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PROBLEMS FOR JUNIORS

JP.241. Let m'_a, m'_b, m'_c be the circumpedal extensions of cevians of centroid in ΔABC. Prove that:

$$m'_a + m'_b + m'_c \geq 3\sqrt[3]{abc}$$

Proposed by Daniel Sitaru - Romania

JP.242. Let m'_a, m'_b, m'_c be the circumpedal extensions of cevians of centroid in ΔABC. Prove that:

$$m'_a m'_b m'_c \geq abc$$

Proposed by Daniel Sitaru - Romania

JP.243. If $x, y, z > 1; xyz = 8$ then:

$$\left(\frac{x}{2}\right)^x + \left(\frac{y}{2}\right)^y + \left(\frac{z}{2}\right)^z \geq 3$$

Proposed by Daniel Sitaru - Romania

JP.244. Prove that the following inequality holds for all real numbers x, y, z:

$$\max\{x, y, z\} \geq \min\{x, y, z\} + \sqrt{x^2 + y^2 + z^2 - xy - yz - zx}$$

When does the equality happen?

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.245. Let x, y, z be non-negative real numbers such that $x^2 + y^2 + z^2 = 1$. Find the minimum and maximum value of:

$$\sqrt{1+x} + \sqrt{1+y} + \sqrt{1+z}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.246. If $a, b, c, d > 0; a^2cd + b^2da + c^2ab + d^2bc = 4abcd$ then:

$$\frac{a^2}{b^2}\left(\frac{a}{b} - 1\right) + \frac{b^2}{c^2}\left(\frac{b}{c} - 1\right) + \frac{c^2}{d^2}\left(\frac{c}{d} - 1\right) + \frac{d^2}{a^2}\left(\frac{d}{a} - 1\right) = 0$$

Proposed by Daniel Sitaru - Romania

JP.247. Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that:

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 + a^2 + b^2 + c^2 \geq 10(ab + bc + ca)$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.248. Prove that the inequality holds for all positive real numbers a, b, c:

$$a^2+b^2+c^2+4abc\left(\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\right) \geq 2(ab+bc+ca)$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.249. Prove that the inequality holds for all positive real numbers a, b, c:

$$\sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} + \sqrt{\frac{a+b}{c}} \geq \sqrt{\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} + 12}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.250. Let a, b, c be positive real numbers such that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{4}{(b+c)^2} + \frac{4}{(c+a)^2} + \frac{4}{(a+b)^2} \leq 6$$

Prove that:

$$ab + bc + ca \geq 3$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.251. If $a, b, c, p > 0; a < b < c$ then solve for real numbers:

$$a^x(b+p)^x+b^x(c+p)^x+c^x(a+p)^x = a^x(c+p)^x+b^x(a+p)^x+c^x(p+b)^x$$

Proposed by Florentin Vişescu - Romania

JP.252. Solve for real numbers:

$$\left(\log\left(\frac{x}{x^2+1}\right) + x\right)^3 = \left(\log\left(\frac{x}{x^2+1}\right) - x\right)^3 + (x - \log(x^3+x))^3 + (x + \log(x^3+x))^3$$

Proposed by Daniel Sitaru - Romania

JP.253. If $n \in \mathbb{N}; n \geq 2; a_k \in \left[\frac{\pi}{12}, \frac{\pi}{6}\right]; k \in \overline{1, n}$ then:

$$\left(\sum_{k=1}^{n} \sin^2 a_k\right)\left(\sum_{k=1}^{n} \cot^2 a_k\right) < n^2$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

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JP.254. If $n \in \mathbb{N}; n \geq 2; a_k \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right); k \in \overline{1,n}$ then:

$$\left(\sum_{k=1}^{n} \sin a_k\right)\left(\sum_{k=1}^{n} \cot a_k\right) < \frac{49n^2}{24}$$

Proposed by D.M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

JP.255. If $m \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$3m + \left(a^2 \cot \frac{A}{2}\right)^{m+1} + \left(b^2 \cot \frac{B}{2}\right)^{m+1} + \left(c^2 \cot \frac{C}{2}\right)^{m+1} \geq 36(m+1)\sqrt{3}r^2$$

Proposed by D.M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

PROBLEMS FOR SENIORS

SP.241. Let w'_a, w'_b, w'_c be the circumpedal extensions of cevian of incentre in ΔABC. Prove that:

$$w'_a w'_b w'_c \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru - Romania

SP.242. Let w'_a, w'_b, w'_c be the circumpedal extensions of cevians of incentre in ΔABC. Prove that:

$$w'_a + w'_b + w'_c \geq \sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

Proposed by Daniel Sitaru - Romania

SP.243. Let P be a polynomial such that $P^4(x) + 16 = 28P^2(x^2 - 4)$, for all x real numbers. Prove that P is constant.

