}\geq\sqrt{h_a+h_b+h_c+\frac{36r^2}{R}}$$

Marin Chirciu

Solution

By squaring:

$$\sum h_a+2\sum\sqrt{h_bh_c}\geq\sum h_a+\frac{36r^2}{R}\Leftrightarrow\sum\sqrt{h_bh_c}\geq\frac{18r^2}{R}$$

$$\sum\sqrt{h_bh_c}\geq 3\sqrt[3]{\sqrt{\prod h_a^2}}=3\sqrt[3]{\prod h_a}=3\sqrt[3]{\frac{2s^2r^2}{R}}\overset{Mitrinovic}{\geq}\frac{18r^2}{R}$$

$$3\sqrt[3]{\frac{2s^2r^2}{R}}\overset{Mitrinovic}{\geq}\frac{18r^2}{R}\Leftrightarrow\sqrt[3]{\frac{2s^2r^2}{R}}\geq\frac{6r^2}{R}\Leftrightarrow\frac{2s^2r^2}{R}\geq\frac{216r^6}{R^3}\Leftrightarrow s^2R^2\geq 108r^4\overset{Euler}{\Leftrightarrow}$$

$$\Leftrightarrow s^2\geq 27r^2\Leftrightarrow s\geq 3r\sqrt{3}.$$

Equality holds for an equilateral triangle.

JP.250. **ABOUT PROBLEM JP.250**

17 RMM Summer Edition 2020

By Marin Chirciu – Romania

1) If $a,b,c>0$ are such that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{4}{(b+c)^2}+\frac{4}{(c+a)^2}+\frac{4}{(a+b)^2}\le 6$, then:

$$ab+bc+ca\ge 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution

By Bergström's inequality $\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{\left(\frac{1}{a}+\frac{1}{b}\right)^2}{2}$, *equality holds if* $a=b$.

By summing: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right)^2$ ***(1)***

$$6\ge\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{4}{(b+c)^2}+\frac{4}{(c+a)^2}+\frac{4}{(a+b)^2}\overset{(1)}{\ge}\frac{1}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right)^2+\sum\frac{4}{(a+b)^2}=$$

$$=\frac{1}{4}\sum\left[\frac{(a+b)^2}{a^2b^2}+\frac{16}{(a+b)^2}\right]\overset{AM-GM}{\ge}\frac{1}{4}\sum 2\sqrt{\frac{(a+b)^2}{a^2b^2}\cdot\frac{16}{(a+b)^2}}=2\sum\frac{1}{ab},$$ *hence* $6\ge 2\sum\frac{1}{ab}$

So, $\sum\frac{1}{ab}\le 3$ ***(2)***

Finally, $\sum ab\overset{AM-HM}{\ge}\frac{9}{\sum\frac{1}{ab}}\overset{(2)}{\ge}\frac{9}{3}=3$. *Equality holds if* $a=b=c$.

EXTENSION FOR 4 VARIABLES

2) If $a,b,c,d>0$ are such that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{4}{(a+b)^2}+\frac{4}{(b+c)^2}+\frac{4}{(c+d)^2}+\frac{4}{(d+a)^2}\le 8,$$ ***then:***

$$ab+bc+cd+da\ge 4$$

Solution

By Bergström's inequality $\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{\left(\frac{1}{a}+\frac{1}{b}\right)^2}{2}$. *Equality holds if* $a=b$.

By summing: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\ge\frac{1}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right)^2$ ***(1)***

$$8\ge\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{4}{(a+b)^2}+\frac{4}{(b+c)^2}+\frac{4}{(c+d)^2}+\frac{4}{(d+a)^2}\overset{(1)}{\ge}$$

$$\geq \frac{1}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right)^2+\sum\frac{4}{(a+b)^2}=$$

$$=\frac{1}{4}\sum\left[\frac{(a+b)^2}{a^2b^2}+\frac{16}{(a+b)^2}\right]\overset{AM-GM}{\geq}\frac{1}{4}\sum 2\sqrt{\frac{(a+b)^2}{a^2b^2}\cdot\frac{16}{(a+b)^2}}=2\sum\frac{1}{ab}, \text{ so } 8\geq 2\sum\frac{1}{ab}$$

Hence $\sum\frac{1}{ab}\leq 4$ ***(2)***

Finally, $\sum ab\overset{AM-HM}{\geq}\frac{16}{\sum\frac{1}{ab}}\overset{(2)}{\geq}\frac{16}{4}=4$

Equality holds if and only if $a=b=c=d$.

EXTENSION FOR n VARIABLES:

3) If $a_1, a_2, \dots, a_n > 0$ are such that:

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\dots+\frac{1}{a_n^2}+\frac{4}{(a_1+a_2)^2}+\frac{4}{(a_2+a_3)^2}+\dots+\frac{4}{(a_n+a_1)^2}\leq 2n$$

then: $a_1a_2+a_2a_3+\dots+a_na_1\geq n$.

Marin Chirciu

Solution

By Bergstöm's inequality $\frac{1}{a_1^2}+\frac{1}{a_2^2}\geq\frac{\left(\frac{1}{a_1}+\frac{1}{a_2}\right)^2}{2}$. *Equality holds if* $a_1=a_2$.

By summing $n-1$ *inequalities:* $\frac{1}{a_1^2}+\frac{1}{a_2^2}+\dots+\frac{1}{a_n^2}\geq\frac{1}{4}\sum\left(\frac{1}{a_1}+\frac{1}{a_2}\right)^2$ ***(1)***

$$2n\geq\frac{1}{a_1^2}+\frac{1}{a_2^2}+\dots+\frac{1}{a_n^2}+\frac{4}{(a_1+a_2)^2}+\frac{4}{(a_2+a_3)^2}+\dots+\frac{4}{(a_n+a_1)^2}\overset{(1)}{\geq}$$

$$\geq\frac{1}{4}\sum\left(\frac{1}{a_1}+\frac{1}{a_2}\right)^2+\sum\frac{4}{(a_1+a_2)^2}=$$

$$=\frac{1}{4}\sum\left[\frac{(a_1+a_2)^2}{a_1^2a_2^2}+\frac{16}{(a_1+a_2)^2}\right]\overset{AM-GM}{\geq}\frac{1}{4}\sum 2\sqrt{\frac{(a_1+a_2)^2}{a_1^2a_2^2}\cdot\frac{16}{(a_1+a_2)^2}}=2\sum\frac{1}{a_1a_2}$$

So, $2n\geq 2\sum\frac{1}{a_1a_2}$, *hence* $\sum\frac{1}{a_1a_2}\leq n$ ***(2)***

Finally, $\sum a_1a_2\overset{AM-HM}{\geq}\frac{n^2}{\sum\frac{1}{a_1a_2}}\overset{(2)}{\geq}\frac{n^2}{n}=n$.

Equality holds if and only if $a_1=a_2=\dots=a_n$.

Remark.

For $n = 3$ we obtain problem JP.250 from 17-RMM SUMMER 2020, proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.251. Let be $a, b, c, p \in (0, \infty)$ with $a < b < c$. Solve the following equation:

$$a^x(b+p)^x + b^x(c+p)^x + c^x(a+p)^x = a^x(c+p)^x + b^x(a+p)^x + c^x(p+b)^x$$

Proposed by Florentin Vișescu – Romania

Solution by proposer

Obviously $x = 0$ is a solution. For $x \neq 0$. Considering the functions $f, g: (0, \infty) \to \mathbb{R}$; $f(t) = t^x$ and $g(t) = (t+p)^x$. Applying Cauchy's theorem on the intervals $[a, b]$ and $[b, c]$. Then $\exists \alpha \in (a, b)$ and $\beta \in (b, c)$

$$\frac{f'(\alpha)}{g'(\alpha)} = \frac{f(b)-f(a)}{g(b)-g(a)} \text{ and } \frac{f'(\beta)}{g'(\beta)} = \frac{f(c)-f(b)}{f(c)-g(b)}$$

$$\text{or } \frac{x\alpha^{x-1}}{x(\alpha+p)^{x-1}} = \frac{b^x-a^x}{(b+p)^x-(a+p)^x} \text{ and } \frac{x\beta^{x-1}}{x(\beta+p)^{x-1}} = \frac{c^x-b^x}{(c+p)^x-(b+p)^x}$$

$$\text{we will prove that } \frac{b^x-a^x}{(b+p)^x-(a+p)^x} = \frac{c^x-b^x}{(c+p)^x-(b+p)^x}$$

$$\Leftrightarrow b^x(c+p)^x - b^x(b+p)^x - a^x(c+p)^x + a^x(b+p)^x = c^x(b+p)^x - c^x(a+p)^x - $$
$$-b^x(b+p)^x + b^x(a+p)^x \Leftrightarrow$$
$$b^x(c+p)^x + a^x(b+p)^x + c^x(a+p)^x = a^x(c+p)^x + c^x(b+p)^x + b^x(a+p)^x \text{ (true)}$$

$$\text{hypothesis. So, } \left(\frac{\alpha}{\alpha+p}\right)^{x-1} = \left(\frac{\beta}{\beta+p}\right)^{x-1} \Leftrightarrow \left(\frac{\alpha\beta+\alpha p}{\alpha\beta+\beta p}\right)^{x-1} = 1$$

(as $\alpha\beta + \alpha p \neq \alpha\beta + \beta p$; $\alpha \neq p$) $\Rightarrow x - 1 = 0; x = 1$. Then, $S = \{0, 1\}$

JP.252. Solve for real numbers:

$$\left(log\left(\frac{x}{x^2+1}\right)+x\right)^3 = \left(log\left(\frac{x}{x^2+1}\right)-x\right)^3 + \left(x - log(x^3+x)\right)^3 + \left(x + log(x^3+x)\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Florentin Vişescu-Romania

$$2x\left(\left(log\left(\frac{x}{x^2+1}\right)+x\right)^2+\left(log\left(\frac{x}{x^2+1}\right)-x\right)^2+log^2\left(\frac{x}{x^2+1}\right)-x^2\right)=$$

$$=2x\left(\left(x-log(x^3+x)\right)^2+\left(x+log(x^3+x)\right)^2-x^2+log^2(x^3+x)\right)$$

$$\left(log\left(\frac{x}{x^2+1}\right)+x\right)^2-\left(x-log(x^3+x)\right)^2+\left(log\left(\frac{x}{x^2+1}\right)-x\right)^2-$$

$$-\left(x+log(x^3+x)\right)^2+log^2\left(\frac{x}{x^2+1}\right)-log^2(x^3+x)=0$$

$$\left(2x+log\left(\frac{x}{x^2+1}\right)-log(x^3+x)\right)\left(log\left(\frac{x}{x^2+1}\right)+log(x^3+x)\right)+$$

$$+\left(log\left(\frac{x}{x^2+1}\right)+log(x^3+x)\right)\left(log\left(\frac{x}{x^2+1}\right)-log(x^3+x)-2x\right)+$$

$$+\left(log\left(\frac{x}{x^2+1}\right)-log(x^3+x)\right)\left(log\left(\frac{x}{x^2+1}\right)+log(x^3+x)\right)=0$$

$$\left(2x+log\frac{1}{(x^2+1)^2}\right)(logx^2)+\left(-2x+log\frac{1}{(x^2+1)^2}\right)(logx^2)+$$

$$+\left(log\frac{1}{(x^2+1)^2}\right)(logx^2)=0$$

$$2x-2log(x^2+1)-2x-2log(x^2+1)-2log(x^2+1)=0$$

$log(x^2+1)=0\rightarrow x=0$***-not solution because*** $x>0$

$$logx^2=0\rightarrow x=1$$

Solution 2 by Orlando Irahola Ortega-La Paz-Bolivia

$$\left[\log\left(\frac{x}{x^2+1}\right)+x\right]^3=\left[\log\left(\frac{x}{x^2+1}\right)-x\right]^3+[x-\log(x^2+x)]^3+[x+\log(x^2+x)]^3$$

$$\frac{x}{x^2+1}>0\wedge x^2+x>0\Rightarrow x\in]0,\infty[$$

$$a=\log\left(\frac{x}{x^2+1}\right)-x\quad\Rightarrow\quad(a+b+c)^3=a^3+b^3+c^3$$

$$b=x-\log(x^2+x)\quad\Rightarrow\quad(a+b+c)(ab+ac+bc)=abc$$

$$c=x+\log(x^2+x)\quad\Rightarrow\quad(a+b)(a+c)(b+c)=0\Rightarrow \begin{matrix}a+b=0 & (1)\\ a+c=0 & (2)\\ b+c=0 & (3)\end{matrix}$$

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$$a+b+c=\log\left(\frac{x}{x^2+1}\right)+x$$

$$\textbf{(1) } a+b=0\Rightarrow \log\left[\frac{x}{(x^2+1)(x^2+x)}\right]=0\Rightarrow x=(x^2+1)(x^2+x)\Rightarrow$$

$$\Rightarrow x=0\wedge x(x^2+x+1)=0\Rightarrow x=0; x^2+x+1\in\mathbb{C}$$

$$\textbf{(2) } a+c=0\Rightarrow \log\left[\frac{x(x^2+x)}{x^2+1}\right]=0\Rightarrow x^2(x+1)=x^2+1\Rightarrow$$

$$\Rightarrow (x-1)(x^2+x+1)=0\Rightarrow x=1$$

$$\textbf{(3) } b+c=0\Rightarrow 2x=0\Rightarrow x=0; S=\{1\}$$

JP.253. If $n\in\mathbb{N}, n\geq 2$ and $a_k\in\left[\frac{\pi}{12},\frac{\pi}{6}\right], \forall k=\overline{1,n}$, then:

$$\left(\sum_{k=1}^{n}\sin^2 a_k\right)\cdot\left(\sum_{k=1}^{n}\cot^2 a_k\right)<n^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by proposers

Because $a_k\in\left[\frac{\pi}{12},\frac{\pi}{6}\right]$ it follows that or $a_k<a_k, \tan a_k>a_k, \forall k=\overline{1,n}\Leftrightarrow$ or $a_k<a_k$

$$\cot a_k=\frac{1}{\tan a_k}<\frac{1}{a_k} \text{ and then:}$$

$$\left(\sum_{k=1}^{n}\sin^2 a_k\right)\cdot\left(\sum_{k=1}^{n}\cot^2 a_k\right)<\left(\sum_{k=1}^{n}a_k^2\right)\cdot\left(\sum_{k=1}^{n}\frac{1}{a_k^2}\right)\leq$$

$$\overset{Polya-Szego}{\leq}\left(\sum_{k=1}^{n}a_k\cdot\frac{1}{a_k}\right)^2\cdot\frac{\left(\frac{\pi^2}{12}+\frac{\pi^2}{6}\right)^2}{4\cdot\frac{\pi^2}{12}\cdot\frac{\pi^2}{6}}\overset{AM\_GM}{\gtrless}n^2\cdot\frac{4\cdot\frac{\pi^2}{12}\cdot\frac{\pi^2}{6}}{4\cdot\frac{\pi^2}{12}\cdot\frac{\pi^2}{6}}=n^2$$

JP.254. Let be $n\in\mathbb{N}, n\geq 2$ and $a_k\in\left[\frac{\pi}{12},\frac{\pi}{2}\right), \forall k=\overline{1,n}$. Prove that:

$$\left(\sum_{k=1}^{n}\sin a_k\right)\cdot\left(\sum_{k=1}^{n}\cot a_k\right)<\frac{49n^2}{24}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by proposers

Because $a_k \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right)$ *it follows that or* $sina_k < a_k$ *and* $\tan a_k > a_k, \forall k = \overline{1,n} \Leftrightarrow$ *or*

$a_k < a_k$ *and* $\cot a_k = \frac{1}{\tan a_k} < \frac{1}{a_k}; \forall k = \overline{1,n}$. *So:*

$$\left(\sum_{k=1}^{n} \sin a_k\right) \cdot \left(\sum_{k=1}^{n} \cot a_k\right) < \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) <$$

$$\overset{POLYA-SZEGO}{\lesssim} \quad n^2 \cdot \frac{\left(\frac{\pi}{12} + \frac{\pi}{2}\right)^2}{4 \cdot \frac{\pi}{12} \cdot \frac{\pi}{2}} = \frac{n^2 \cdot \frac{7^2\pi^2}{12^2}}{2 \cdot \frac{\pi^2}{12}} = \frac{49}{2 \cdot 12} = \frac{49}{24}$$

JP.255. If $m \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$3m + \left(a^2 \cot\frac{A}{2}\right)^{m+1} + \left(b^2 \cot\frac{B}{2}\right)^{m+1} + \left(c^2 \cot\frac{C}{2}\right)^{m+1} \geq 36(m+1)\sqrt{3}r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \Omega = 3m + \sum\left(a^2 \cot\frac{A}{2}\right)^{(m+1)} = 3m + \sum\left[1 + \left(a^2 \cot\frac{A}{2} - 1\right)\right]^{m+1} \overset{Bernoulli}{\geq}$$

$$\geq 3m + \sum\left[1 + (m+1)a^2 \cot\frac{A}{2} - (m+1)\right]$$

$$\Rightarrow \Omega \geq 3m + 3 - 3m - 3 + (m+1)\sum a^2 \cot\frac{A}{2}$$

$$\Rightarrow \Omega \geq (m+1)\left(\sum a^2 \cot\frac{A}{2}\right) \overset{AM-GM}{\geq} 3(m+1)\left[(abc)^2 \prod \cot\frac{A}{2}\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3(m+1)\left[(4Rrs) \cdot \left(\frac{s^3}{r_a r_b r_c}\right)\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3(m+1)\left[\left(4 \cdot 2r \cdot r \cdot 3\sqrt{3}r\right)^2 \cdot \frac{s^3}{s^2 r}\right]^{\frac{1}{3}} \quad [\because R \geq 2r, s \geq 3\sqrt{3}r]$$

$$\Rightarrow \Omega \ge 3(m+1)\cdot\left[(64)^{\frac{1}{3}}\cdot(r^3)^{\frac{2}{3}}\cdot(27)^{\frac{1}{3}}\cdot\left(\frac{3\sqrt{3}r}{r}\right)^{\frac{1}{3}}\right]\ [\because s \ge 3\sqrt{3}r]$$

$$\Rightarrow \Omega \ge 3(m+1)\cdot 4\cdot 3\cdot r^2\sqrt{3}$$

$$\Leftrightarrow 3m + \sum\left(a^2\cot\frac{A}{2}\right)^{(m+1)} \ge 36(m+1)\sqrt{3}r^2 \quad \textit{(proved)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = 3m + \sum\left(1+\left(a^2\cot\frac{A}{2}-1\right)\right)^{m+1}$$

$$\overset{Bernoulli}{\ge} 3m + \sum\left(1+(m+1)\left(a^2\cot\frac{A}{2}-1\right)\right)\ (\because m+1 \ge 1)$$

$$= 3m + \sum\left(1+(m+1)a^2\cot\frac{A}{2}-m-1\right)$$

$$= 3m - 3m + (m+1)\sum\frac{a^2 s}{s\tan\frac{A}{2}} = \left(\sum\frac{a^2 s}{r_a}\right)(m+1)$$

$$= \left(\sum\frac{a^2 s(s-a)}{rs}\right)^{(m+1)} = \left(\frac{s\sum a^2 - \sum a^3}{r}\right)(m+1)$$

$$= \left[\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{r}\right](m+1)$$

$$= 4s(R+r)(m+1) \overset{Euler}{\ge} 4s\cdot 3r(m+1) \overset{Mitrinovic}{\ge} 4(3\sqrt{3}r)\cdot 3r(m+1)$$

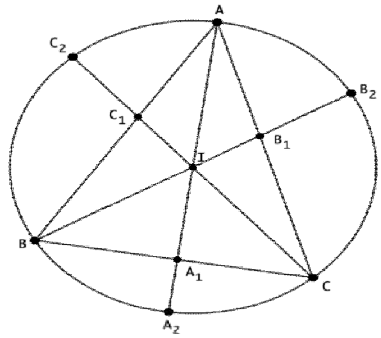
$$= 36\sqrt{3}(m+1)r^2 \quad \textit{(Proved)}$$

SP.241 Let w'_a, w'_b, w'_c be the circumpedal extensions of cevian of incentre in ΔABC. Prove that:

$$w'_a w'_b w'_c \ge \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1 = w_a; BB_1 = w_b; CC_1 = w_c; AA_1 \cap BB_1 \cap CC_1 = \{I\}$$

$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{b+c}$$

$$\frac{A_1B}{a} = \frac{c}{b+c} \Rightarrow A_1B = \frac{ac}{b+c}; A_1C = \frac{ab}{b+c}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2$$

$$\frac{ac}{b+c} \cdot \frac{ab}{b+c} = w_a \cdot A_1A_2$$

$$A_1A_2 = \frac{a^2bc}{w_a(b+c)^2}$$

$$w'_a = AA_2 = AA_1 + A_1A_2 = w_a + \frac{a^2bc}{w_a(b+c)^2} \overset{AM-GM}{\geq} 2\sqrt{w_a \cdot \frac{a^2bc}{w_a(b+c)^2}} = \frac{2a}{b+c}\sqrt{bc}$$

Analogous: $w'_b = \frac{2b}{c+a}\sqrt{ca}; w'_c = \frac{2c}{a+b}\sqrt{ab}$

By multiplying: $w'_a w'_b w'_c \geq \frac{2a}{b+c}\sqrt{bc} \cdot \frac{2b}{c+a}\sqrt{ca} \cdot \frac{2c}{a+b}\sqrt{ab} =$

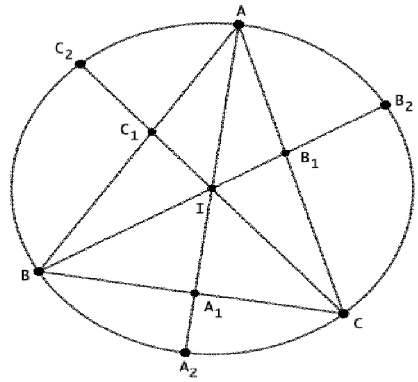
$$= \frac{8abc \cdot abc}{(a+b)(b+c)(c+a)} \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

SP. 242 Let w'_a, w'_b, w'_c be the circumpedal extensions of cevians of incentre in ΔABC. Prove that:

$$w'_a + w'_b + w'_c \geq 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1=w_a, BB_1=w_b, CC_1=w_c, AA_1\cap BB_1\cap CC_1=\{I\}$$

$$\frac{A_1B}{A_1C}=\frac{c}{b}\Rightarrow\frac{A_1B}{A_1B+A_1C}=\frac{c}{b+c}$$

$$\frac{A_1B}{a}=\frac{c}{b+c}\Rightarrow A_1B=\frac{ac}{b+c};A_1C=\frac{ab}{b+c}$$

$$\rho(A_1)=BA_1\cdot A_1C=AA_1\cdot A_1A_2$$

$$\frac{ac}{b+c}\cdot\frac{ab}{b+c}=w_a\cdot A_1A_2$$

$$A_1A_2=\frac{a^2bc}{w_a(b+c)^2}$$

$$w'_a=AA_2=AA_1+A_1A_2=w_a+\frac{a^2bc}{w_a(b+c)^2}$$

Analogous: $w'_b=w_b+\frac{ab^2c}{w_b(a+c)^2};w'_c=w_c+\frac{abc^2}{w_c(b+a)^2}$

By summing: $w'_a+w'_b+w'_c=w_a+\frac{a^2bc}{w_a(b+c)^2}+w_b+\frac{ab^2c}{w_b(a+c)^2}+w_c+\frac{abc^2}{w_c(b+a)^2}\geq$

$$\overset{AM-GM}{\geq}6\sqrt[6]{w_aw_bw_c\cdot\frac{a^2bc\cdot ab^2c\cdot abc^2}{w_aw_bw_c(b+c)^2(a+c)^2(b+a)^2}}=$$

$$=6\sqrt[6]{\frac{a^4b^4c^4}{(a+b)^2(b+c)^2(c+a)^2}}=6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

SP.243. Let P be a polynomial such that $P^4(x)+16=28P^2(x^2-4)$, for all x real numbers. Prove that P is constant.

Proposed by Pedro H. O. Pantoja – Natal/RN – Brazil

Solution 1 by Khaled Abd Imouti-Damascus-Syria

Suppose $\deg P(x)=x$

$$\left(P^2(x)-14+6\sqrt{5}\right)\left(P^2(x)-14-6\sqrt{5}\right)=P^4(x)+16-28P^2(x),\forall x\in\mathbb{R}$$

$$\left(P^2(x)-14+6\sqrt{5}\right)\left(P^2(x)-14-6\sqrt{5}\right)=28P^2(x^2-4)-28P^2(x)$$

$$\underbrace{\left(P^2(x)-14+6\sqrt{5}\right)\left(P^2(x)-14-6\sqrt{5}\right)}_{\deg=4n}=\underbrace{28[P^2(x^2-4)-P^2(x)]}_{\deg=2n-1},\forall x\in\mathbb{R}$$

So: it must $a_n=a_{n-1}=\cdots=a_1=0$

So: $\left(P^2(x)-14+6\sqrt{5}\right)\cdot\left(P^2(x)-14-6\sqrt{5}\right)=0$

$$P^2(x)=14-6\sqrt{5}=\left(3-\sqrt{5}\right)^2\left\langle\begin{array}{l}P(n)=3-\sqrt{5}\\P(n)=-3+\sqrt{5}\end{array}\right.$$

$$\text{or } P^2(x)=14+6\sqrt{5}=\left(3+\sqrt{5}\right)^2\left\langle\begin{array}{l}P(n)=3+\sqrt{5}\\P(n)=-3-\sqrt{5}\end{array}\right.$$

So: $P(n)$ *is constant*

Solution 2 by proposer

Suppose that P is not a constant. Fixing $\deg(P)=n$ and comparing coefficients of both sides we deduce that the coefficients of polynomial P must be rational. On the other hand, setting $x=a$ with $a=a^2-4$, that is, $a=\frac{1\pm\sqrt{17}}{2}$, we obtain $P(a)=b$, where $b^4-28b^2+16=0$, i.e. $b=\pm\sqrt{14\pm6\sqrt{5}}=\pm\left(3\pm\sqrt{5}\right)$. However, this is impossible because $P(a)$ must be of the form $p+q\sqrt{17}$ for some rational p,q for the coefficients of P are rational. It follows that $P(x)$ is constant.

SP.244. If $0<y<x<2y$ then:

$$x(x+y)\sqrt{4y^2-x^2}<3y^3\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$x(x+y)\sqrt{4y^2-x^2}\overset{(1)}{<}3y^3\sqrt{3}$$

$$(1)\Leftrightarrow x^2(x+y)^2(4y^2-x^2)<27y^6$$

$$\Leftrightarrow y^6(t^6+2t^5-3t^4-8t^3-4t^2+27)>0\ \left(t=\frac{x}{y}\right)$$

$$\Leftrightarrow (t^6-8t^3+16)+2t^5-3t^4-4t^2+11>0$$

$$\Leftrightarrow (t^3-4)^2+2t^5-3t^4-4t^2+11\overset{(2)}{>}0$$

$$\because (t^3-4)^2\ge 0, \therefore \text{ it suffices to prove: } 2t^5-3t^4-4t^2+11\overset{(3)}{>}0$$

$$\because 0<y<x<2y \therefore 0<1<\frac{x}{y}<2\Rightarrow 1<t<2$$

$$\text{Let } t-1=m \text{ of course, } m>0$$

$$\text{Then, (3)}\Leftrightarrow 2(m+1)^5-3(m+1)^4-4(m+1)^2+11>0$$

$$\Leftrightarrow 2m^5+7m^4+8m^3-2m^2-10m+6\overset{(4)}{>}0$$

$$\text{Now, } 8m^3+2\cdot 5+2\cdot 5\overset{A-G}{\ge}3\sqrt[3]{8m^3\times(2\cdot 5)^2}>11m\Rightarrow 8m^3-11m+5\overset{(i)}{>}0$$

$$\text{Also, discriminant of } 7m^4-2m^2+1=4-4(7)<0\Rightarrow 7m^4-2m^2+1\overset{(ii)}{>}0$$

$$\text{and of course, } \because m>0, 2m^5+m\overset{(iii)}{>}0$$

(i)+(ii)+(iii)⇒(4)⇒(3)⇒(2)⇒(1) is true

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$x(x+y)\sqrt{4y^2-x^2}\overset{?}{<}3y^3\sqrt{3},\ x\cdot y\left(\frac{x}{y}+1\right)\sqrt{y^2\left(4-\frac{x^2}{y^2}\right)}\overset{?}{<}3y^3\sqrt{3}$$

$$xy^2\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2}\overset{?}{<}3y^3\sqrt{3},\ x\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2}\overset{?}{<}3y\sqrt{3}$$

$$\frac{x}{y}\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2}\overset{?}{<}3\sqrt{3}.\ \text{Suppose: } t=\frac{x}{y}$$

$$t(t+1)\sqrt{4-t^2}\overset{?}{<}3\sqrt{3},\ y<x<2y\Rightarrow 1<\frac{x}{y}<2$$

$$1<t<2, t\in\,]1,2[.\ \text{ Now, let us prove:}$$

$$t(t+1)\sqrt{4-t^2}\overset{?}{<}3\sqrt{3}, t\in\,]1,2[.\ \text{ Let be the function:}$$

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$$f(t)=(t^2+t)\cdot\sqrt{4-t^2},\ t\in\,]1,2[,\ \lim_{\substack{t\to 1\\ >}}(f(t))=2\sqrt{3},\ \lim_{\substack{t\to 1\\ <}}(f(t))=0$$

$$f'(t)=(2t+1)\sqrt{4-t^2}+\frac{-2t}{2\sqrt{4-t^2}}\cdot(t^2+t),\ f'(t)=\frac{(2t+1)(4-t^2)-t(t^2+t)}{\sqrt{4-t^2}}$$

$$f'(t)=\frac{8t-2t^3+4-t^2-t^3-t^2}{\sqrt{4-t^2}},\ f'(t)=\frac{-3t^3-2t^2+8t+4}{\sqrt{4-t^2}};\ f'(t)=0\Rightarrow -3t^3-2t^2+8t+4=0$$

Let be the function: $g(x)=-3t^3-2t^2+3t+4,\ D=\,]1,2[$

$$\lim_{\substack{x\to 1\\ >}}(g(x))=7,\ \lim_{\substack{x\to 2\\ <}}(g(x))=-12$$

$$g'(t)=-9t^2-6t+8,\ g'(t)=0\Rightarrow -9t^2-6t+8=0$$

$$\Delta=36-4(-9)(8)=36+36(8);\ \Delta=36(1+8)=36\cdot 9\Rightarrow\sqrt{\Delta}=18$$

$$t=\frac{+6-18}{2(-9)}=\frac{-12}{-18}=+\frac{2}{3}>0,\ t=\frac{6+18}{2(-9)}=\frac{+24}{-18}=-\frac{4}{3}<0$$

$$g(\sqrt{3})=-9\sqrt{3}-6+8\sqrt{3}+4=-\sqrt{3}-2<0$$

$$g\left(\frac{3}{2}\right)=-3\left(\frac{27}{8}\right)-2\left(\frac{9}{4}\right)+8\left(\frac{3}{2}\right)+4=\frac{-31}{8}-\frac{9}{2}+16$$

$$g\left(\frac{3}{2}\right)=\frac{-81-36}{8}+16=\frac{11}{8}>0;\ g(\sqrt{3})=\sqrt{3}-2<0,\ g\left(\frac{3}{2}\right)=\frac{11}{8}>0$$

x	1 $\frac{3}{2}$ x_0 $1,616$ $\sqrt{3}$ 2
$g'(x)$	$7 + + + + + + + + \ 0 - - - - - - - \ -12$
$g(x)$	

x	1	$\frac{3}{2}$	$1,558$	x_0	$\frac{\sqrt{3}+\frac{3}{2}}{2}$	$\sqrt{3}$	2
$f'(x)$				0			
$f(x)$	$2\sqrt{3}$ ↗	$\frac{15}{8}\sqrt{7}$	$4,99$	$f(x_0)$	$4,22$ ↘	$\sqrt{3}(\sqrt{3}+1)$	0
		$4,96$			$4,73$		

Not: Local max of $f(x)$ ***on the interval*** $[1,2]$ ***is nearly*** 5.

So: $\forall x \in]1,2[: f(x) \le 5,\ x(x+1)\sqrt{4-x^2} < 5,5 < 3\sqrt{3},\ f(x) < 5\sqrt{3}$

If $0 < y < x < 2y$ ***then:*** $x(x+y)\sqrt{4y^2-x^2} \overset{?}{<} 3y^3\sqrt{3}$

$$xy\left(\frac{x}{y}+1\right)\sqrt{y^2\left(4-\frac{x^2}{y^2}\right)} \overset{?}{<} 3y^3\sqrt{3},\ xy^2\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2} \overset{?}{<} 3y^3\sqrt{3}$$

$x\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2} \overset{?}{<} 3y\sqrt{3}; \frac{x}{y}\left(\frac{x}{y}+1\right)\sqrt{4-\left(\frac{x}{y}\right)^2} \overset{?}{<} 3\sqrt{3}$. ***Suppose:*** $t = \frac{x}{y}$

$$t\cdot(t+1)\sqrt{4-t^2} \overset{?}{<} 3\sqrt{3},\ \ y < x < 2y \Rightarrow 1 < \frac{x}{y} < 2$$

$1 < t < 2, t \in]1,2[$. ***Now, let us prove:*** $t(t+1)\sqrt{4-t^2} \overset{?}{<} 3\sqrt{3},\ \ \sqrt{4-t^2} \overset{?}{<} \frac{3\sqrt{3}}{t(t+1)}$

Let be the function: $f(t) = \sqrt{4-t^2}, t \in]1,2[$

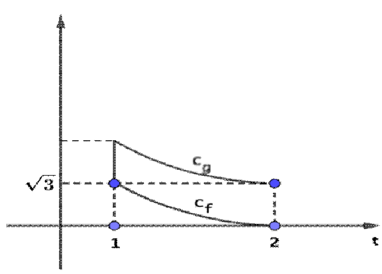
$$\lim_{\substack{t\to 1\\ >}}[f(t)] = \sqrt{3}, \lim_{\substack{t\to 2\\ <}}[f(t)] = 0;\ f'(t) = \frac{-t}{\sqrt{4-t^2}}; f'(t) = 0 \Rightarrow t = 0 \notin]1,2[$$

t	1	2
$f'(t)$	− − − − − − − − − − − − − − − − − − − −	
$f(t)$	$\sqrt{3}$ ↘	0

Let be the function: $g(t) = \frac{3\sqrt{3}}{t^2+t},]1,2[, \lim_{\substack{t\to 1\\ >}}[g(t)] = \frac{3\sqrt{3}}{2}, \lim_{\substack{t\to 2\\ <}}[g(t)] = \frac{3\sqrt{3}}{5}$

$$g'(t) = \frac{-3\sqrt{3}(2t+1)}{(t^2+t)^2}, g'(t) = 0 \Rightarrow t = -\frac{1}{2} \notin]1,2[$$

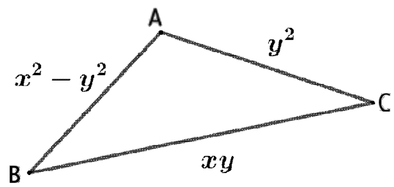
x	1	2
$g'(t)$	− − − − − − − − − − − − − − − − − − − −	
$g(t)$	$\frac{3\sqrt{3}}{2}$ ↘	$\frac{3\sqrt{3}}{5}$



$$\forall t \in]1,2[: f(t) < g(t)$$

Solution 3 by proposer

Let's consider $\Delta ABC; BC = a = xy; AC = b = y^2; AB = c = x^2 - y^2$



$$a < b + c \Leftrightarrow xy < x^2 - y^2 + y^2 \Leftrightarrow y < x \quad \textbf{(true)}$$

$$b < a + c \Leftrightarrow y^2 < xy + x^2 - y^2 \Leftrightarrow x(x + y) > 2y^2$$

But: $x(x + y) > y(y + y) = 2y^2$

$$c < a + b \Leftrightarrow x^2 - y^2 < xy + y^2 \Leftrightarrow x^2 < y(x + 2y)$$

$$\Leftrightarrow 2y^2 + xy - x^2 > 0 \Leftrightarrow 2t^2 - t - 1 > 0; t = \frac{y}{x}$$

$$\Delta = 9; t_1 = 2; t_2 = -1; \min(2t^2 - t - 1) = -\frac{9}{8}$$

$$t > 1 \Rightarrow 2t^2 - t - 1 > 2 \cdot 1^2 - 1 - 1 = 0 \quad \textbf{(true)}$$

Denote s **– semiperimeter;** R **– circumradii**

$$s = \frac{a+b+c}{2} = \frac{xy+y^2+x^2-y^2}{2} = \frac{x^2+xy}{2} \quad \textbf{(1)}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{x^2y^2+(x^2-y^2)^2-y^4}{2xy(x^2-y^2)} = \frac{x^4-x^2y^2+y^4-y^4}{2xy(x^2-y^2)} = \frac{x^2(x^2-y^2)}{2xy(x^2-y^2)} = \frac{x}{2y} < 1 \quad \textbf{(true)}$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{x^2}{4y^2}} = \frac{\sqrt{4y^2 - x^2}}{2y}$$

$$R = \frac{b}{2\sin B} = \frac{y^2}{2 \cdot \frac{\sqrt{4y^2 - x^2}}{2y}} = \frac{y^3}{\sqrt{4y^2 - x^2}}$$

By Mitrinovic's inequality: $s \le \frac{3\sqrt{3}}{2}R$

By (1); (2): $\frac{x(x+y)}{2} < \frac{3\sqrt{3}}{2} \cdot \frac{y^3}{\sqrt{4y^2-x^2}}$

$$x(x + y)\sqrt{4y^2 - x^2} < 3y^3\sqrt{3}$$

(Equality doesn't hold because ΔABC **can't be an equilateral one:**

$x < y \Rightarrow xy < y^2 \Rightarrow a < b$**)**

SP.245. If $0 < y < x < 2y$ then:

$$x(x+y) > 3(x-y)\sqrt{3(4y^2-x^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

If $0 < y < x < 2y$ ***then:*** $x(x+y) > 3(x-y)\sqrt{3(4y^2-x^2)}$

$$xy\left(\frac{x}{y}+1\right) > 3y\left(\frac{x}{y}-1\right)\sqrt{3y^2\left(4-\frac{x^2}{y^2}\right)}$$

$$xy\left(\frac{x}{y}+1\right) > 3y^2\left(\frac{x}{y}-1\right)\sqrt{3\left(4-\frac{x^2}{y^2}\right)}$$

$$\frac{x}{y}\left(\frac{x}{y}+1\right) > 3\left(\frac{x}{y}-1\right)\sqrt{3\left(4-\frac{x^2}{y^2}\right)}$$