Proposed by Pedro H. O. Pantoja - Natal/RN - Brazil

SP.244. If $0 < y < x < 2y$ then:

$$x(x+y)\sqrt{4y^2 - x^2} < 3y^3\sqrt{3}$$

Proposed by Daniel Sitaru - Romania

SP.245. If $0 < y < x < 2y$ then:

$$x(x+y) > 3(x-y)\sqrt{3(4y^2 - x^2)}$$

Proposed by Daniel Sitaru - Romania

SP.246. If $ABCD$ bicentric quadrilateral; I − incenter then:

$$(IA^2+IC^2)(IB^2+ID^2)\geq AB\cdot BC\cdot CD\cdot DA$$

Proposed by Daniel Sitaru - Romania

SP.247. In $\Delta ABC; I$ - incenter; O_A, O_B, O_C - circumcenters of $\Delta BIC, \Delta CIA; \Delta AIB$. Prove that:

$$\frac{5}{2}+\frac{r}{R}\leq\frac{BC}{O_BO_C}+\frac{AC}{O_AO_C}+\frac{AB}{O_AO_B}\leq\sqrt{8+\frac{2r}{R}}$$

Proposed by Marian Ursărescu - Romania

SP.248. If $A\in M_5(\mathbb{R}); A\cdot A^T=I_5; \mathrm{Tr}\,A=\mathrm{Tr}\,A^2=0$ then find: $\Omega=A^{2020}$

Proposed by Marian Ursărescu - Romania

SP.249. Let $\Delta A'B'C'$ be the circumcevian triangle of centroid in ΔABC. Prove that:

$$\frac{S[A'B'C']}{S[ABC]}\leq\left(\frac{R}{2r}\right)^6$$

Proposed by Marian Ursărescu - Romania

SP.250. Let be $z_1, z_2, z_3\in\mathbb{C}\setminus\{0\}$ different in pairs;

$$|z_1|=|z_2|=|z_3|=1; A(z_1); B(z_2); C(z_3).$$

If $|z_1-z_2-z_3|+|z_2-z_1-z_3|+|z_3-z_2-z_1|=6$ then $AB=BC=CA$.

Proposed by Marian Ursărescu - Romania

SP.251. In ΔABC the following relationship holds:

$$m_a^6+m_b^6+m_c^6\geq 9\sqrt{3}F^3; (F\text{ - area})$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

SP.252. If $m,n\geq 1$ then in ΔABC the following relationship holds:

$$\frac{\sqrt{a(mb+nc-a)}}{mb+nc}+\frac{\sqrt{b(mc+na-b)}}{mc+na}+\frac{\sqrt{c(ma+nb-c)}}{ma+nb}\leq\frac{3}{2}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

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SP.253. If $n \in \mathbb{N}; n \geq 2; a_k \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right); k \in \overline{1,n}$ then:

$$\left(\sum_{k=1}^{n} k \sin a_k\right)\left(\sum_{k=1}^{n} k \cot a_k\right) < \frac{49n^2(n+1)^2}{24}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

SP.254. Let $A_1A_2...A_n; n \geq 3$ be a convexe polygon with area $F = [A_1A_2...A_n]$ with sides $a_k = A_kA_{k+1}; k \in \overline{1,n}; a_{n+1} = a_1$; $x_k, y_k \in \left(0, \frac{\pi}{2}\right); k \in \overline{1,n}$ then:

$$\sum_{k=1}^{n} \frac{a_k^4}{\sin x_k + \sin y_k} > \frac{16F^2 \tan^2(\frac{\pi}{n})}{\sum_{k=1}^{n}(x_k + y_k)}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

SP.255. If $n \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$3^n\left(\left(a^2 \cot \frac{A}{2}\right)^{n+1} + \left(b^2 \cot \frac{B}{2}\right)^{n+1} + \left(c^2 \cot \frac{C}{2}\right)^{n+1}\right) \geq$$

$$\geq 4^{n+1} \cdot 3^{\frac{5(n+1)}{2}} \cdot r^{2n+2}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.241. Let be $n \in \mathbb{N}; n \geq 3; m \geq 0; A_1A_2\ldots A_n$; a convexe polygon inscribed in a circle with radius R. If $a_k = A_kA_{k+1}$; $k \in \overline{1,n}; A_{n+1} = A_1; s$ - semiperimeter then:

$$\sum_{k=1}^{n} \frac{1}{a_k^m} \geq \frac{n}{2^m R^m \sin^m \frac{\pi}{n}}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.242. If $a, b \in \mathbb{R}$; $A = \begin{pmatrix} \sin^2 a & \cos^2 a \sin^2 b & \cos^2 a \cos^2 b \\ \cos^2 b \sin^2 c & \sin^2 b & \cos^2 b \cos^2 c \\ \cos^2 c \sin^2 a & \cos^2 c \cos^2 a & \sin^2 c \end{pmatrix}$

$$A^n = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}; n \geq \mathbb{N}; n \geq 2; x_i \in \mathbb{R}, i \in \overline{1,9}$$

then find $\Omega = \sum_{i=1}^{9} x_i$

Proposed by Daniel Sitaru - Romania

UP.243. Find:

$$\Omega = \lim_{n\to\infty}\left(H_n + \log_2\frac{1}{3} + \log_3\frac{1}{5} + \ldots + \log_n\frac{1}{2n-1}\right)$$