Suppose: $t = \frac{x}{y} > 0$

$$t(t+1) > 3(t-1)\sqrt{3(4-t^2)}$$

$$0 < y < x < 2y \Rightarrow 1 < \frac{x}{y} < 2 \Rightarrow 1 < t < 2$$

So: let us prove $\forall t \in]1,2[$***:***

$$t(t+1) \overset{?}{>} 3(t-1)\sqrt{3(4-t^2)}$$

$$\frac{t^2+t}{t-1} \overset{?}{>} 3\sqrt{3}\cdot\sqrt{4-t^2}$$

Let be the function: $f(t) = \frac{t^2+t}{t-1},]1,2[$

$$\lim_{\substack{t\to 1\\ >}}[f(t)] = +\infty, \lim_{\substack{t\to 2\\ <}}[f(t)] = 6$$

$$f'(t) = \frac{(2t+1)(t-1)-(1)(t^2+t)}{(t-1)^2} = \frac{2t^2-t-1-t^2-t}{(t-1)^2}$$

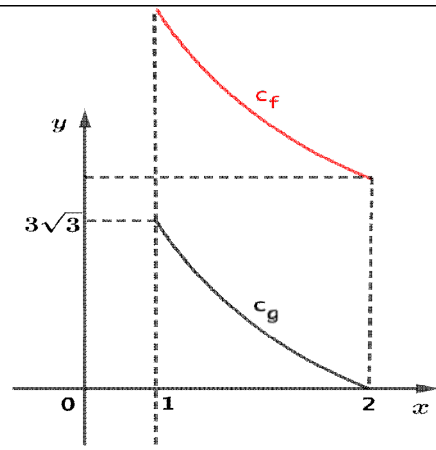
$$f'(t) = \frac{t^2-2t-1}{(t-1)^2}, f'(t) = 0 \Rightarrow t^2-2t-1 = 0$$

$$\Delta = 4-4(1)(-1) = 4+4 = 8 > 0, \sqrt{\Delta} = 2\sqrt{2}$$

$$x = \frac{2 - 2\sqrt{2}}{2} = 1 - \sqrt{2} \notin]1,2[$$

$$x = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2} \notin]1,2[$$

x	1	2
$f'(x)$	$- -$	
$f(x)$	$+\infty$ ↘	6



Let be the function: $g(t) = 3\sqrt{3} \cdot \sqrt{4 - t^2}, t \in]1,2[$

$$\lim_{\substack{t \to 1 \\ >}} g(t) = 3\sqrt{3}, \lim_{\substack{t \to 2 \\ <}} [g(t)] = 0$$

$$g'(t) = 3\sqrt{3} \cdot -\frac{2t}{2\sqrt{4 - t^2}}; g'(t) = \frac{-3\sqrt{3}t}{\sqrt{4 - t^2}}$$

$$g'(t) = 0 \Rightarrow t = 0 \notin]1,2[$$

x	1	2
$g'(x)$	$- -$	
$g(x)$	$3\sqrt{3}$ ↘	0

Note: $\forall t \in]1,2[; f(t) > g(t)$

$$\frac{t^2 + t}{t - 1} > 3\sqrt{3} \cdot \sqrt{4 - t^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$x(x + y) \overset{(1)}{>} 3(x - y)\sqrt{3(4y^2 - x^2)}$$

$$\because x, x+y, x-y>0 \therefore \textbf{(1)} \Leftrightarrow$$

$$27(4y^2-x^2)(x-y)^2-x^2(x+y)^2>0$$

$$\Leftrightarrow y^4(7t^4-13t^3-20t^2+54t-27)>0\ \left(t=\frac{x}{y}\right)$$

$$\Leftrightarrow 7t^4-13t^3-20t^2+54t-27 \overset{(2)}{>} 0$$

$$\because 0<y<x<2y, \therefore 0<1<\frac{x}{y}<2 \Rightarrow t>1$$

Let $t-1=m$ ***of course,*** $m>0$

Then (2) $\Leftrightarrow 7(m+1)^4-13(m+1)^3-20(m+1)^2+54(m+1)-27>0$

$$\Leftrightarrow 7m^4+15m^3+3m+1 \overset{(3)}{>} 17m^2$$

Now, $7m^4+1 \overset{A-G}{\geq} 2\sqrt{7}m^2 \overset{(i)}{>} 2\left(\frac{5}{2}\right)m^2=5m^2$

$$\left(\because 28>25, \therefore 7>\frac{25}{4} \Rightarrow \sqrt{7}>\frac{5}{2}\right)$$

Also, $15m^3+3m \overset{A-G}{\geq} 2\sqrt{45}m^2 \overset{(ii)}{>} 2\left(\frac{13}{2}\right)m^2=13m^2$

$$\left(\because 180>169, \therefore 45>\frac{169}{4} \Rightarrow \sqrt{45}>\frac{13}{2}\right)$$

(i)+(ii)$\Rightarrow 7m^4+15m^3+3m+1>18m^2>17m^2$

$\Rightarrow$ ***(3)$\Rightarrow$(2)$\Rightarrow$(1) is true (Proved)***

Solution 3 by proposer

Let's consider $\Delta ABC; BC=a=xy; AC=b=y^2; AB=c=x^2-y^2.$

$a<b+c \Leftrightarrow xy<x^2-y^2+y^2 \Leftrightarrow y<x$ ***(true)***

$$b<a+c \Leftrightarrow y^2<xy+x^2-y^2 \Leftrightarrow x(x+y)>2y^2$$

But: $x(x+y)>y(y+y)=2y^2$

$$c<a+b \Leftrightarrow x^2-y^2<xy+y^2 \Leftrightarrow x^2<y(x+2y)$$

$$\Leftrightarrow 2y^2+xy-x^2>0 \Leftrightarrow 2t^2-t-1>0; t=\frac{y}{x}$$

$$\Delta=9; t_1=\frac{1+3}{2}=2; t_2=\frac{1-3}{2}=-1$$

$$\min(2t^2-t-1)=-\frac{9}{8}$$

$t>1\Rightarrow 2t^2-t-1>2\cdot 1^2-1-1=0$ ***(true)***

Denote s – semiperimeter; F – area; r – inradii

$$s=\frac{a+b+c}{2}=\frac{xy+y^2+x^2-y^2}{2}=\frac{x^2+xy}{2}\quad \textit{(1)}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{x^2y^2+(x^2-y^2)^2-y^4}{2xy(x^2-y^2)}=$$

$$=\frac{x^4-x^2y^2+y^4-y^4}{2xy(x^2-y^2)}=\frac{x^2(x^2-y^2)}{2xy(x^2-y^2)}=\frac{x}{2y}<1\quad \textit{(true)}$$

$$\sin B=\sqrt{1-\cos^2 B}=\sqrt{1-\frac{x^2}{4y^2}}=\frac{\sqrt{4y^2-x^2}}{2y}\quad \textit{(2)}$$

$$F=\frac{1}{2}ac\sin B=\frac{1}{2}\cdot xy\cdot(x^2-y^2)\cdot\frac{\sqrt{4y^2-x^2}}{2y}$$

$$F=\frac{x(x^2-y^2)\sqrt{4y^2-x^2}}{4}\quad \textit{(2)}$$

By Mitrinovic's inequality:

$s>3\sqrt{3}r$ ***(Equality doesn't hold because ΔABC can't be an equilateral one***

$x<y\Rightarrow xy<y^2\Rightarrow a<b$***)***

$$s>3\sqrt{3}\cdot\frac{F}{s}\Rightarrow s^2>3\sqrt{3}F$$

By (1); (2): $\frac{x^2(x+y)^2}{4}>3\sqrt{3}\cdot\frac{x(x-y)(x+y)\sqrt{4y^2-x^2}}{4}$

$$x(x+y)>3\sqrt{3}(x-y)\sqrt{4y^2-x^2}$$

$$\frac{x(x+y)}{3(x-y)}>\sqrt{3(4y^2-x^2)}$$

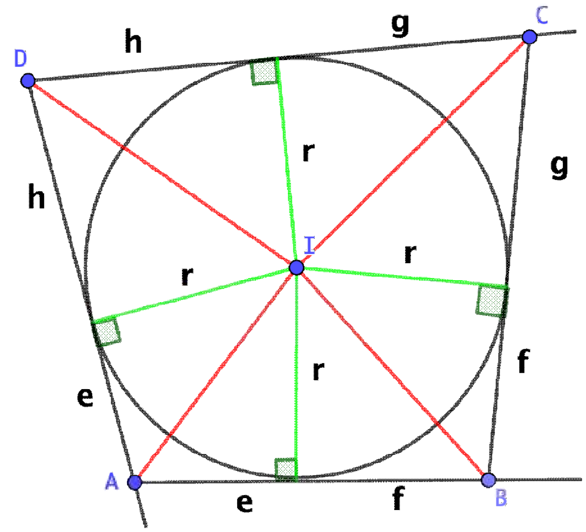
$$x(x+y)>3(x-y)\sqrt{3(4y^2-x^2)}$$

SP.246. If $ABCD$ bicentric quadrilateral; I – incircle then:

$$(IA^2+IC^2)(IB^2+ID^2)\geq AB\cdot BC\cdot CD\cdot DA$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



$$IA^2 \overset{(1)}{=} r^2 + e^2,\ IC^2 \overset{(2)}{=} r^2 + g^2,\ IB^2 \overset{(3)}{=} r^2 + f^2,\ ID^2 \overset{(4)}{=} r^2 + h^2$$

(1), (2), (3), (4) $\Rightarrow (IA^2 + IC^2)(IC^2 + ID^2) =$

$$= (2r^2 + e^2 + g^2)(2r^2 + f^2 + h^2)$$

$$= r^2 \cdot 2(e^2 + g^2) + r^2 \cdot 2(f^2 + h^2) + (e^2 + g^2)(g^2 + h^2) + 4r^4$$

$$\overset{(i)}{\geq} r^2(e+g)^2 + r^2(f+h)^2 + \frac{1}{4}(e+g)^2(f+h)^2 + 4r^4$$

Now, $\frac{e+g}{f+h} \underset{(a)}{\overset{Grinberg}{=}} \frac{IAIC}{IBIC} \Rightarrow \frac{e+g+f+h}{f+h} = \frac{IAIC+IBIC}{IBIC} \Rightarrow \frac{s}{f+h} = \frac{\sqrt{abcd}}{IBIC}$ **(China, IMO TST, 2003)**

$= \frac{rs}{IBIC}$ **($\because$ $ABCD$ is bicentric)**$\Rightarrow f + h \overset{(m)}{=} \frac{IB \cdot ID}{r}$

Similarly, using (a), $e + g \overset{(n)}{=} \frac{IA \cdot IC}{r}$

(m), (n), (i)$\Rightarrow (IA^2 + IC^2)(IB^2 + ID^2) \geq$

$$\geq r^2 \cdot \frac{IA^2 \cdot IC^2}{r^2} + r^2 \cdot \frac{IB^2 \cdot ID^2}{r^2} + \left(\frac{1}{4r^2}(IAICIBID)^2 + 4r^4\right) \geq$$

$$\overset{A-G}{\geq} IA^2IC^2 + IB^2ID^2 + 2IA \cdot IC \cdot IB \cdot ID = (IA \cdot IC + IB \cdot ID)^2$$

$= AB \cdot BC \cdot CD \cdot DA$ **(China IMO TST, 2003)**

Solution 2 by proposer

Let a,b,c,d *be sides in quadrilateral;* $s=\frac{a+b+c+d}{2}$ *– semiperimeter;* $ABCD$ *– tangential quadrilateral* $\Rightarrow s-a=c; s-b=d; s-c=a; s-d=b.$

By Brahmagupta's formula ($ABCD$ *– cyclic):*

$$[ABCD]=\sqrt{(s-a)(s-b)(s-c)(s-d)}=\sqrt{abcd}\quad (1)$$

$$\sqrt{AB\cdot BC\cdot CD\cdot DA}=\sqrt{abcd}\overset{(1)}{=}[ABCD]=[AIB]+[BIC]+[CID]+[DIA]=$$

$$=\frac{AI\cdot BI\cdot\sin(\widehat{AIB})}{2}+\frac{BI\cdot CI\cdot\sin(\widehat{BIC})}{2}+\frac{CI\cdot DI\cdot\sin(\widehat{CID})}{2}+\frac{DI\cdot AI\cdot\sin(\widehat{DIA})}{2}\le$$

$$\le\frac{IA\cdot IB}{2}+\frac{IB\cdot IC}{2}+\frac{IC\cdot ID}{2}+\frac{ID\cdot IA}{2}=\frac{IA(IB+ID)+IC(IB+ID)}{2}=$$

$$=\frac{(IB+ID)(IA+IC)}{2}=2\cdot\frac{IA+IC}{2}\cdot\frac{IB+ID}{2}\le$$

$$\overset{AM-QM}{\le}2\cdot\sqrt{\frac{IA^2+IC^2}{2}}\cdot\sqrt{\frac{IB^2+ID^2}{2}}=\sqrt{(IA^2+IC^2)(IB^2+ID^2)}$$

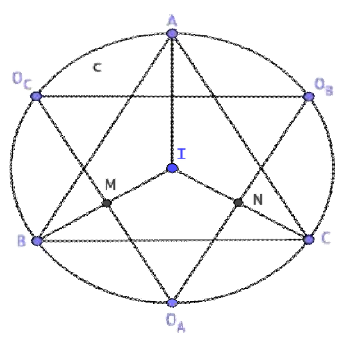
$$\sqrt{(IA^2+IC^2)(IB^2+ID^2)}\ge\sqrt{AB\cdot BC\cdot CD\cdot DA}$$

$$(IA^2+IC^2)(IB^2+ID^2)\ge AB\cdot BC\cdot CD\cdot DA$$

SP.247. In ΔABC; I – incenter; O_A,O_B,O_C – circumcenters of $\Delta BIC,\Delta CIA,\Delta AIB$. Prove that: $\frac{5}{2}+\frac{r}{R}\le\frac{BC}{O_BO_C}+\frac{AC}{O_AO_C}+\frac{AB}{O_AO_B}\le\sqrt{8+\frac{2r}{R}}$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer



$$m(O_A)=\frac{B+C}{2},\ m(\widehat{BIC})=\pi-\frac{B+C}{2}\Rightarrow$$

$$O_AMIN\text{ cyclic}\Rightarrow m(\widehat{BO_AC})=2m(O_A)=B+C$$

$$\Rightarrow A+BO_AC=\pi\Rightarrow ABO_AC\text{ cyclic}\Rightarrow A,B,C,O_A,O_B,O_C\text{ are concyclic}$$

We have: $O_BO_C=2R\sin O_A=2R\sin\frac{BC}{2}=2R\cos\frac{A}{2}\Rightarrow$

$$\frac{BC}{O_BO_C}=\frac{2R\sin A}{2R\cos\frac{A}{2}}=\frac{\sin A}{\cos\frac{A}{2}}=2\sin\frac{A}{2}\Rightarrow\frac{5}{4}+\frac{r}{2R}\le\sum\sin\frac{A}{2}\le\sqrt{2+\frac{r}{2R}}$$

For the left-hand side, we apply Popoviciu's inequality for the concave function:

$$f(x)=\cos x\Rightarrow\sum\cos A=1+\frac{r}{R}\Rightarrow$$

$$\Rightarrow\frac{\cos A+\cos B+\cos C}{3}+\cos\frac{A+B+C}{3}\le 2\sum\cos\frac{A+B}{2}=\frac{2}{3}\sin\frac{C}{2}\Rightarrow$$

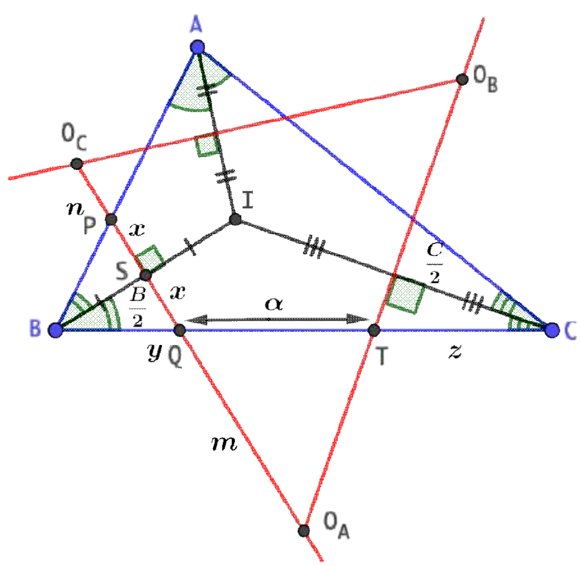
$$\sum\cos A+\frac{3}{2}\le 2\sum\sin\frac{A}{2}\Leftrightarrow 2\sum\sin\frac{A}{2}\ge\frac{3}{2}+1+\frac{r}{R}\Rightarrow\sum\sin\frac{A}{2}\ge\frac{5}{4}+\frac{r}{2R}$$

For the right-hand side, we have: $2\sum\sin\frac{B}{2}\sin\frac{C}{2}\le\sum\cos A$

and $\sum\sin^2\frac{A}{2}=\frac{1}{2}\sum(1+\cos A)=\frac{3}{2}-\frac{1}{2}\sum\cos A$

$$\left(\sum\sin\frac{A}{2}\right)^2\le\frac{3}{2}+\frac{1}{2}\sum\cos A=2+\frac{r}{2R}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



From $\Delta QO_AT, \angle O_A = 180° - \left(90° - \frac{B}{2} + 90° - \frac{C}{2}\right) = \frac{B+C}{2} = \frac{180°-A}{2} = 90° - \frac{A}{2}$

From $\Delta BSQ, \tan\frac{B}{2} = \frac{x}{\frac{1}{2}BI} = \frac{2x\sin\frac{B}{2}}{r} \Rightarrow 2x\tan\frac{B}{2}\cos\frac{B}{2} = r\tan\frac{B}{2} \Rightarrow 2x = PQ \overset{(1)}{=} \frac{r}{\cos\frac{B}{2}}$

Again, using $\Delta BSQ, \cos\frac{B}{2} = \frac{\frac{1}{2}BI}{y} = \frac{r}{2y\sin\frac{B}{2}} \Rightarrow y\sin B = r \Rightarrow y\left(\frac{b}{2R}\right) = r$

$\Rightarrow y \overset{(2)}{=} \frac{2Rr}{b}$*. Similarly,* $z \overset{(3)}{=} \frac{2Rr}{c} \therefore \alpha = a - y = z \overset{by\,(2),(3)}{=} a - 2Rr\left(\frac{1}{b} + \frac{1}{c}\right)$

$$= a - \frac{2Rr(b+c)a}{4Rrs} = a - \frac{a(b+c)}{2s} = a\left[1 - \frac{b+c}{2s}\right] = \frac{2^2}{2s} \therefore \alpha \overset{(4)}{=} \frac{a^2}{2s}$$

Applying sine rule on ΔQO_AT*,* $\frac{\alpha}{\sin\left(90°-\frac{A}{2}\right)} = \frac{m}{\sin\left(90°-\frac{C}{2}\right)}$

$\Rightarrow \frac{a^2}{2s\cos\frac{A}{2}} = \frac{m}{\cos\frac{C}{2}}$ *(by (4))*$\Rightarrow m \overset{(5)}{=} \frac{a^2\cos\frac{C}{2}}{2s\cos\frac{A}{2}}$*. Similarly,* $n \overset{(6)}{=} \frac{c^2\cos\frac{A}{2}}{2s\cos\frac{C}{2}}$

(1)+(5)+(6)$\Rightarrow O_AO_C$

$$= \frac{1}{2s}\left(\frac{a^2\cos^2\frac{C}{2} + c^2\cos^2\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}}\right) + \left(\frac{r}{\sin\frac{B}{2}}\right)\left(\frac{s\tan\frac{B}{2}}{s}\right)$$

$$= \left(\frac{1}{2s}\right)\sqrt{\frac{bc\cdot ab}{s(s-a)s(s-c)}}\left(a^2\cdot\frac{s(s-c)}{ab} + c^2\cdot\frac{s(s-a)}{bc}\right) + BI\left(\frac{r_b}{s}\right)$$

$$= \frac{1}{2s}\sqrt{\frac{ca}{(s-a)(s-b)}}\left(a(s-c) + c(s-a)\right) + \frac{BI}{s}\left(\frac{rs}{s-b}\right)$$

$$= \frac{1}{2s\sin\frac{B}{2}}\{s(a+c) - 2ac\} + BI\left(\frac{r}{s-b}\right) = BI\left[\frac{s(a+c) - 2ac}{2rs} + \frac{r}{s-b}\right]$$

$$= BI\left[\frac{s(s-b)(a+c) - 2ac(s-b) + 2r^2s}{2rs(s-b)}\right]$$

$$= BI\left[\frac{(a+c)(\sum a)(c+a-b) - 4ac(c+a-b) + \pi(a+b-c)}{8rs(s-b)}\right]$$

$$= BI\cdot\frac{b\{(a^2+2ac+c^2) - b^2\}}{8rs(s-b)} = BI\cdot\frac{b\{(a+c)^2 - b^2\}}{8rs(s-b)}$$

$$= BI\cdot\frac{b(2s)2(s-b)}{8rs(s-b)} = BI\cdot\left(\frac{b}{2r}\right) \Rightarrow \frac{O_AO_C}{b} = \frac{BI}{2r} \Rightarrow \frac{AC}{O_AO_C} \overset{(i)}{=} \frac{2r}{BI}$$

Similarly, $\frac{BC}{O_BO_C} \stackrel{(ii)}{=} \frac{2r}{AI}$ ***and*** $\frac{AB}{O_AO_B} \stackrel{(iii)}{=} \frac{2r}{CI}$

$$\textit{(i)+(ii)+(iii)} \Rightarrow \sum \frac{BC}{O_BO_C} = 2r\sum\frac{1}{AI} = 2r\sum\frac{\sin\frac{A}{2}}{r} \stackrel{(iv)}{=} 2\sum\sin\frac{A}{2} = 2\sum\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\stackrel{CBS}{\leq} 2\sqrt{\sum(s-b)(s-c)}\sqrt{\sum\frac{1}{bc}} = 2\sqrt{\sum(s^2 - s(b+c) + bc)}\sqrt{\frac{2s}{4Rrs}}$$

$$= 2\sqrt{3s^2 - 4s^2 + s^2 + 4Rr + r^2}\sqrt{\frac{1}{2Rr}}$$

$$= 2\sqrt{\frac{4Rr + r^2}{2Rr}} = 2\sqrt{\frac{4R + r}{2R}} = \sqrt{\frac{16R + 4r}{2R}} = \sqrt{8 + \frac{2r}{R}}$$

$\therefore \sum\frac{BC}{O_BO_C} \leq \sqrt{8 + \frac{2r}{R}}$***. Again, (iv)*** $\Rightarrow \sum\frac{BC}{O_BO_C} = 2\sum\sin\frac{A}{2} \geq \frac{5}{2} + \frac{r}{R}$

$$\Leftrightarrow 2\sum\sin\frac{A}{2} \geq \frac{5R + 2r}{2R} = \frac{(4R + r) + (R + r)}{2R} = \frac{4R + r}{2R} + \frac{1}{2}\left(1 + \frac{r}{R}\right)$$

$$\Leftrightarrow 2\sum\sin\frac{A}{2} \geq \sum\cos^2\frac{A}{2} + \sum\cos A = \frac{1}{2}\sum(1 + \cos A) + \left(\frac{1}{2}\right)\sum\cos A = \frac{3}{2} + \sum A$$

$\Leftrightarrow 2\sum\cos A + 3 \stackrel{(a)}{\leq} 4\sum\sin\frac{A}{2}$***. Now,*** $\because \Delta ABC$ ***is acute,*** $\therefore f(x) = \cos x\ \forall x \in \left(0, \frac{\pi}{2}\right)$ ***is concave,*** $\therefore$ ***applying Popoviciu's inequality:*** $\frac{\sum\cos A}{3} + \cos\left(\frac{A+B+C}{3}\right) \leq \frac{2}{3}\sum\cos\left(\frac{B+C}{2}\right)$

$\Rightarrow \sum\cos A + \frac{3}{2} \leq 2\sum\sin\frac{A}{2} \Rightarrow 2\sum\cos A + 3 \leq 4\sum\sin\frac{A}{2} \Rightarrow$ ***(a) is true*** $\therefore \sum\frac{BC}{O_BO_C} \geq \frac{5}{2} + \frac{r}{R}$

(Hence proved)

SP.248. Let be $A \in M_5(\mathbb{R})$ such that $AA^T = I_5$ and $Tr\ A = Tr\ A^2 = 0$. Find A^{2020}.

Proposed by Marian Ursărescu – Romania

Solution by proposer

The matrix A ***being orthogonal one*** $\Rightarrow |\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = |\lambda_5| = 1$

$$P_A(x) = x^5 - ax^4 + bx^3 - cx^2 + dx - \det A = 0 \quad \textit{(1)}$$

$$a = Tr\ A = 0 \quad \textit{(2)}$$

$$B = Tr\,A^* = \frac{1}{2}((Tr\,A)^2 - Tr\,A^2) = 0 \quad \textit{(3)}$$

$$d = \sum \lambda_1\lambda_2\lambda_3\lambda_4 = \sum \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}+\frac{1}{\lambda_5}\right)$$

$$= \det A \cdot \sum(\overline{\lambda_1}+\overline{\lambda_2}+\overline{\lambda_3}+\overline{\lambda_4}+\overline{\lambda_5}) = \det A \cdot \overline{\sum(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)}$$

$$= \det A \cdot \overline{Tr\,A} = 0 \; \textit{(A)}$$

$$C = \sum \lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \sum \frac{1}{\lambda_1\lambda_2} = \det A \cdot \sum \overline{\lambda_1\lambda_2}$$

$$= \det A \cdot \left(\overline{Tr\,A^2}\right) = 0 \quad \textit{(5)}$$

From (1)+(2)+(3)+(4)+(5)$\Rightarrow P_A(x) = x^5 - \det A \Rightarrow$

$$\left.\begin{array}{c} A^5 = \det A I_5 \\ But\ \det(AA^T) = \det I_5 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1 \end{array}\right\} \Rightarrow$$

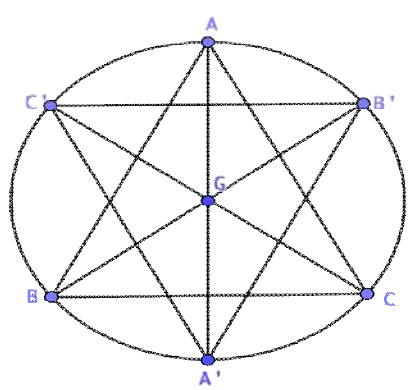
$$\Rightarrow A^5 = \pm I_5 \Rightarrow \left(A^5\right)^{404} = I_5^{404} \Rightarrow A^{2020} = I_5$$

SP.249. Let $\Delta A'B'C'$ be the circumcevian triangle of centroid in ΔABC. Prove that:

$$\frac{S[A'B'C']}{S[ABC]} \le \left(\frac{R}{2r}\right)^6$$

Proposed by Marian Ursărescu – Romania

Solution by proposer



We have the following relationship: $\frac{S_{A'B'C'}}{S_{ABC}} = \frac{(\rho(G))^3}{GA^2\cdot GB^2\cdot GC^2} = \frac{(R^2-OG^2)^3}{\frac{64}{9^3}m_a^2m_b^2m_c^2} =$

$$\frac{\frac{1}{9^3}(a^2+b^2+c^2)^3}{\frac{64}{9^3}(m_a^2m_b^2m_c^2)} = \frac{(a^2+b^2+c^2)^3}{64m_a^2m_b^2m_c^2} \quad (1)$$

But $a^2 + b^2 + c^2 \le 9R^2$ ***(2)***

$$m_a \ge \sqrt{bc}\cdot\cos\frac{A}{2} \Rightarrow m_a^2m_b^2m_c^2 \ge a^2b^2c^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} \quad (3)$$

From (1)+(2)+(3) $\Rightarrow \frac{S_{A'B'C'}}{S_{ABC}} \le \frac{9^3\cdot R^6}{64a^2b^2c^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}$ ***(4)***

But $abc = 4sRr$ ***and*** $\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = \frac{s^2}{16R^2}$ ***(5)***

From (4)+(5)$\Rightarrow \frac{S_{A'B'C'}}{S_{ABC}} \le \frac{3^6R^6}{64\cdot s^4r^2}$ ***(6)***

From Mitrinovic's inequality $s^2 \ge 2 + r^2$ ***(7)***

From (6)+(7)$\Rightarrow \frac{S_{A'B'C'}}{S_{ABC}} \le \frac{3^6R^6}{64\cdot 3^6\cdot r^6} = \left(\frac{R}{2r}\right)^6$

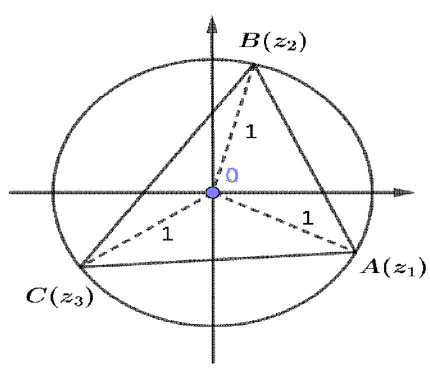
SP.250. Let be $z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$ different in pairs:

$$|z_1| = |z_2| = |z_3| = 1; A(z_1); B(z_2); C(z_3).$$

If $|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_2 - z_1| = 6$ then $AB = BC = CA$.

Proposed by Marian Ursărescu – Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria



$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$$

$$|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_2 - z_1| = 6$$

$$|z_1-(z_2+z_3)|+|z_2-(z_1+z_3)|+|z_3-(z_2+z_1)|=6$$

$$|z_1-(z_1+z_2+z_3-z_1)|+|z_2-(z_1+z_2+z_3-z_2)|+|z_3-(z_1+z_2+z_3-z_3)|=6$$

$$|2z_1-3z_G|+|2z_2-3z_G|+|2z_3-3z_G|=6$$

$$|2z_1-3z_G|+|2z_2-3z_G|+|2z_3-3z_G|=2|z_1|+2|z_2|+2|z_3|$$

$$(|2z_1-3z_G|-2|z_1|)+(|2z_2-3z_G|-2|z_2|)+(|2z_3-3z_G|-2|z_3|)=0 \quad (*)$$

Suppose $|2z_3-3z_G|\neq 2$ ***and*** $|2z_2-3z_G|\neq 2, |2z_3-3z_G|\neq 2$

$$(2z_3-3z_G)(2\overline{z_3}-3\overline{z_G})\neq 4$$

$$4-6z_3\overline{z_G}-6z_G\overline{z_3}+9z_G\overline{z_G}\neq 4$$

$$2(z_3\cdot\overline{z_G}+z_G\cdot\overline{z_3})\neq 3\cdot z_G\cdot\overline{z_G} \quad (I)$$

Similarly, $2(z_1\overline{z_G}+z_G\overline{z_1})\neq 3z_G\cdot\overline{z_G}$ ***(II)***

and $2(z_2\overline{z_G}+z_G\cdot\overline{z_2})\neq 3\cdot z_G\cdot\overline{z_G}$ ***(III)***

By adding (I), (II), (III): $6\overline{z_G}\cdot z_G+6z_G\cdot\overline{z_G}+6z_G\overline{z_G}\neq 9z_G\cdot\overline{z_G}$

$$18z_G\cdot\overline{z_G}-9z_G\cdot\overline{z_G}\neq 0$$

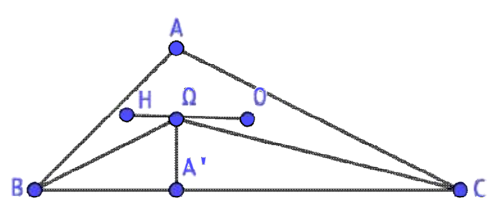
$$9z_G\cdot\overline{z_G}\neq 0\Rightarrow 9|z_G|^2\neq 0\Rightarrow |z_G|\neq 0$$

$z_G\neq 0, G\not\equiv 0$ ***and hence:*** $|2z_1-3z_G|-2|z_1|\neq 0, |2z_3-3z_G|-2|z_2|\neq 0$

and $|2z_3-3z_G|-2|z_3|\neq 0$ ***this is in contradiction with relation (*)***

So: $z_G=0\Rightarrow G\equiv 0$ ***and the triangle*** ABC ***is equilateral.***

Solution 2 by proposer



$$A(z_1), B(z_2), C(z_3)\ \Delta ABC; C\subset(0,1)$$

$$|z_1-z_2-z_3|=|2z_1-z_1-z_2-z_3|=2\left|z_1-\frac{z_1+z_2+z_3}{2}\right|=$$

$=2\left|z_1-\frac{z_0+z_H}{2}\right|=2|z_1-z_\Omega|=2A\Omega$***, where*** Ω ***is Euler's ninepointcenter. The relationship from hypothesis can be written:*** $A\Omega+B\Omega+C\Omega=3$ ***(1)***

Applying median theorem in $\Delta B\Omega C\Rightarrow$

$\Omega A'^2 = \frac{2(B\Omega+C\Omega^2)-a^2}{4} \Rightarrow R^2 = 2(A\Omega^2 + B\Omega^2) - C^2$, ***because the radius of Euler's circle***

$= \frac{R}{2}$, ***and the analogs.*** $R^2 = 2(A\Omega^2 + C\Omega^2) - b^2, R^2 = 2(A\Omega^2 + B\Omega^2) - C^2 \Rightarrow$

$$\left.\begin{array}{c} A\Omega^2 + B\Omega^2 + C\Omega^2 = \dfrac{3R^2 + a^2 + b^2 + c^2}{4} \\ But\ a^2 + b^2 + c^2 = 9R^2 \end{array}\right\} \Rightarrow A\Omega^2 + B\Omega^2 + C\Omega^2 \le 3R^2 \Rightarrow$$

$$\left.\begin{array}{c} 3(A\Omega^2 + B\Omega^2 + C\Omega^2) \le 9R^2 \\ From\ Cauchy's\ inequality: (A\Omega + B\Omega + C\Omega)^2 \le 3(A\Omega^2 + B\Omega^2 + C\Omega^2) \end{array}\right\} \Rightarrow$$

$(A\Omega + B\Omega + C\Omega)^2 \le 9R^2 \Rightarrow A\Omega + B\Omega + C\Omega \le 3R$, ***but in our case*** $R = 1 \Rightarrow$

$\Rightarrow A\Omega + B\Omega + C\Omega \le 3$ ***with equality when the triangle is equilateral.***

SP.251. ABOUT PROBLEM SP.251

17 RMM SUMMER EDITION 2020

By Marin Chiriciu – Romania

1) In ΔABC the following relationship holds:

$$m_a^6 + m_b^6 + m_c^6 \ge 9S^3\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu and Neculai Stanciu – Romania

Solution

We can give a refinement:

2) In ΔABC the following relationship holds:

$$m_a^6 + m_b^6 + m_c^6 \ge 9S^2(2R - r)^2$$

Marin Chirciu

Proof.

By Chebyshev: $\sum m_a^6 \ge \frac{1}{3}\sum m_a^4 \sum m_a^2$ ***(1)***

Known: $\sum m_a^2 = \frac{3}{4}\sum a^2$ ***and*** $\sum m_a^4 = \frac{9}{16}\sum a^4$

(1) can be written: $\sum m_a^6 \ge \frac{1}{3}\cdot\frac{9}{16}\sum a^4 \cdot \frac{3}{4}\sum a^2 = \frac{9}{64}\sum a^4 \sum a^2$ ***(2)***

By CBS: $\sum a^4 \sum a^2 \ge (\sum a^3)^2$ ***(3)***

By $\sum a^3 \ge 8S(2R - r)$, ***(S.G. Guba, 1963), from (1), (2), (3):***

$$\sum m_a^6 \geq \frac{9}{64} \cdot 64S^2(2R-r)^2 = 9S^2(2R-r)^2$$

Equality holds for an equilateral triangle. Inequality 2) it's stronger than 1)

3) In ΔABC the following relationship holds:

$$m_a^6 + m_b^6 + m_c^6 \geq 9S^2(2R-r)^2 \geq 9S^3\sqrt{3}$$

Solution

By 2) and $9S^2(2R-r)^2 \geq 9S^3\sqrt{3} \Leftrightarrow (2R-r)^2 \geq rs\sqrt{3}$, ***(Doucet*** $4R+r \geq s\sqrt{3}$***). It remains:*** $(2R-r)^2 \geq r(4R+r) \Leftrightarrow 4R^2 \geq 8Rr \Leftrightarrow R \geq 2r$, ***(Euler). Equality holds for an equilateral triangle.***

Guba's inequality:

4) In ΔABC the following relationship holds:

$$\sum a^3 \geq 8S(2R-r)$$

S.G. Guba, 1963

Proof.

By $\sum a^3 = 2s(s^2-3r^2-6Rr)$, ***the inequality can be written:***

$2s(s^2-3r^2-6Rr) \geq 8sr(2R-r) \Leftrightarrow s^2 \geq 14Rr - r^2$, ***(Gerretsen*** $s^2 \geq 16Rr - 5r^2$***)***

It remains: $16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow R \geq 2r$, ***(Euler)***

Equality holds for an equilateral triangle.

Remark.