Proposed by Daniel Sitaru - Romania

UP.244. If $A \in M_4(\mathbb{Q}); \det((1-i)A + \sqrt{2}I_4) = 0$ then:

$$\det(A + xI_4) \geq 2x^2; x \in \mathbb{R}$$

Proposed by Marian Ursărescu - Romania

UP.245. If $x_0 = 1; x_{n+1} = n + \frac{1}{x_n}; n \in \mathbb{N}$ then find:

$$\Omega = \lim_{n\to\infty}(n(x_n - n))$$

Proposed by Marian Ursărescu - Romania

UP.246. Prove that:

$$\Psi\left(\frac{3+\sqrt{3}}{2}\right) - \Psi\left(\frac{3-\sqrt{3}}{2}\right) = 2\sqrt{3} + \pi\tan\left(\frac{\sqrt{3}}{2}\pi\right)$$

where $\Psi(x)$ is the digamma function.

Proposed by Vasile Mircea Popa - Romania

UP.247. If $0 < a \leq b$ then:

$$\int_a^b e^{-x^2}dx \geq \frac{\sqrt{a}(\sqrt{b}-\sqrt{a})}{e^{ab}} + \frac{(\sqrt{b}-\sqrt{a})^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{(b-a)^2}{2e^{b^2}}$$

Proposed by Daniel Sitaru - Romania

UP.248. If $a_1, a_2, \ldots, a_n \geq 1, a_1a_2\ldots a_n = 2^n, n \geq 1$ then:

$$a_1 + a_2 + \ldots + a_n - \frac{2}{a_1} - \frac{2}{a_2} - \ldots - \frac{2}{a_n} \geq n$$

Proposed by Marin Chirciu - Romania

UP.249. If $\alpha > 0, r \in (0,\alpha), z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $|z_k - \alpha| < r, k = \overline{1,n}$, then:

$$|z_{x_k1} + z_2 + \ldots + z_n|\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right| > \frac{n^2(\alpha^2 - r^2)}{\alpha^2}$$

Proposed by Marian Voinea - Romania

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UP.250. If $a, b, c > 0$ then:

$$\sum_{cyc}\left(1+\frac{1}{a}\right)^b > e^{\frac{b}{2a+1}+\frac{c}{2b+1}} + e^{\frac{c}{2b+1}+\frac{a}{2c+1}} + e^{\frac{a}{2c+1}+\frac{b}{2a+1}}$$

Proposed by Daniel Sitaru - Romania

UP.251. If $t, u, x, y > 0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2}{(t\cos^2\frac{A}{2}+u\cos^2\frac{B}{2})(xb+yc)^2} \geq \frac{18R}{(t+u)(x+y)^2(4R+r)}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.252. If $m, n, t > 0$ then in ΔABC the following relationship holds:

$$m^3\left(\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^4 + n^3\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^4 + t^3\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^4 \geq$$

$$\geq \frac{mntr^2}{s^2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.253. If $m \in \mathbb{N}; m \geq 1; a, b, x_k > 0; (y_n)_{n\geq 0}; y_0, y_1 > 0;$ $y_{n+2} = ay_{n+1} + by_n, (\forall)n \in \mathbb{N}$ then:

$$\left(mn+\sum_{k=1}^{n}\frac{1}{x_k^{m+1}}\right)\left(mn+\sum_{k=1}^{n}\left(\frac{x_k x_{k+1}}{ay_{n+1}x_{k+1}+by_n x_k}\right)^{m+1}\right) \geq$$

$$\geq \frac{(m+1)^2 n^2}{y_{n+2}}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.254. If $a_n, b_n > 0; n \in \mathbb{N}; n \geq 1$

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a > 0; \lim_{n\to\infty}\frac{b_n}{\sqrt[n]{a_n}} = b > 0$$

then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1} b_k} - \sqrt[n]{\prod_{k=1}^{n} b_k}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.255. If $t > 0; a_n, b_n > 0; n \in \mathbb{N}; n \geq 1;$

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{n^t \cdot b_n} = b > 0$$

then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^{t+2}}{\sqrt[n+1]{a_{n+1} \cdot b_{n+1}}} - \frac{n^{t+2}}{\sqrt[n]{a_n \cdot b_n}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

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