The same kind of problems can be proposed:

5) In ΔABC the following relationship holds:

$$\sum m_a^6 \geq \frac{27}{64}\sum a^6$$

Marin Chirciu

Solution

Lemma:

6) In ΔABC the following holds:

$$\sum m_a^6 = \frac{51s^6 + s^4(81r^2 - 612Rr) + s^2r^2(9r^2 + 288Rr - 288R^2) - 21r^3(4R+r)^3}{64}$$

Proof.

$$\sum m_a^6=\sum\left(\frac{2b^2+2c^2-a^2}{4}\right)^3=\frac{1}{64}\left[15\sum a^6+18\sum b^2c^2\left(b^2+c^2\right)-72a^2b^2c^2\right]=$$

$$=\frac{51s^6+s^4(81r^2-612Rr)+s^2r^2(9r^2+288Rr-288R^2)-21r^3(4R+r)^3}{64}, \text{ from}$$

$$abc=4Rrs, \sum a^6=s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4Rr+r)^3$$

$$\sum b^2c^2\left(b^2+c^2\right)=2[s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3]$$

Back to the main problem:

By Lemma:

$$\frac{51s^6+s^4(81r^2-612Rr)+s^2r^2(9r^2+288Rr-288R^2)-21r^3(4R+r)^3}{64}\ge$$

$$\ge\frac{27}{64}[s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4Rr+r)^3]\Leftrightarrow$$

$$\Leftrightarrow s^6-12Rrs^4-3s^2r^2(2R-r)^2-2r^3(4Rr+r)^3\ge 0\Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(s^2-12Rr)-3r^2(2R-r)^2]\ge 2r^3(4R+r)^3$$

(Gerretsen: $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$***). It remains:***

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(16Rr-5r^2-12Rr)-3r^2(2R-r)^2]\ge 2r^3(4R+r)^3\Leftrightarrow$$

$$\Leftrightarrow 22R^2-49Rr+10r^2\ge 0\Leftrightarrow(R-2r)(22R-5r)\ge 0$$

Obviously by Euler $R\ge 2r$***. Equality holds for an equilateral triangle.***

SP.252. If $m,n\ge 1$ then in ΔABC the following relationship holds:

$$\frac{\sqrt{a(mb+nc-a)}}{mb+nc}+\frac{\sqrt{b(mc+na-b)}}{mc+na}+\frac{\sqrt{c(ma+nb-c)}}{ma+nb}\le\frac{3}{2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Florentin Vișescu

$$\sqrt{a(mb+nc-a)}\overset{AM-GM}{\le}\frac{a+mb+nc-a}{2}=\frac{mb+nc}{2}$$

$$\sqrt{b(mc+na-b)}\overset{AM-GM}{\le}\frac{b+mc+na-b}{2}=\frac{mc+na}{2}$$

$$\sqrt{c(ma+nb-c)} \overset{AM-GM}{\leq} \frac{c+ma+na-c}{2} = \frac{ma+nb}{2}$$

$$\frac{\sqrt{a(mb+nc-a)}}{mb+nc} \leq \frac{1}{2}$$

$$\frac{\sqrt{b(mc+na-b)}}{mc+na} \leq \frac{1}{2}$$

$$\frac{\sqrt{c(ma+nb-c)}}{ma+nb} \leq \frac{1}{2}$$

$$\frac{\sqrt{a(mb+nc-a)}}{mb+nc} + \frac{\sqrt{b(mc+na-b)}}{mc+na} + \frac{\sqrt{c(ma+nb-c)}}{ma+nb} \leq \frac{3}{2}$$

Equality holds for:

$$\begin{cases} mb+nc-a=a \\ mc+na-b=b \\ ma+nb-c=c \end{cases} \rightarrow m(a+b+c)+n(a+b+c)=2(a+b+c)$$

$$m,n \geq 1, m+n=2 \rightarrow m=n=1$$

$$\begin{cases} b+c=2a \\ c+a=2b \\ a+b=2c \end{cases} \rightarrow a=b=c$$

SP.253. If $n \in \mathbb{N}; n \geq 2; a_k \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right); k \in \overline{1,n}$ then:

$$\left(\sum_{k=1}^{n} k \sin a_k\right)\left(\sum_{k=1}^{n} k \cot a_k\right) < \frac{49n^2(n+1)^2}{24}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution by Florentin Vișescu – Romania

$$n \geq 2, a_k \in \left[\frac{\pi}{12}; \frac{\pi}{2}\right), \forall k = \overline{1,n}$$

$$S = \left(\sum_{k=1}^{n} k \sin a_k\right)\left(\sum_{k=1}^{n} k \cot a_k\right) < \frac{49n^2(n+1)^2}{24}$$

Let be $\{a_1, a_2, \dots, a_n\} = \{x_1, x_2, \dots, x_n\}$ ***with*** $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n$

Then $(\sum_{k=1}^{n} k \sin a_k)(\sum_{k=1}^{n} k \cot a_k) \overset{arangements\ inequality}{\leq}$

$$\le\left(\sum_{k=1}^{n}k\sin x_k\right)\left(\sum_{k=1}^{n}k\cot x_{n-k+1}\right)\overset{Cebyshev}{\le} n\left(\sum_{k=1}^{n}k^2\sin x_k\cot x_{n-k+1}\right)$$

But $\sin x_k\cot x_{n-k+1} < \sin\frac{\pi}{2}\cot\frac{\pi}{12} = \cot\frac{\pi}{12}$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow \cot\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} =$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{8+2\sqrt{12}}{4} = 2+\sqrt{3}$$

$$S < n\sum_{k=1}^{n}n^2(2+\sqrt{3}) = (2+\sqrt{3})n\cdot\frac{n(n+1)(2n+1)}{6}$$

$$S < \frac{2+\sqrt{3}}{6}n^2(n+1)(2n+1)$$

$$\frac{2+\sqrt{3}}{6}n^2(n+1)(2n+1) < \frac{49n^2(n+1)^2}{24}$$

$$(8+4\sqrt{3})(2n+1) < 49(n+1)$$

$$(16+8\sqrt{3})n + 8 + 4\sqrt{3} < 49n + 49$$

$$n(49-16-8\sqrt{3}) > 8 + 4\sqrt{3} - 49$$

$$n\underbrace{(33-8\sqrt{3})}_{+} > \underbrace{4\sqrt{3}-41}_{-}\ \textit{(True)}$$

SP.254. Let $A_1A_2\ldots A_n; n\ge 3$ be a convexe polygon with area $F = [A_1A_2\ldots A_n]$ with sides $a_k = A_kA_{k+1}; k\in\overline{1,n}; a_{n+1} = a_1$; $x_k, y_k \in \left(0,\frac{\pi}{2}\right); k\in\overline{1,n}$ then:

$$\sum_{k=1}^{n}\frac{a_k^4}{\sin x_k+\sin y_k} > \frac{16F^2\tan^2\left(\frac{\pi}{n}\right)}{\sum_{k=1}^{n}(x_k+y_k)}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Because $x_k, y_k\in\left(0,\frac{\pi}{2}\right)\Rightarrow \sin x_k < x_k$ *and* $\sin y_k < y_k \Rightarrow$ *we must show:*

$$\sum_{k=1}^{n}\frac{a_k^4}{x_k+y_k}\ge\frac{16F^2\tan\frac{2\pi}{n}}{\sum_{k=1}^{n}(x_k+y_k)}\quad (1)$$

From Bergström's inequality we have: $\sum_{k=1}^{n}\left(\frac{(a_k^2)^2}{x_k+y_k}\right)\ge\frac{\left(\sum_{k=1}^{n}a_k^2\right)^2}{\sum_{k=1}^{n}(x_k+y_k)}$ *(2)*

From (1)+(2) we must show:

$$\left(\sum_{k=1}^{n}a_k^2\right)^2\ge 16F^2\tan\frac{\pi}{n}\Leftrightarrow\sum_{k=1}^{n}a_k^2\ge 4F\tan\frac{\pi}{n}$$

Relation which is true by Schanmberger's inequality.

SP.255. If $n\in\mathbb{N}$ then in ΔABC the following relationship holds:

$$3^n\left(\left(a^2\cot\frac{A}{2}\right)^{n+1}+\left(b^2\cot\frac{B}{2}\right)^{n+1}+\left(c^2\cot\frac{C}{2}\right)^{n+1}\right)\ge$$

$$\ge 4^{n+1}\cdot 3^{\frac{5(n+1)}{2}}\cdot r^{2n+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

From Hölder's inequality we have:

$$\left(a^2\cot\frac{A}{2}\right)^{n+1}+\left(b^2\cot\frac{B}{2}\right)^{n+1}+\left(c^2\cot\frac{C}{2}\right)^{n+1}\ge\frac{\left(a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}\right)^{n+1}}{3^n}$$

$\Rightarrow$ *we must show:* $\left(a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}\right)^{n+1}\ge 4^{n+1}\cdot 3^{\frac{5(n+1)}{2}}\cdot r^{2(n+1)}\Leftrightarrow$

$$\Leftrightarrow\left(a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}\right)\ge 4\cdot 9\sqrt{3}r^2\quad (1)$$

$$a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}\ge 3\sqrt[3]{(abc)^2\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}}\quad (2)$$

But $abc=4sRr$ *and* $\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\frac{s}{r}$ *(3)*

From (2)+(3)$\Rightarrow a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}\ge 3\sqrt[3]{16s^2R^2r^2\cdot\frac{s}{r}}=$

$$=3\sqrt[3]{16s^3\cdot R^2\cdot r}\ge 3\sqrt[3]{16\cdot 81\sqrt{3}r^3\cdot 4R^2\cdot r}=$$

$$= 3\sqrt[3]{64 \cdot 27\sqrt{27}r^6} = 4 \cdot 9\sqrt{3}r^2 \Rightarrow$$ ***(1) it is true.***

Solution 2 by Avishek Mitra-West Bengal-India

$$3^n\left[\sum\left(a^2\cot\frac{A}{2}\right)^{n+1}\right] \overset{AM-GM}{\geq} 3 \cdot 3^n\left[\left(\prod a^2\cot\frac{A}{2}\right)^{n+1}\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3^{n+1}\left[(abc)^{2(n+1)} \cdot \left(\frac{s^3}{r_a r_b r_c}\right)^{n+1}\right]^{\frac{1}{3}} \Rightarrow \Omega \geq 3^{n+1}\left[(4Rrs)^{2(n+1)} \cdot \left(\frac{s^3}{s^2 r}\right)^{(n+1)}\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3^{n+1}\left[\left(4 \cdot 2r \cdot r \cdot 3\sqrt{3}r\right)^{2(n+1)} \cdot \left(\frac{3\sqrt{3}r}{r}\right)^{(n+1)}\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3^{n+1}\left[(8)^{2(n+1)} \cdot (r^3)^{2(n+1)} \cdot 3^{\frac{3}{2}\cdot 2(n+1)} \cdot 3^{\frac{3(n+1)}{2}}\right]^{\frac{1}{3}}$$

$$\Rightarrow \Omega \geq 3^{n+1}\left[(64)^{\frac{(n+1)}{3}} \cdot (r^3)^{\frac{2(n+1)}{3}} \cdot 3^{\frac{3(n+1)}{3}} \cdot 3^{\frac{(n+1)}{2}}\right]$$

$$\Rightarrow \Omega \geq 3^{n+1} \cdot 4^{n+1} \cdot r^{2(n+1)} \cdot 3^{n+1} \cdot 3^{\frac{(n+1)}{2}} \Leftrightarrow \Omega \geq 4^{n+1} \cdot 3^{\frac{5(n+1)}{2}} \cdot r^{2n+2}$$ ***(Proved)***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS = 3^n\sum_{cyc}\frac{\left(a^2 cot\frac{A}{2}\right)^{n+1}}{1^n} \overset{RADON}{\geq} 3^n \cdot \frac{\left(\sum_{cyc} a^2 cot\frac{A}{2}\right)^{n+1}}{3^n} =$$

$$= \left(\sum_{cyc}\frac{a^2 s}{s\tan\frac{A}{2}}\right)^{n+1} = \left(\sum_{cyc}\frac{a^2 s}{r_a}\right)^{n+1} = \left(\sum_{cyc}\frac{a^2 s(s-a)}{rs}\right)^{n+1} =$$

$$= \left(\frac{s\sum_{cyc} a^2 - \sum_{cyc} a^3}{r}\right)^{n+1} = \left(\frac{2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2)}{r}\right)^{n+1} =$$

$$= 4^{n+1}(R+r)^{n+1}s^{n+1} \overset{EULER;MITRINOVIC}{\geq} 4^{n+1}(2r+r)^{n+1}(3\sqrt{3}r)^{n+1} =$$

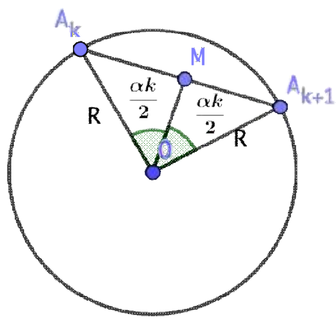
$$= 4^{n+1} \cdot 3^{\frac{5(n+1)}{2}} \cdot r^{2n+2}$$

UP.241. Let be $n \in \mathbb{N}; n \geq 3; m \geq 0; A_1A_2 \dots A_n$; a convexe polygon inscribed in a circle with radius R. If $a_k = A_kA_{k+1}; k \in \overline{1,n}; A_{n+1} = A_1$; s – semiperimeter then:

$$\sum_{k=1}^{n} \frac{1}{a_k^m} \geq \frac{n}{2^m R^m \sin^m \frac{\pi}{n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by Adrian Popa – Romania



$$\Delta OMA_{k+1}: MA_{k+1} = \frac{a_k}{2}$$

$$\sin\frac{\alpha_k}{2} = \frac{MA_{k+1}}{OA_{k+1}} = \frac{\frac{a_k}{2}}{R} = \frac{a_k}{2R}$$

$$a_k = 2R\sin\frac{\alpha k}{2}$$

$$\sum_{k=1}^{n}\frac{1}{a_k^m} \overset{J.Radon}{>} \frac{(1+1+\dots+1)^{m+1}}{(a_1+a_2+\dots+a_n)^m} = \frac{n^{m+1}}{(2s)^m} = \frac{n^{m+1}}{2^m \cdot s^m}$$

$$s = \frac{a_1+a_2+\dots+a_n}{2} = \frac{2R\left(\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\dots+\sin\frac{\alpha_n}{2}\right)}{2} =$$

$$= R\left(\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\dots+\sin\frac{\alpha_n}{2}\right)$$

But $\alpha_1+\alpha_2+\dots+\alpha_n = 2\pi \Rightarrow \frac{\alpha_1+\alpha_2+\dots+\alpha_n}{2} = \frac{2\pi}{2} = \pi$

Let be $f(x) = \sin x \Rightarrow f''(x) = \cos x \Rightarrow f''(x) = -\sin x < 0 \Rightarrow f \rightarrow$ *concave*

We apply Jensen's inequality to function f*:*

$$\frac{f\left(\frac{\alpha_1}{2}\right)+f\left(\frac{\alpha_2}{2}\right)+\dots+f\left(\frac{\alpha_n}{2}\right)}{n}\le f\left(\frac{\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\dots+\frac{\alpha_n}{2}}{n}\right)\Rightarrow$$

$$\Rightarrow \sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\dots+\sin\frac{\alpha_n}{2}\le n\sin\frac{\pi}{n}$$

So, $\sum_{k=1}^{n}\frac{1}{a_k^m}\ge\frac{n^{m+1}}{2^m\cdot R^m\cdot\sin^m\frac{\pi}{m}\cdot n^m}=\frac{n}{2^m\cdot R^m\cdot\sin^m\frac{\pi}{n}}$

UP.242. $A=\begin{pmatrix}\sin^2 a & \cos^2 a\cdot\sin^2 b & \cos^2 a\cdot\cos^2 b\\ \cos^2 b\cdot\sin^2 c & \sin^2 b & \cos^2 b\cdot\cos^2 c\\ \cos^2 c\cdot\sin^2 a & \cos^2 c\cdot\cos^2 a & \sin^2 c\end{pmatrix}$

$$A^{2019}=\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}$$

If $a,b,c\in\mathbb{R}$ then find:

$$\Omega=\sum_{i=1}^{9}x_i$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $A=\begin{bmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{bmatrix}=\begin{bmatrix}R_1\\ R_2\\ R_3\end{bmatrix}$

$$B=\begin{bmatrix}b_1 & b_2 & b_3\\ b_4 & b_5 & b_6\\ b_7 & b_8 & b_9\end{bmatrix}=\begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}$$

where $b_1+b_2+b_3=b_4+b_5+b_6=b_7+b_8+b_9=1$

$$AB=\begin{bmatrix}R_1\cdot c_1 & R_1\cdot c_2 & R_1\cdot c_3\\ R_2\cdot c_1 & R_2\cdot c_2 & R_2\cdot c_3\\ R_3\cdot c_1 & R_3\cdot c_2 & R_3\cdot c_3\end{bmatrix}$$

Where $R_1\cdot c_1=a_1b_1+a_2b_4+a_3b_6$, *etc*

Sum of the elements of AB

$$=R_1\cdot c_1+R_1\cdot c_2+R_1\cdot c_3+R_2\cdot c_1+R_2\cdot c_2+R_2\cdot c_3+R_3\cdot c_1+R_3\cdot c_2+R_3\cdot c_3$$

$$= (a_1 + a_4 + a_7)(b_1 + b_2 + b_3) + (a_2 + a_5 + a_8)(b_4 + b_5 + b_6) +$$

$$+(a_3 + a_6 + a_9)(b_7 + b_8 + b_9) =$$

$$= a_1 + a_4 + a_7 + a_2 + a_5 + a_8 + a_3 + a_6 + a_9$$

$$[\because b_1 + b_2 + b_3 = 1, etc]$$

= *sum of the elements of* A.

Thus, if sum of the elements of each row of B is 1, sum of the elements of (AB)

= *sum of the elements of* A

We have: $A = \begin{pmatrix} \sin^2 a & \cos^2 a \sin^2 b & \cos^2 a \cos^2 b \\ \cos^2 b \sin^2 c & \sin^2 b & \cos^2 b \cos^2 c \\ \cos^2 c \sin^2 a & \cos^2 c \cos^2 a & \sin^2 c \end{pmatrix}$

Note that the sum of the elements of each row is 1.

Thus, the sum of the elements of A^2

= *the sum of the elements* (AA)

= *then sum of the elements of* A

$$= 3$$

Assume the sum of the elements of A^m *is* 3 *for the same* $m \in \mathbb{N}$

$$m \geq 1$$

The sum of the elements of A^{m+1}

= *sum of the elements of* $A^m A$

= *sum of the elements of* A^m

$$= 3$$

$\therefore$ *the sum of the elements of* A^n *is* $3; \forall n \geq 1 \Rightarrow \sum_{k=1}^{a} x_k = 3$

Solution 2 by Ravi Prakash-New Delhi-India

Let $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$AX = \begin{pmatrix} \sin^2 a + \cos^2 a \sin^2 b + \cos^2 a \cos^2 b \\ \cos^2 b \sin^2 c + \sin^2 b + \cos^2 b \cos^2 c \\ \cos^2 c \sin^2 a + \cos^2 c \sin^2 a + \sin^2 c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x$$

Thus $A^2(x) = A(AX) = AX = x$. *Continuing in this way, we get*

$$A^{2019}X = X \Rightarrow \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 + x_5 + x_6 \\ x_7 + x_8 + x_9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore \sum_{k=1}^{9} x_k = 3$$

UP.243. Find:

$$\Omega = \lim_{n\to\infty}\left(H_n + \log_2\frac{1}{3} + \log_3\frac{1}{5} + \cdots + \log_n\frac{1}{2n-1}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rohan Shinde-India

$$\Omega = \lim_{n\to\infty}\left(H_n + \sum_{k=2}^{n}\log_k\left(\frac{1}{2k-1}\right)\right) = \lim_{n\to\infty}\left(H_n - \sum_{k=2}^{n}\log_k(2k-1)\right)$$

$\forall k \geq 2$ ***we know that*** $\log_k(2k-1) > 1$

$$\Rightarrow \sum_{k=2}^{n}\log_2(2k-1) > \sum_{k=2}^{n} 1 = n-1$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\left(H_n - \sum_{k=2}^{n}\log_k(2k-1)\right) < \lim_{n\to\infty}(H_n - n + 1)$$

As $n \to \infty; H_n \sim \ln(n) + \gamma \Rightarrow \Omega < \lim_{n\to\infty}(\ln(n) - n + \gamma + 1)$

But as $n \to \infty$***, the linear function grows more rapidly than the logarithmic function.***

$\Rightarrow$ ***as*** $n \to \infty; \ln(n) - n \to -\infty \Rightarrow \Omega \leq -\infty \Rightarrow \Omega = -\infty$

Solution 2 by Srinivasa Raghava-AIRMC-India

We have: $\log_n\left(\frac{1}{2n+1}\right) = -\frac{\log(2n+1)}{\log(n)} = O(n^2) - \frac{(2n)}{\log(n)}$

It is easy to see that: $\sum_{n=2}^{m}\frac{\log(2n+1)}{\log(n)} = O(m)$***. Then, we have:***

$$\lim_{m\to\infty}\left(\sum_{n=2}^{m}\log_n\left(\frac{1}{2n+1}\right) + H_m\right) = \lim_{n\to\infty}(H_m - m) = -\infty$$

Solution 3 by Naren Bhandari-Bajura-Nepal

We write:

$$\Omega = \lim_{n\to\infty}\left(H_n - \sum_{k=1}^{n}\frac{\log(2k-1)}{\log k}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \sum_{k=2}^{n}\left(1 + \frac{\log\frac{2k-1}{k}}{\log k}\right)\right) = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \sum_{k=2}^{n}1 - \sum_{k=2}^{n}\frac{\log\frac{2k-1}{k}}{\log k}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{m=1}^{n}\left(\frac{1}{m} - \frac{\log\frac{2m+1}{m+1}}{\log(m+1)}\right) - \sum_{k=1}^{n-1}1\right)$$

$$= \lim_{n\to\infty}\left(1 - n + \sum_{m=1}^{n}\frac{1}{m} - \log(m+1) + \sum_{m=1}^{n}\frac{\log(2m+1) - \log^2(m+1)}{\log(m+1)}\right)$$

$$\le \lim_{n\to\infty}\left(1 - n + \gamma - \sum_{m=1}^{n}\log(m+1)\right) \le \lim_{n\to\infty}(1 + \gamma + \log(n!) - n)$$

$$= \lim_{n\to\infty}(1 + \gamma - n\log n - n + O\log(n) - n) = 1 + \gamma - \lim_{n\to\infty}(2n + \log n^n) = -\infty$$

UP.244. If $A \in M_4(\mathbb{Q})$, $\det\left((1-i)A + \sqrt{2}I_4\right) = 0$ then:

$$\det(A + xI_4) \ge 2x^2, x \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

In these conditions: $\det\left((1-i)A + \sqrt{2}I_4\right) = 0 \Leftrightarrow \det\left(A + \frac{\sqrt{2}+i\sqrt{2}}{2}I_4\right) = 0$

We denote $\theta = \frac{\sqrt{2}+i\sqrt{2}}{2} \Rightarrow \theta$ *root for*

$\det(A + xI_4)$. *But* $\det(A + xI_4)$ *has rational coefficients* $\Rightarrow \overline{\theta}$ *root of it*

As $\det(A + xI_4)$ *has the grade IV, we obtain:*

$$\det(A + xI_4) = (x - \theta)(x - \overline{\theta})(x^2 + ax + b) = \left(x^2 - \sqrt{2}x + 1\right)(x^2 + ax + b) =$$

$$= x^4 + ax^3 + bx^2 - \sqrt{2}x^3 - a\sqrt{2}x^2 - \sqrt{2}xb + x^2 + ax + b$$

$$= x^4 + x^3\left(a - \sqrt{2}\right) + x^2\left(b - a\sqrt{2} + 1\right) + x\left(a - b\sqrt{2}\right) + b \in \mathbb{Q}[x] \Rightarrow b \in \mathbb{Q}$$

$$a-\sqrt{2}\in\mathbb{Q}\Rightarrow a-\sqrt{2}=q\in\mathbb{Q}\Rightarrow a=q+\sqrt{2}$$

$$b-a\sqrt{2}+1\in\mathbb{Q}\Rightarrow b-a\sqrt{2}\in\mathbb{Q}\Rightarrow a-b\sqrt{2}\in\mathbb{Q}\Rightarrow$$

$$\Rightarrow b-\left(q+\sqrt{2}\right)\sqrt{2}\in\mathbb{Q}\Rightarrow b-q\sqrt{2}-2\in\mathbb{Q}$$

$$\Rightarrow b-q\sqrt{2}\in\mathbb{Q}\Rightarrow b-q\sqrt{2}=p\in\mathbb{Q}$$

or $-q\sqrt{2}=p-b$ **or** $q\sqrt{2}=b-p\in\mathbb{Q}$

If $q\neq 0\Rightarrow\sqrt{2}=\frac{b-p}{q}$ ***(False) so,*** $q=0\Rightarrow a=\sqrt{2}\Rightarrow\sqrt{2}-b\sqrt{2}\in\mathbb{Q}$

$$\sqrt{2}(1-b)\in\mathbb{Q}\Rightarrow 1-b=0\Rightarrow b=1$$

Then, $\det(A+xI_4)=\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)$

$$\det(A+xI_4)=(x^2+1)^2-2x^2=x^4+2x^2+1-2x^2=x^4+1$$

$$x^4+1\geq 2x^2$$

$x^4-2x^2+1\geq 0\ (x^2-1)^2\geq 0$ ***(True)***

UP.245. If $x_0=1,\ x_{n+1}=n+\frac{1}{x_n},\ n\in\mathbb{N}$ **then find:**

$$\Omega=\lim_{n\to\infty}\left(n(x_n-n)\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Florentin Vișescu-Romania

$$x_0=1;x_{n+1}=n+\frac{1}{x_n};\Omega=\lim_{n\to\infty}n(x_n-n)$$

$$x_1=0+\frac{1}{x_0}=1;x_2=1+\frac{1}{x_1}=2;x_3=2+\frac{1}{2}=\frac{5}{2};x_4=3+\frac{2}{5}=\frac{17}{5}$$

We suppose $x_k>\frac{k-1}{(k>2)}$ ***and we prove that*** $x_{k+1}>k$

$x_{k+1}=k+\frac{1}{x_{k-1}}>k$ ***obviously.***

So, $x_n>n-1,\forall n>2,n\in\mathbb{N}\Rightarrow\lim_{n\to\infty}x_n=\infty$

As $x_{n+1}=n+\frac{1}{x_n}\Rightarrow x_n=n-1+\frac{1}{x_{n-1}}\Rightarrow$

$$\Rightarrow x_n-n=\frac{1}{x_{n-1}}\Rightarrow n(x_n-n)=n\left(\frac{1}{x_{n-1}}-1\right)$$

As $\lim_{n\to\infty} x_n = \infty \Rightarrow \lim_{n\to\infty} x_{n-1} = \infty$

$$\Rightarrow \lim_{n\to\infty} n(x_n - n) = \lim_{n\to\infty} n\left(\frac{1}{x_{n-1}} - 1\right) = \infty(0-1) = -\infty$$

Solution 2 by Remus Florin Stanca-Romania

We prove by using the Mathematical induction that $x_n > 0; \forall n \in \mathbb{N}$:

$P(0): x_0 > 0$ ***is true*** $(x_0 = 1)$

We suppose that $P(n): x_n > 0$ ***is true and we prove that*** $P(n+1): x_{n+1} > 0$ ***is true by using the fact that*** $P(n)$ ***is true:*** $x_n > 0$ ***and*** $n > 0 \overset{by\ adding}{\Rightarrow} n + \frac{1}{x_n} > 0 \Rightarrow x_{n+1} > 0 \Rightarrow$

$\Rightarrow$ ***proved*** $\Rightarrow x_n > 0; \forall n \in \mathbb{N}$.

$$x_0 \ge 1, x_1 = 1 \ge 1, x_{n+1} = n + \tfrac{1}{x_n} \text{ for } n \ge 1 \Rightarrow x_{n+1} \ge 1; \forall n \ge 1 \Rightarrow x_n \ge 1; \forall n \in \mathbb{N}$$

$$x_n \ge 1 \Rightarrow \frac{1}{x_n} \le 1 \Rightarrow n + \frac{1}{x_n} \le n + 1 \Rightarrow x_{n+1} \le n+1, x_{n+1} - n = \frac{1}{x_n} > 0 \Rightarrow$$

$$\Rightarrow x_{n+1} - n > 0 \Rightarrow x_{n+1} > n \Rightarrow n < x_{n+1} \le n+1 \Rightarrow \lim_{n\to\infty} x_n = \infty$$

$$x_n = n - 1 + \tfrac{1}{x_{n-1}} \Rightarrow x_n - n = \tfrac{1}{x_{n-1}} - 1 \Rightarrow n(x_n - n) = n\left(\tfrac{1}{x_{n-1}} - 1\right) \quad (1)$$

$$x_{n+1} = n + \frac{1}{x_n} \Rightarrow \frac{x_{n+1}}{n+2} = \frac{n}{n+2} + \frac{1}{(n+2)x_n} \Rightarrow \lim_{n\to\infty} \frac{x_{n-1}}{n} = 1 + \frac{1}{\infty} = 1 \overset{(1)}{\Rightarrow}$$

$$\Rightarrow \lim_{n\to\infty}\big(n(x_n - n)\big) = 1 - \infty = -\infty \Rightarrow \Omega = -\infty$$

UP.246.Prove that:

$$\psi\left(\frac{3+\sqrt{3}}{2}\right) - \psi\left(\frac{3-\sqrt{3}}{2}\right) = 2\sqrt{3} + \pi tan\left(\frac{\sqrt{3}}{2}\pi\right)$$

where $\psi(x)$ is the digamma function.

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Rohan Shinde-India

$$\psi(1+z) = \psi(z) + \frac{1}{z}$$

$$\psi\left(\frac{3+\sqrt{3}}{2}\right)=\psi\left(1+\frac{1+\sqrt{3}}{2}\right)=\frac{2}{1+\sqrt{3}}+\psi\left(\frac{1+\sqrt{3}}{2}\right)$$

$$\psi\left(\frac{3-\sqrt{3}}{2}\right)=\psi\left(1+\frac{1-\sqrt{3}}{2}\right)=\frac{2}{1-\sqrt{3}}+\psi\left(\frac{1-\sqrt{3}}{2}\right)$$

$$\Omega=2\left(\frac{1}{\sqrt{3}+1}-\frac{1}{1-\sqrt{3}}\right)+\psi\left(\frac{1+\sqrt{3}}{2}\right)-\psi\left(\frac{1-\sqrt{3}}{2}\right)$$

$$\Omega=2\sqrt{3}+\psi\left(\frac{1+\sqrt{3}}{2}\right)-\psi\left(\frac{1-\sqrt{3}}{2}\right)=$$

$$=2\sqrt{3}+\pi cot\left(\frac{\pi}{2}-\frac{\pi\sqrt{3}}{2}\right)=2\sqrt{3}+\pi tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

Solution 2 by Remus Florin Stanca-Romania

$$\psi\left(\frac{3+\sqrt{3}}{2}\right)-\psi\left(\frac{3-\sqrt{3}}{2}\right)=\psi\left(1+\frac{1+\sqrt{3}}{2}\right)-\psi\left(1+\frac{1-\sqrt{3}}{2}\right)=$$

$$=\frac{2}{\sqrt{3}+1}+\psi\left(1-\frac{1+\sqrt{3}}{2}\right)-\frac{2}{1-\sqrt{3}}-\psi\left(1-\frac{1-\sqrt{3}}{2}\right)=$$

$$=2\left(\frac{1}{\sqrt{3}+1}-\frac{1}{1-\sqrt{3}}\right)+\left(\psi\left(1-\frac{1-\sqrt{3}}{2}\right)-\psi\left(\frac{1-\sqrt{3}}{2}\right)\right)=$$

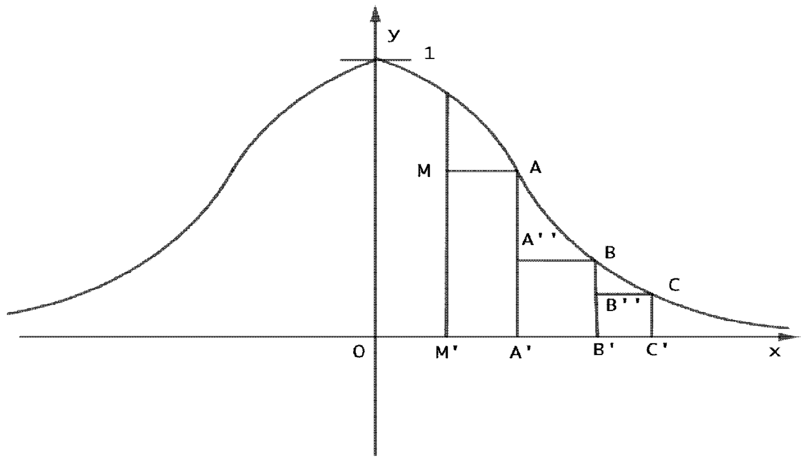
$$=2\cdot\frac{2\sqrt{3}}{(\sqrt{3}+1)(\sqrt{3}-1)}+\pi cot\left(\frac{\pi}{2}-\frac{\pi\sqrt{3}}{2}\right)=2\sqrt{3}+\pi tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\psi\left(\frac{3+\sqrt{3}}{2}\right)-\psi\left(\frac{3-\sqrt{3}}{2}\right)=$$

$$=\psi\left(1+\frac{1+\sqrt{3}}{2}\right)-\psi\left(1-\frac{1+\sqrt{3}}{2}\right)-\left(\psi\left(1+\frac{1-\sqrt{3}}{2}\right)-\psi\left(\frac{1-\sqrt{3}}{2}\right)\right)=$$

$$=\frac{2}{\sqrt{3}+1}+\pi cot\left(\frac{\pi}{2}-\frac{\pi\sqrt{3}}{2}\right)-\frac{2}{1-\sqrt{3}}=2\sqrt{3}+\pi tan\left(\frac{\pi\sqrt{3}}{2}\right)$$



UP.247. If $0 < a \leq b$ then:

$$\int_a^b e^{-x^2}\,dx \geq \frac{\sqrt{a}(\sqrt{b}-\sqrt{a})}{e^{ab}} + \frac{(\sqrt{b}-\sqrt{a})^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{(b-a)^2}{2e^{b^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

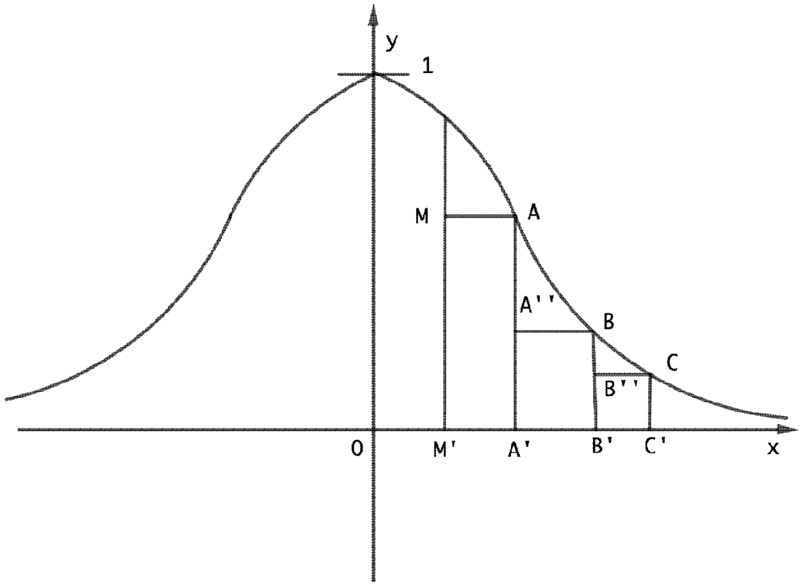
$$\int_a^b e^{-x^2}\,dx = \int_a^{\sqrt{ab}} e^{-x^2}\,dx + \int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}\,dx + \int_{\frac{a+b}{2}}^{b} e^{-x^2}\,dx$$

$$\geq (\sqrt{ab}-a)e^{-ba} + \left(\frac{a+b}{2}-\sqrt{ab}\right)e^{-\left(\frac{a+b}{2}\right)^2} + \left(b-\frac{a+b}{2}\right)e^{-b^2}$$

$$\geq \sqrt{a}(\sqrt{b}-\sqrt{a})e^{-ab} + \frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2 e^{-\left(\frac{a+b}{2}\right)^2} + \frac{b-a}{2}e^{-b^2}$$

$$\geq \frac{\sqrt{a}(\sqrt{b}-\sqrt{a})}{e^{ab}} + \frac{(\sqrt{b}-\sqrt{a})^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{b-a}{2e^{b^2}}$$

Solution 2 by proposer



$$\int_a^b e^{-x^2}dx \geq A[AMM'A'] + A[BA''A'B'] + A[CB''B'C']$$

$$M'(a,0); A'(\sqrt{ab},0); B'\left(\frac{a+b}{2},0\right); C'(b,0)$$

$$M(a,f(a)); A''\left(\sqrt{ab},f(\sqrt{ab})\right); B''\left(\frac{a+b}{2},f\left(\frac{a+b}{2}\right)\right);$$

$$\int_a^b e^{-x^2}dx \geq (\sqrt{ab}-a)e^{-ab} + \left(\frac{a+b}{2}-\sqrt{ab}\right)e^{-\frac{(a+b)^2}{4}} + \left(b-\frac{a+b}{2}\right)e^{-b^2} =$$

$$= \frac{\sqrt{a}(\sqrt{b}-\sqrt{a})}{e^{ab}} + \frac{(\sqrt{b}-\sqrt{a})^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{b-a}{2e^{b^2}}$$

UP.248. If $a_1, a_2, \dots, a_n \geq 1, a_1 a_2 \dots a_n = 2^n, n \geq 1$ then:

$$a_1 + a_2 + \dots + a_n - \frac{2}{a_1} - \frac{2}{a_2} - \dots - \frac{2}{a_n} \geq n$$

Proposed by Marin Chirciu – Romania

Solution 1 by Florentin Vișescu-Romania

Let $f:\left[\frac{\ln 2}{2};\infty\right)\to\mathbb{R};\ f(x)=e^x-\frac{2}{e^x};f'(x)=e^x+\frac{2}{e^x};f''(x)=e^x-\frac{2}{e^x}=\frac{e^{2x}-2}{e^x}$

For $e^{2x}-2\ge 0;e^{2x}\ge 2;2x\ge\ln 2\,;x\ge\frac{\ln 2}{2}$ *f is convexe. Then:*

$$f\left(\frac{x_1+x_2+x_3+\cdots+x_n}{n}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$$

For $x_1,x_2,\dots,x_n\in\left[\frac{\ln 2}{2};\infty\right)$. *Let be* $x_1=\ln a\,,x_2=\ln a_2\,,\dots,x_n=\ln a_n$

Then we obtain: $f\left(\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}\right)\le\frac{f(\ln a_1)+\cdots+f(\ln a_n)}{n}$

$$f\left(\frac{\ln a_1a_2\dots a_n}{n}\right)\cdot n\le f(\ln a_1)+\cdots+f(\ln a_n)$$

$$f\left(\frac{\ln 2^n}{n}\right)\cdot n\le e^{\ln a_1}-\frac{2}{e^{\ln a_1}}+\cdots+e^{\ln a_n}-\frac{2}{e^{\ln a_n}}$$

$$f(\ln 2)n\le a_1-\frac{2}{a}+\cdots+a_n-\frac{2}{a_n}$$

$$n\left(e^{\ln 2}-\frac{2}{e^{\ln 2}}\right)\le a_1+a_2+\cdots+a_n-\frac{2}{a_1}-\frac{2}{a_2}-\cdots-\frac{2}{a_n}$$

$$n\left(2-\frac{2}{2}\right)\le a_1+a_2+\cdots+a_n-\frac{2}{a_1}+\frac{2}{a_2}\dots\frac{2}{a_n}$$

$$n\le a_1+a_2+\cdots+a_n-\frac{2}{a_1}-\frac{2}{a_2}\dots\frac{-2}{a_n}$$

The inequality holds for $\ln a_i\ge\ln\sqrt{2}\,;a_i\ge\sqrt{2};i=\overline{1,n}$

Solution 2 by Michael Sterghiou-Greece

$$\prod_{i=1}^{n}\alpha_i=2^n,n\ge 1$$

$$\sum_{k=1}^{n}\alpha_k-\sum_{\lambda=1}^{n}\frac{2}{\alpha_\lambda}\ge n\ \ \textit{(1)}$$

$$\sum_{k=1}^{n}\alpha_k\overset{AM-GM}{\ge}n\cdot\sqrt[n]{\prod_{k=1}^{n}\alpha_k}=2n\ \ \textit{(2)}$$

$$\sum_{k=1}^{n}\frac{1}{\alpha_k}\ge n\sqrt[n]{\prod_{k=1}^{n}\frac{1}{\alpha_k}}=\frac{n}{2}\ \ \textit{(3)}$$

$$\frac{1}{\alpha_\lambda}\le 1\to\sum_{\lambda=1}^{n}\frac{1}{\alpha_\lambda}\le n\ \ \textit{(4)}$$

The function $f(t) = t^2 - 2$ is concave and (1)$\rightarrow \sum_{k=1}^{n} \frac{\alpha_k^2 - 2}{\alpha_k} \geq n$

By generalized Jensen on $f(t) = t^2 - 2$ with "weights" $\frac{1}{\alpha_\lambda}, \lambda = \overline{1,n}$ we get:

$$\sum_{\lambda=1}^{n} \frac{\alpha_\lambda^2 - 2}{\lambda} \geq \sum_{\lambda=1}^{n} \frac{1}{\alpha_\lambda}\left[\left(\frac{\sum_1^n \alpha_\lambda}{\sum_1^n \frac{1}{\alpha_\lambda}}\right)^2 - 2\right] \overset{(3),(2)}{\geq} \frac{n}{2} \cdot \left[\frac{4n^2}{\left(\sum_1^n \frac{1}{\alpha_\lambda}\right)^2} - 2\right]$$

$\geq \frac{2n^3}{\left(\sum_{\lambda=1}^n \frac{1}{\alpha_\lambda}\right)^2} - n$ ***which suffices to $b \geq n$. This reduces to*** $\frac{2n^3}{\left(\sum_{\lambda=1}^n \frac{1}{\alpha_\lambda}\right)^2} - n \geq n \rightarrow \sum_{\lambda=1}^{n} \frac{1}{\alpha_\lambda} \leq n$

which is (4). Done!

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a_1, a_2, a_3, \ldots, a_n \geq 1$ and $a_1 a_2 a_3 \ldots a_n = 2^n$, where $n \geq 1$ we have:

$$a_1 + a_2 + a_3 + \cdots + a_n \geq 2n$$

$$\Rightarrow (a_1 + 2) + (a_2 + 2) + \cdots + (a_n + 2) \geq 4n$$

and $(a_1 a_2 a_3 \ldots a_n)^2 = 2^{(2n)} \Rightarrow a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 \geq 4n$

and since $a_1, a_2, a_3, \ldots, a_n \geq 1$

Hence $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 \geq (a_1 + 2) + (a_2 + 2) + (a_3 + 2) + \cdots + (a_n + 2)$ ***and***

since $a_2 a_3 \ldots a_n + a_1 a_3 \ldots a_n + a_1 a_2 a_4 \ldots a_n + \cdots + a_1 a_2 \ldots a_{n-1}$

$$= a_2 a_3 \ldots a_n + a_1 a_3 \ldots a_n + a_1 a_2 a_4 \ldots a_n + \cdots + a_1 a_2 \ldots a_{n-1}$$

Hence $(a_1^2 + a_2^2 + \cdots a_n^2)(a_2 a_3 \ldots a_n + a_1 a_3 \ldots a_n + \cdots a_1 a_2 \ldots a_{n-1}) \geq$

$$\geq ((a_1 + 2) + (a_2 + 2) + \cdots (a_n + 2))(a_2 a_3 \ldots a_n + a_1 a_3 + \cdots + a_n + \cdots + a_1 a_2 \ldots a_{n-1})$$

By matching we have:

$$a_1^2(a_2 a_3 \ldots a_n) + a_2^2(a_1 a_3 \ldots a_n) + a_3^2(a_1 a_2 a_4 \ldots a_n) + \cdots + a_n^2(a_1 a_2 \ldots a_{n-1})$$

$$\geq (a_1 + 1)(a_2 a_3 \ldots a_n) + (a_2 + 2)(a_1 a_3 \ldots a_n) + (a_3 + 2)(a_1 a_2 a_4 \ldots a_n) + \cdots +$$

$$+(a_n + 2)(a_1 a_2 a_3 \ldots a_{n-1})$$

Hence $a_1(a_1 a_2 \ldots a_n) + a_2(a_1 a_2 \ldots a_n) + a_3(a_1 a_2 a_3 \ldots a_n) + \cdots + a_n(a_1 a_2 \ldots a_n)$

$$\geq a_1 a_2 - a_n + 2a_2 a_3 \ldots a_n + a_1 a_2 - a_n + 2a_1 a_3 a_4 \ldots a_n + a_1 a_2 a_3 \ldots a_n +$$

$$+2a_1 a_2 a_4 \ldots a_n + \cdots + a_1 a_2 \ldots a_n + 2a_1 a_2 \ldots a_{n-1}$$

$$\Rightarrow a_1 + a_2 + a_3 + \cdots + a_n \geq 1 + \frac{2}{a_1} + 1 + \frac{2}{a_2} + 1 + \frac{2}{a_3} + 1 + \frac{2}{a_4} + 1 + \frac{2}{a_5} + \cdots + 1 + \frac{2}{a_n}$$

$$=\frac{2}{a_1}+\frac{2}{a_2}+\frac{2}{a_3}+\dots+\frac{2}{a_n}+n$$

Therefore, it is true.

UP.249 If $\propto>0$, $r\in(0,\propto)$, $z_1,z_2,\dots,z_n\in C$ such that $|z_k-\propto|<r, k=\overline{1,n}$ then:

$$\left|z_{x_k1}+z_2+\dots+z_n\right|\left|\frac{1}{z_1}+\frac{1}{z_2}+\dots+\frac{1}{z_n}\right|>\frac{n^2(\propto^2-r^2)}{\propto^2}$$

Proposed by Marian Voinea-Romania

Solution by proposer

Let be $z_k=x_k+iy_k$, *with* $x_k, y_k\in R, k=\overline{1,n}$.

By $|z_k-\propto|<r$, *we obtain* $(x_k-\propto)^2+y_k^2<r^2$

$$2\propto x_k>x_k^2+y_k^2+\propto^2-r^2\geq 2\sqrt{(x_k^2+y_k^2)(\propto^2-r^2)}\Rightarrow x_k>0, k=\overline{1,n}.$$

$$\propto x_k>\sqrt{(x_k^2+y_k^2)(\propto^2-r^2)}, \propto, x_k>0,$$

By squaring: $\frac{x_k^2}{x_k^2+y_k^2}>\frac{\propto^2-r^2}{\propto^2}$

$$\frac{x_k}{x_k^2+y_k^2}>\frac{\propto^2-r^2}{\propto^2x_k}\ (1)$$

$$z_1+z_2+\dots+z_n=x_1+x_2+\dots+x_n+i(y_1+y_2+\dots+y_n)$$

$$\frac{1}{z_1}+\frac{1}{z_2}+\dots+\frac{1}{z_n}=\frac{x_1-iy_1}{x_1^2+y_1^2}+\frac{x_2-iy_2}{x_2^2+y_2^2}+\dots+\frac{x_n-iy_n}{x_n^2+y_n^2}$$

$$|z_1+z_2+\dots+z_n|\left|\frac{1}{z_1}+\frac{1}{z_2}+\dots+\frac{1}{z_n}\right|=$$

$$=\sqrt{(x_1+\dots+x_n)^2+(y_1+\dots+y_n)^2}\sqrt{\left(\frac{x_1}{x_1^2+y_1^2}+\dots+\frac{x_n}{x_n^2+y_n^2}\right)^2+\left(\frac{y_1}{x_1^2+y_1^2}+\dots+\frac{y_n}{x_n^2+y_n^2}\right)^2}$$

$$\geq\sqrt{(x_1+\dots+x_n)^2}\sqrt{\left(\frac{x_1}{x_1^2+y_1^2}+\dots+\frac{x_n}{x_n^2+y_n^2}\right)^2}=(x_1+\dots+x_n)\left(\frac{x_1}{x_1^2+y_1^2}+\dots+\frac{x_n}{x_n^2+y_n^2}\right)\overset{(1)}{>}$$

$$(x_1+\dots+x_n)\left(\frac{\propto^2-r^2}{\propto^2x_1}+\dots+\frac{\propto^2-r^2}{\propto^2x_n}\right)=\frac{\propto^2-r^2}{\propto^2}(x_1+\dots+x_n)\left(\frac{1}{x_1}+\dots+\frac{1}{x_n}\right)\geq\frac{n^2(\propto^2-r^2)}{\propto^2}$$

$$(x_1+\dots+x_n)\left(\frac{1}{x_1}+\dots+\frac{1}{x_n}\right)\geq n^2$$

Hence:

$$|z_1+z_2+\cdots+z_n|\left|\frac{1}{z_1}+\frac{1}{z_2}+\cdots+\frac{1}{z_n}\right|>\frac{n^2(\propto^2-r^2)}{\propto^2}.$$

UP.250 If $a,b,c>0$ then:

$$\sum_{cyc}\left(1+\frac{1}{a}\right)^b>e^{\frac{b}{2a+1}+\frac{c}{2b+1}}+e^{\frac{c}{2b+1}+\frac{a}{2c+1}}+e^{\frac{a}{2c+1}+\frac{b}{2a+1}}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

Let be $f:(0,\infty)\to\mathbb{R};\ f(x)=\ln\left(1+\frac{1}{x}\right)-\frac{2}{2x+1}$

$$f'(x)=\frac{-\frac{1}{x^2}}{\frac{x+1}{x}}+\frac{4}{(2x+1)^2}=\frac{-4(x+1)}{x(x+1)(2x+1)^2}=\frac{-4}{x(2x+1)^2}<0$$

f ***decreasing*** $\Rightarrow \inf f(x)=\lim_{x\to\infty}f(x)=0$

$$\Rightarrow f(x)>0\Rightarrow \ln\left(1+\frac{1}{x}\right)-\frac{2}{2x+1}>0$$

$$\ln\left(1+\frac{1}{a}\right)-\frac{2}{2a+1}>0\Rightarrow \ln\left(1+\frac{1}{a}\right)>\frac{2}{2a+1}$$

$$b\ln\left(1+\frac{1}{a}\right)>\frac{2b}{2a+1}\Rightarrow \ln\left(1+\frac{1}{a}\right)^b>\frac{2b}{2a+1}$$

$$\left(1+\frac{1}{a}\right)^b>e^{\frac{2b}{2a+1}}$$

$$\sum_{cyc}\left(1+\frac{1}{a}\right)^b>\sum_{cyc}e^{\frac{2b}{2a+1}}\geq\sum_{cyc}\left(e^{\frac{b}{2a+1}}\cdot e^{\frac{c}{2b+1}}\right)=\sum_{cyc}\left(e^{\frac{b}{2a+1}+\frac{c}{2b+1}}\right)$$

UP.251. If $t,u,x,y>0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2}{\left(t\cos^2\frac{A}{2}+u\cos^2\frac{B}{2}\right)(xb+yc)^2}\geq\frac{18R}{(t+u)(x+y)^2(4R+r)}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$\frac{\sum\left(\frac{a}{xb+yc}\right)^2}{t\cos^2\frac{A}{2}+y\cos^2\frac{B}{2}} \overset{Bergstrom}{\geq} \frac{\left(\sum\frac{a}{xb+yc}\right)^2}{(t+y)\cdot\sum\cos^2\frac{A}{2}} \quad (1)$$

$$\text{But } \sum\cos^2\frac{A}{2} = \frac{4R+r}{2R} \quad (2)$$

$$\text{From (1)+(2) we must show: } \frac{\left(\sum\frac{a}{xb+yc}\right)\cdot 2R}{(t+y)(4R+r)} \geq \frac{18R}{(t+y)(x+y)^2(4R+r)} \Leftrightarrow$$

$$\left(\sum\frac{a}{xb+yc}\right)^2 \geq \frac{9}{(x+y)^2} \Leftrightarrow \sum\frac{a}{xb+yc} \geq \frac{3}{x+y} \quad (3)$$

$$\text{But } \sum\frac{a}{xb+yc} = \sum\frac{a^2}{abx+acy} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{(x+y)(ab+ac+bc)} \quad (4)$$

$$\text{But } (a+b+c)^2 \geq 3(ab+ac+bc) \quad (5)$$

$$\text{From (4)+(5)} \Rightarrow \sum\frac{a}{xb+yc} \geq \frac{3}{x+y} \Rightarrow \text{(3) it is true.}$$

UP.252. If $m, n, t > 0$ then in ΔABC the following relationship holds:

$$m^3\left(\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^4 + n^3\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^4 + t^3\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^4 \geq \frac{mntr^2}{s^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by proposers

According to AM-GM inequality:

$$V = m^3\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^4 + n^3\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^4 + t^3\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^4 \geq$$

$$\geq 3\cdot\sqrt[3]{m^3n^3t^3\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^8} = 3mn\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2\cdot\sqrt[3]{\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2} \quad (1)$$

$$\text{But } \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{s} \quad (2)$$

$$\text{So, } V \geq 3mnt\cdot\frac{r^2}{s^2}\sqrt[3]{\frac{r^2}{s^2}} \overset{Mitrinovic}{\geq} 3mnt\cdot\frac{r^2}{s^2}\sqrt[3]{\frac{r^2}{\left(3\sqrt{3}r\right)^2}} = 9mnt - \frac{r^2}{s^2}\cdot\frac{1}{3} =$$

$$= mnt\cdot\frac{r^2}{s^2} \text{ q.e.d with equality} \Leftrightarrow \text{the triangle is equilateral, } m = n = t.$$

UP.253. If $m \in \mathbb{N}; m \geq 1; a, b, x_k > 0; (y_n)_{n\geq 0}; y_0, y_1 > 0;$

$y_{n+2} = ay_{n+1} + by_n, (\forall)n \in \mathbb{N}$ **then:**

$$\left(mn + \sum_{k=1}^{n} \frac{1}{x_k^{m+1}}\right)\left(mn + \sum_{k=1}^{n}\left(\frac{x_kx_{k+1}}{ay_{n+1} + x_{k+1} + by_nx_k}\right)^{m+1}\right) \geq \frac{(m+1)^2n^2}{y_{n+2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca – Romania

The inequality can be written as:

$$\left(1^2 + 1^2 + \cdots + 1^2 + \sum_{k=1}^{n} \frac{1}{x_k^{m+1}}\right)\left(1^2 + 1^2 + \cdots + 1^2 + \sum_{k=1}^{n}\left(\frac{x_kx_{k+1}}{ay_{n+1}x_{k+1} + by_nx_k}\right)^{m+1}\right) \geq$$

$\geq \frac{(m+1)^2n^2}{y_{n+2}}$***. We use Cauchy's inequality:***

$$\left(1^2 + \cdots + 1^2 + \sum_{k=1}^{n} \frac{1}{x_k^{m+1}}\right)\left(1^2 + \cdots + 1^2 + \sum_{k=1}^{n}\left(\frac{x_kx_{k+1}}{ay_{n+1}x_{k+1} + by_nx_k}\right)^{m+1}\right) \geq$$

$$\geq \left(mn + \sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\right)^{\frac{m+1}{2}}\right)^2 \quad \textbf{(1)}$$

We use Bernoulli's inequality: $(x+1)^a \geq ax + 1$ ***for*** $x \geq -1$ ***and*** $a \geq 1$ ***and in our particular case, we have:***

$$\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k} - 1 + 1\right)^{\frac{m+1}{2}} \geq \left(\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\right) \cdot \frac{m+1}{2} + 1 \quad \textbf{(2) because}$$

$\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k} - 1 \geq -1$ ***and*** $\frac{m+1}{2} \geq 1$ ***because*** $m \geq 1| + 1 \Rightarrow \frac{m+1}{2} \geq 1 \overset{(2)}{\Rightarrow}$

$$\overset{(2)}{\Rightarrow} \sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1} + by_nx_k}\right)^{\frac{m+1}{2}} \geq \sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1} + by_nx_k} - 1\right) \cdot \frac{m+1}{2} + n \overset{(1)}{\Rightarrow}$$

$$\overset{(1)}{\Rightarrow} \left(mn + \sum_{k=1}^{n} \frac{1}{x_k^{m+1}}\right)\left(mn + \sum_{k=1}^{n}\left(\frac{x_{k+1}x_k}{ay_{n+1}x_{k+1} + by_nx_k}\right)^{m+1}\right) \geq$$

$$\geq \left(\frac{m+1}{2}\sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1} + by_nx_k} - 1\right) + n + mn\right)^2$$

We need to prove that: $\left(\frac{m+1}{2}\sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}-1\right)+n+mn\right)^2\ge\frac{(m+1)^2n^2}{y_{n+2}}\Leftrightarrow$

$$\Leftrightarrow\frac{m+1}{2}\sum_{k=1}^{n}\left(\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}-1\right)+n+mn\ge\frac{(m+1)n}{\sqrt{y_{n+2}}}\Leftrightarrow$$

$$\Leftrightarrow\frac{m+1}{2}\sum_{k=1}^{n}\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}-\frac{(m+1)n}{2}+n+mn\ge\frac{(m+1)n}{\sqrt{y_{n+2}}}\Leftrightarrow$$

$$\Leftrightarrow\frac{m+1}{2}\sum_{k=1}^{n}\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\ge\frac{(m+1)n}{\sqrt{y_{n+2}}}-\frac{(m+1)n}{2}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{2}\sum_{k=1}^{n}\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\ge n\left(\frac{1}{\sqrt{ay_{n+1}+by_n}}-\frac{1}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{n}\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\ge n\left(\frac{2}{\sqrt{ay_{n+1}+by_n}}-1\right)\quad (3)$$

We need to prove that

$$\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}-1\Leftrightarrow\frac{1}{ay_{n+1}+by_n\frac{x_k}{x_{k+1}}}\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}-1\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{ay_{n+1}+by_n\frac{x_k}{x_{k+1}}}+1\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}$$

Case I: $x_k\le x_{k+1}\Rightarrow\frac{x_k}{x_{k+1}}\le1\Rightarrow by_n\frac{x_k}{x_{k+1}}\le by_n\Rightarrow ay_{n+1}+by_n\frac{x_k}{x_{k+1}}\le by_n+ay_{n+1}\Rightarrow$

$$\Rightarrow\frac{1}{ay_{n+1}+by_n\frac{x_k}{x_{k+1}}}\ge\frac{1}{by_n+ay_{n+1}}\quad (4)$$

Let $by_n=B$ ***and*** $ay_{n+1}=A$ ***we prove that:*** $\frac{1}{A+B}+1\ge\frac{2}{\sqrt{A+B}}\Leftrightarrow\frac{A+B+1}{A+B}\ge\frac{2}{\sqrt{A+B}}\Leftrightarrow$

$$\Leftrightarrow\frac{A+B+1}{\sqrt{A+B}}\ge2\Leftrightarrow\sqrt{A+B}+\frac{1}{\sqrt{A+B}}\ge2\ \textbf{(true)}\Rightarrow\frac{1}{ay_{n+1}+by_n}+1\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}\overset{(4)}{\Rightarrow}$$

$$\overset{(4)}{\Rightarrow}\frac{1}{ay_{n+1}+by_n\frac{x_k}{x_{k+1}}}+1\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}\Rightarrow\frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\ge\frac{2}{\sqrt{ay_{n+1}+by_n}}-1\quad (*)$$

Case II: $x_k>x_{k+1}\Rightarrow\frac{x_k}{x_{k+1}}>1\Rightarrow1<\frac{x_k}{x_{k+1}}\Rightarrow ay_{n+1}<ay_{n+1}\frac{x_k}{x_{k+1}}\Rightarrow ay_{n+1}+by\frac{x_k}{x_{k+1}}<$

$<ay_{n+1}\frac{x_k}{x_{k+1}}+by_n\frac{x_k}{x_{k+1}}\Rightarrow\frac{1}{ay_{n+1}+by_n\frac{x_k}{x_{k+1}}}>1\frac{1}{\frac{x_k}{x_{k+1}}(ay_{n+1}+by_n)}$ ***(5), let*** $y_0,y_1\ge4$ ***and***

$a,b\ge1$:

We prove by using the Mathematical induction that $y_n\ge4$:

1. we suppose that $P(n)$:" $y_n \geq 4$" is true

2. we prove that $P(n+1)$: "$y_{n+1} \geq 4$" is true by using the fact that $P(n)$ is true:

$$y_{n+1} = ay_n + by_{n-1}, a \geq 1 \Rightarrow ay_n \geq y_n \geq 4 \Rightarrow ay_n \geq 4, b \geq 1 \Rightarrow by_{n-1} \geq y_{n-1} \geq 4 \Rightarrow$$

$$\Rightarrow by_{n-1} \geq 4$$

So: $ay_n \geq 4$

$by_{n-1} \geq 4$

$------ \text{"+"}$

$$\overset{"+"}{\Rightarrow} ay_n + by_{n-1} \geq 4 \Rightarrow y_{n+1} \geq 4 \Rightarrow y_n \geq 4 \forall n \in \mathbb{N}$$

$y_n \geq 4$, *let* $\frac{x_{k+1}}{x_k} = \alpha \Rightarrow \alpha \leq 1$, *we prove that* $1 + \frac{\alpha}{ay_{n+1}+by_n} > \frac{2}{\sqrt{ay_{n+1}+by_n}}$ *(6)*

Let $A = ay_{n+1}$ *and* $B = by_n \Leftrightarrow 1 + \frac{\alpha}{A+B} > \frac{2}{\sqrt{A+B}} \Leftrightarrow \frac{A+B+\alpha}{A+B} > \frac{2}{\sqrt{A+B}} \Leftrightarrow$

$\Leftrightarrow \sqrt{A+B} + \frac{\alpha}{\sqrt{A+B}} > 2 | \cdot \sqrt{A+B} \Leftrightarrow \left(\sqrt{A+B}\right)^2 - 2\sqrt{A+B} + \alpha > 0$, *let* $\sqrt{A+B} = x \Leftrightarrow$

$\Leftrightarrow x^2 - 2x + \alpha > 0, \Delta = 4 - 4\alpha \Rightarrow \sqrt{\Delta} = 2\sqrt{2-\alpha} \Rightarrow$ *we prove that* $\sqrt{A+B} > 1 +$

$\sqrt{1-\alpha}$ *such that* $x^2 - 2x + \alpha > 0 \Leftrightarrow \sqrt{y_{n+2}} > 1 + \sqrt{1 - \frac{x_{k+1}}{x_k}}, y_n > 4 \Rightarrow \sqrt{y_{n+2}} > 2 >$

$> 1 + \sqrt{1 - \frac{x_{k+1}}{x_k}} \Rightarrow$ *true* $\Rightarrow$ *(6) is true* $\Rightarrow$ *(5) is true* $\Rightarrow \frac{x_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k} > \frac{2}{\sqrt{y_{n+2}}} - 1$ *(**)*

$\overset{(*);(**)}{\Rightarrow}$ *the given inequality is true* $\Rightarrow$

$$\Rightarrow \left(mn + \sum_{k=1}^n \frac{1}{x_k^{m+1}}\right)\left(mn + \sum_{k=1}^n \left(\frac{x_kx_{k+1}}{ay_{n+1}x_{k+1}+by_nx_k}\right)^{m+1}\right) \geq \frac{(m+1)^2n^2}{y_{n+2}} \quad \text{(Q.E.D.)}$$

Solution 2 by proposers

We have:

$$U_m = mn + \sum_{k=1}^n \frac{1}{x_k^{m+1}} = \sum_{k=1}^n \left(m + \frac{1}{x_k^{m+1}}\right) \overset{MA \geq MG}{\geq} \sum_{k=1}^n (m+1)\sqrt[m+1]{\underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{"m" \text{ times}} \cdot \frac{1}{x_k^{m+1}}} =$$

$$= (m+1)\sum_{k=1}^n \frac{1}{x_k} = (m+1)V_0 \quad (1)$$

Also, we have:

$$V_m = mn + \sum_{k=1}^n \left(\frac{x_kx_{k+1}}{ay_{n+1}x_{k+1} + by_kx_k}\right)^{m+1} = \sum_{k=1}^n \left(m + \left(\frac{x_ky_{k+1}}{ay_{n+1}x_{k+1} + by_kx_k}\right)^{m+1}\right) =$$

$$=\sum_{k=1}^{n}\left(m+\left(\frac{1}{\frac{ay_{k+1}}{x_k}+\frac{by_k}{x_{k+1}}}\right)^{m+1}\right)\overset{HA\geq HG}{\geq}$$

$$\geq\sum_{k=1}^{n}(m+1)\cdot\sqrt[m+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1\cdot 1}_{\text{"m" times}}\frac{1}{\left(\frac{ay_{n+1}}{x_k}+\frac{by_n}{x_{k+1}}\right)^{m+1}}}=$$

$$=(m+1)\sum_{k=1}^{n}\frac{1}{\frac{ay_{n+1}}{m}+\frac{by_n}{x_{k+1}}}\overset{Bergstrom}{\geq}(m+1)\frac{n^2}{\sum_{k=1}^{n}\left(\frac{ay_{k+1}}{x_k}+\frac{by_k}{x_{k+1}}\right)}=$$

$$=\frac{(m+1)n^2}{(ay_{n+1}+by_n)\sum_{k=1}^{n}\frac{1}{x_k}}=\frac{(m+1)n^2}{y_{n+2}\cdot V_0}\quad (2)$$

So, relationships (1) and (2) we deduce that:

$$W_m=U_m\cdot V_m\geq(m+1)V_0\cdot\frac{(m+1)n^2}{V_0\cdot y_{n+2}}=\frac{(m+1)^2n^2}{y_{n+2}},\forall n\in\mathbb{N}^*$$

UP.254. If $a_n,b_n>0; n\in\mathbb{N}; n\geq 1$

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a>0;\ \lim_{n\to\infty}\frac{b_n}{\sqrt[n]{a_n}}=b>0$$

then find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1}b_k}-\sqrt[n]{\prod_{k=1}^{n}b_k}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\Omega=\lim_{n\to\infty}\sqrt[n]{b_1b_2\ldots b_n}\left(\frac{\sqrt[n+1]{b_1b_2\ldots b_{n+1}}}{\sqrt[n]{b_1\ldots b_n}}-1\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{b_1b_2\ldots b_n}}{n}\cdot n\left(e^{\lim_{n\to\infty}\ln\left(\frac{\sqrt[n+1]{b_1\ldots b_{n+1}}}{\sqrt[n]{b_1\ldots b_n}-1}\right)}-1\right)\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_1b_2\dots b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_1b_2\dots b_n}{n^n}}\overset{C.D.}{=}$$

$$=\lim_{n\to\infty}\frac{b_1b_2\dots b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_1b_2\dots b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{n+1}\cdot\left(\frac{n}{n+1}\right)^n$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{b_n}{n}=\frac{1}{e}\lim_{n\to\infty}\frac{b_n}{\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n}=\frac{b}{e}\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=$$

$$\overset{C.D.}{=}\frac{b}{e}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\frac{b}{e}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)a_n}\cdot\frac{n^n}{(n+1)^n}$$

$$=\frac{b}{e}\lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\frac{n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{ab}{e^2}\quad\textbf{(2)}$$

$$=\lim_{n\to\infty}\frac{n\left(e^{\ln\frac{\sqrt[n+1]{b_1\dots b_{n+1}}}{\sqrt[n]{b_1\dots b_n}}}-1\right)}{\ln\left(\frac{\sqrt[n+1]{b_1\dots b_{n+1}}}{\sqrt[n]{b_1\dots b_n}}\right)}\cdot\ln\left(\frac{\sqrt[n+1]{b_1\dots b_{n+1}}}{\sqrt[n]{b_1\dots b_n}}\right)=$$

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(b_1\dots b_{n+1})^{n+1}}}{b_1\dots b_n}=\ln\left(\lim_{n\to\infty}\frac{b_1\dots b_{n+1}}{b_1\dots b_n}\cdot\frac{1}{\sqrt[n+1]{b_1\dots b_{n+1}}}\right)$$

$$=\ln\left(\lim_{n\to\infty}b_{n+1}\cdot\frac{1}{\sqrt[n+1]{b_1\dots b_{n+1}}}\right)=\ln\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{b_1\cdot\dots\cdot b_{n+1}}}\right)\right)$$

$$=\ln\left(\lim_{n\to\infty}\frac{b_n}{n}\cdot\frac{n}{\sqrt[n]{b_1\dots b_n}}\right)=\ln\left(\lim_{n\to\infty}\frac{b_n}{\sqrt[n]{a_n}}\cdot\frac{\sqrt[n]{a_n}}{n}\cdot\frac{n}{\sqrt[n]{b_1\dots b_n}}\right)\overset{(2)}{=}$$

$$=\ln\left(b\cdot\frac{a}{e}\cdot\frac{e^2}{ab}\right)=\ln e=1\quad\textbf{(3)}$$

From (1)+(2)+(3) $\Rightarrow\Omega=\frac{ab}{e^2}\cdot e=\frac{ab}{e^2}$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^n b_k}\cdot\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1}b_k}}{\sqrt[n]{\prod_{k=1}^n b_k}}-1\right)\quad\textbf{(1)}$$

$$\lim_{n\to\infty}\frac{b_n}{n}\cdot\frac{n}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\left(\frac{b_n}{n}\right)\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}=\lim_{n\to\infty}\left(\frac{b_n}{n}\right)\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{a_n}{a_{n+1}}=$$

$$= e \cdot \lim_{n\to\infty}\left(\frac{na_n}{a_{n+1}}\right) \cdot \lim_{n\to\infty}\left(\frac{b_n}{n}\right) =$$

$$= \frac{e}{a} \cdot \lim_{n\to\infty}\left(\frac{b_n}{n}\right) = b \Rightarrow \lim_{n\to\infty}\left(\frac{b_n}{n}\right) = \frac{ab}{e} \quad (a)$$

$$\overset{(1)}{\Rightarrow} \Omega = \lim_{n\to\infty} \frac{\sqrt[n]{\prod_{k=1}^{n} b_k}}{n} \cdot n \cdot \left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}} - 1\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[n]{\frac{\prod_{k=1}^{n} b_k}{n^n}}\right) \cdot n \cdot \left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}} - 1\right) =$$

$$= \lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n+1} b_k}{\prod_{k=1}^{n} b_k} \cdot \frac{n^n}{(n+1)^{n+1}}\right) \cdot \lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}} - 1\right) =$$

$$= \frac{1}{e} \cdot \lim_{n\to\infty} \frac{b_{n+1}}{n+1} \cdot \lim_{n\to\infty} n\left(\frac{\sqrt[n1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}} - 1\right) \overset{(a)}{=}$$

$$= \frac{ab}{e^2} \cdot \lim_{n\to\infty} n \cdot \frac{\left(e^{\ln\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}}\right)} - 1\right)}{\ln\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k}^{n} b_k}}\right)} \cdot \ln\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}}\right) \quad (2)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{\prod_{k=1}^{n} b_k}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{\prod_{k=1}^{n} b_k}{n^n}} = \lim_{n\to\infty} \frac{\prod_{k=1}^{n+1} b_k}{(n+1)^{n+1}} \cdot \frac{n^n}{\prod_{k=1}^{n} b_k} = \frac{1}{e} \cdot \lim_{n\to\infty}\left(\frac{b_{n+1}}{n+1}\right) =$$

$$\overset{(a)}{=} \frac{ab}{e^2} \Rightarrow \lim_{n\to\infty} \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{n} = \frac{ab}{e^2}$$

So, $\lim_{n\to\infty} \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{n} = \frac{ab}{e^2}$

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{\prod_{k=1}^{n} b_k}} = \frac{e^2}{ab}$$

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$$\overset{"."}{\Rightarrow} \lim_{n\to\infty} \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}} = 1 \overset{(2)}{\Rightarrow} \Omega = \frac{ab}{e^2} \cdot \lim_{n\to\infty} \ln\left(\frac{\prod_{k=1}^{n+1} b_k}{\prod_{k=1}^{n} b_k} \cdot \frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}\right) =$$

$$= \frac{ab}{e^2} \lim_{n\to\infty} \ln\left(\frac{b_n}{n} \cdot \frac{n}{\sqrt[n]{\prod_{k=1}^{n} b_k}}\right) =$$

$$= \frac{ab}{e^2} \ln\left(\frac{ab}{e} \lim_{n\to\infty} \frac{(n+1)^{n+1}}{\prod_{k=1}^{n+1} b_k} \cdot \frac{\prod_{k=1}^{n} b_k}{n^n}\right) = \frac{ab}{e^2} \ln\left(\frac{ab}{e} \cdot e \cdot \lim_{n\to\infty} \frac{n+1}{b_{n+1}}\right) =$$

$$= \frac{ab}{e^2} \cdot \ln\left(ab \cdot \frac{e}{ab}\right) = \frac{ab}{e^2} \Rightarrow \Omega = \frac{ab}{e^2}$$

Solution 3 by Soumitra Mandal-Chandar Nagore- India

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{CAUCHY\ D'ALEMBERT}{=} \lim_{n\to\infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n}\right)$$

$= \lim_{n\to\infty} \frac{a_{n+1}}{na_n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{n}{n+1} = \frac{a}{e}$. ***Now,*** $\lim_{n\to\infty} \frac{\sqrt[n]{\prod_{k=1}^{n} b_k}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{\prod_{k=1}^{n} b_k}{n^n}}$

$$\overset{CAUCHY\ D'ALEMBERT}{=} \lim_{n\to\infty} \frac{\prod_{k=1}^{n+1} b_k}{(n+1)^{n+1}} \cdot \frac{n^n}{\prod_{k=1}^{n} b_k} = \lim_{n\to\infty} \frac{b_{n+1}}{n+1} \cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} =$$

$= \frac{1}{e} \lim_{n\to\infty} \frac{b_n}{\sqrt[n]{a_n}} \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \frac{ab}{e^2}$. ***Let*** $u_n = \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{\sqrt[n]{\prod_{k=1}^{n} b_k}}$ ***for all*** $n \in \mathbb{N}$***. Then***

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}{n+1}}{\frac{\sqrt[n]{\prod_{k=1}^{n} b_k}}{n}} \cdot \frac{n+1}{n} = 1$$

Hence, $\frac{u_n - 1}{\ln u_n} \to 1$***, for all*** $n \to \infty$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n+1} b_k}{\prod_{k=1}^{n} b_k} \cdot \frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}}\right) = \lim_{n\to\infty} \left(\frac{b_{n+1}}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n+1}{\sqrt[n+1]{\prod_{k=1}^{n+1} b_k}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{n+1}\right)$$

$$= b \cdot \frac{e^2}{ab} \cdot \frac{a}{e} = e$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1} b_k} - \sqrt[n]{\prod_{k=1}^{n} b_k}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{\prod_{k=1}^{n+1} b_k}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n\right)$$

$$= \frac{ab}{e^2} \cdot 1 \cdot \ln e = \frac{ab}{e^2} \quad \textit{(Answer)}$$

UP.255. If $t > 0; a_n, b_n > 0; n \in \mathbb{N}; n \geq 1;$

$$\lim_{n\to\infty} \frac{a_{n+1}}{n a_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{n^t \cdot b_n} = b > 0 \textbf{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^{t+2}}{\sqrt[n+1]{a_{n+1} \cdot b_{n+1}}} - \frac{n^{t+2}}{\sqrt[n]{a_n \cdot b_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$\Omega = \lim_{n\to\infty} \frac{n^{t+2}}{\sqrt[n]{a_n b_n}}\left(\frac{(n+1)^{t+2}}{n^{t+2}} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} \cdot b_{n+1}}} - 1\right) =$$

$$= \lim_{n\to\infty} \frac{n^{t+1}}{\sqrt[n]{a_n b_n}} \cdot n\left[e^{\ln\left(\frac{(n+1)^{t+2}}{n^{t+2}} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} b_{n+1}}}\right)} - 1\right] \quad \textit{(1)}$$

$$\lim_{n\to\infty} \frac{n^{t+1}}{\sqrt[n]{a_n b_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{n(t+1)}}{a_n b_n}} \overset{C.D.}{=} \lim_{n\to\infty} \frac{(n+1)^{(n+1)(t+1)}}{a_{n+1} \cdot b_{n+1}} \cdot \frac{a_n \cdot b_n}{n^{n(t+1)}} =$$

$$= \lim_{n\to\infty}\left(\left[\left(\frac{n+1}{n}\right)^n\right]^{t+1} \cdot \frac{n a_n}{a_{n+1}} \cdot \frac{n^t \cdot b_n}{b_{n+1}} \cdot \frac{(n+1)^{t+1}}{n^{t+1}}\right) = \frac{e^{t+1}}{ab} \quad \textit{(2)}$$

$$\lim_{n\to\infty} \frac{\left(e^{\ln\left(\frac{(n+1)^{t+2}}{n^{t+2}} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} b_{n+1}}} - 1\right)}\right)}{\ln\left[\frac{(n+1)^{t+2}}{n^{t+2}} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} b_{n+1}}}\right]} \ln\left[\frac{(n+1)^{t+2}}{n^{t+2}} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} b_{n+1}}}\right]$$

$$= \lim_{n\to\infty} n \ln\left(\left(\frac{n+1}{n}\right)^{t+2} \cdot \frac{\sqrt[n]{a_n b_n}}{\sqrt[n+1]{a_{n+1} b_{n+1}}}\right) =$$

$$= \lim_{n\to\infty} \ln\left(\frac{a_n b_n}{\sqrt[n]{(a_{n+1}\cdot b_{n+1})^n}}\cdot\left(\frac{n+1}{n}\right)^{t+2}\right) \quad (3)$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{t+2} = 1 \quad (4)$$

$$\lim_{n\to\infty} \ln\left(\frac{a_n b_n}{\sqrt[n+1]{(a_{n+1}b_{n+1})^n}}\right) = \lim_{n\to\infty} \ln\left(\frac{a_n b_n}{a_{n+1}b_{n+1}}\cdot\sqrt[n+1]{a_{n+1}b_{n+1}}\right) =$$

$$\ln\left[\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{n^t b_n}{b_{n+1}}\cdot\frac{\sqrt[n]{a_n b_n}}{n^{t+1}}\right] \overset{(2)}{=} \ln\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{ab}{e^{t+1}}\right)$$

$$= \ln e^{-(t+1)} = -(t+1) \quad (5)$$

From (1)+(2)+(3)+(4)+(5) $\Rightarrow \Omega = \frac{e^{t+1}}{ab} - (t+1) = \frac{-(t+1)e^{t+1}}{ab}$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru