SOME LIMITS OF DEFINITE INTEGRALS OF LALESCU'S TYPE

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Abstract. In this paper we present some limits of definite integrals.

Theorem 1.

If $x_n = \sum_{k=1}^{n} \frac{1}{k}$, and $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continue function, then:

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right)dx = e^{\gamma} f(\gamma)$$

Proof.

We have: $e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(e^{\frac{1}{n+1}} - 1\right) = e^{x_n - \ln n} \cdot n\left(e^{\frac{1}{e^{n+1}}} - 1\right) =$

$$= e^{\gamma_n} \cdot \frac{e^{\frac{1}{e^{n+1}}} - 1}{\frac{1}{n+1}} \cdot \frac{n}{n+1}, \text{ so, } \lim_{n\to\infty}\left(e^{x_{n+1}} - e^{x_n}\right) = e^{\gamma} \cdot 1 \cdot 1 = e^{\gamma} \tag{1}$$

where we have used the facts that $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{n}$ and $\lim_{n\to\infty} \gamma_n = \gamma$

By the mean value theorem for definite integrals we have that

there exist $\zeta_n \in \lfloor e^{x_n}, e^{x_{n+1}} \rfloor$ such that:

$$\int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right)dx = \left(e^{x_{n+1}} - e^{x_n}\right) f\left(\frac{\zeta_n}{n}\right) \tag{2}$$

Since $e^{x_n} \le \zeta_n \le e^{x_{n+1}} \Leftrightarrow \frac{e^{x_n}}{n} \le \frac{\zeta_n}{n} \le \frac{e^{x_{n+1}}}{n+1} \cdot \frac{n+1}{n}$, yields

$$\lim_{n\to\infty} \frac{\zeta_n}{n} = \lim_{n\to\infty} \frac{e^{x_n}}{n} = \lim_{n\to\infty} e^{x_n - \ln n} = \lim_{n\to\infty} e^{\gamma_n} = e^{\gamma} \tag{3}$$

Then by (1), (2) and (3), we obtain:

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right)dx = e^{\gamma} f\left(\lim_{n\to\infty} \frac{\zeta_n}{n}\right) = e^{\gamma} f(e^{\gamma}).$$

□

1991 *Mathematics Subject Classification.* 97D50,97I30.
Key words and phrases. integrals, limits.

Theorem 2.

If $a \in (0,1) \cup (1,\infty), (E_n)_{n\geq 0}, E_n = \sum_{k=0}^{n} \frac{1}{k!}, (x_n)_{n\geq 1} > 0$, such that $\lim_{n\to\infty} \frac{x_n}{n!} = b > 0$ and $f : (0,\infty) \to (0,\infty)$ is a continue function, then:

$$\lim_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)dx = ba^e f(a^e).$$

Proof. By the mean value theorem for definite integrals we have that there exists $\zeta_n \in \lfloor e^{E_n}, e^{E_{n+1}} \rfloor, \forall n \in \mathbb{N}^*$ such that:

$$\int_{a^{E_n}}^{a^{E_{n+1}}} f(x)dx = (a^{E_{n+1}} - a^{E_n}) f(\zeta_n) = a^{E_n}(a^{E_{n+1}-E_n} - 1) f(\zeta_n) =$$

$$= a^{E_n}\left(a^{\frac{1}{(n+1)!}} - 1\right) f(\zeta_n). \text{ So,}$$

$$(1) \qquad x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)dx = \frac{x_{n+1}}{(n+1)!} \cdot a^{E_n} \cdot \frac{a^{\frac{1}{(n+1)!}} - 1}{\frac{1}{(n+1)!}} \cdot f(\zeta_n)$$

Since, $a^{E_n} \leq \zeta_n \leq a^{E_{n+1}}$ and $\lim_{n\to\infty} E_n = e$ yields

$$(2) \qquad \lim_{n\to\infty} a^{E_n} \leq \lim_{n\to\infty} \zeta_n \leq \lim_{n\to\infty} a^{E_{n+1}} \Leftrightarrow a^e \leq \lim_{n\to\infty} \zeta_n \leq a^e \Rightarrow \lim_{n\to\infty} \zeta_n = a^e$$

By (1) and (2) and hypothesis we obtain:

$$\lim_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)dx = b \cdot a^e \cdot 1 \cdot f\left(\lim_{n\to\infty} a^e\right) = ba^e f(a^e).$$

□

Theorem 3.

Let $(s_n)_{n\geq 1}, s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $f : (0,\infty) \to (0,\infty)$ a continue function, then:

$$\lim_{n\to\infty} \sqrt[n]{n!} \int_{s_n}^{\frac{\pi^2}{6}} f(x)dx = \frac{f(\frac{\pi^2}{6})}{e}.$$

Proof.

It is well-known that $\lim_{n\to\infty} s_n = \frac{\pi^2}{6}$.

By the mean value theorem for definite integrals we have that there exists:

$$(1) \qquad \zeta_n \in \left(s_n, \frac{\pi^2}{6}\right) \text{ such that } \int_{s_n}^{\frac{\pi^2}{6}} f(x)dx = \left(\frac{\pi^2}{6} - s_n\right) f(\zeta_n), \forall n \in \mathbb{N}^*$$

$$\text{So, } \sqrt[n]{n!} \int_{s_n}^{\frac{\pi^2}{6}} f(x)dx = \frac{\sqrt[n]{n!}}{n} \cdot f(\zeta_n) \cdot n\left(\frac{\pi^2}{6} - s_n\right) = \frac{\sqrt[n]{n!}}{n} \cdot f(\zeta_n) \cdot \frac{\frac{\pi^2}{6} - s_n}{\frac{1}{n}}.$$

Since, $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, we deduce that:

$$\lim_{n\to\infty} \sqrt[n]{n!} \int_{s_n}^{\frac{\pi^2}{6}} f(x)dx = \frac{1}{e} \cdot f\left(\lim_{n\to\infty} \zeta_n\right) \cdot \lim_{n\to\infty} \frac{\frac{\pi^2}{6} - s_n}{\frac{1}{n}} \overset{\text{Stolz}}{\underset{(\frac{0}{0})}{=}}$$

$$= \frac{1}{e} f\Big(\frac{\pi^2}{6}\Big) \lim_{n\to\infty} \frac{-s_{n+1}+s_n}{\frac{1}{n+1}-\frac{1}{n}} = \frac{f(\frac{\pi^2}{6})}{e} \lim_{n\to\infty} \frac{s_{n+1}-s_n}{\frac{1}{n(n+1)}} =$$

$$= \frac{f(\frac{\pi^2}{6})}{e} \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n(n+1)}} = \frac{f(\frac{\pi^2}{6})}{e} \lim_{n\to\infty} \frac{n}{n+1} = \frac{f(\frac{\pi^2}{6})}{e}.$$

□

Theorem 4.
Let $f:(0,\infty)\to(0,\infty)$ be a continuous function, then:

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\Big(\frac{x}{n}\Big)dx = \frac{1}{e} f\Big(\frac{1}{e}\Big)$$

Proof. There are well-known the following two facts:
(1)
$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, respectively $\lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) = \frac{1}{e}$ (Traian Lalescu's limit)
By the mean value theorem for definite integrals we have that there exist $\zeta_n \in (\sqrt[n]{n!}, \sqrt[n+1]{(n+1)!}$ such that:

$$\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\Big(\frac{x}{n}\Big)dx = (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) \cdot f\Big(\frac{\zeta_n}{n}\Big) \tag{2}$$

Since $\sqrt[n]{n!} \le \zeta_n \le \sqrt[n+1]{(n+1)!} \Leftrightarrow \frac{\sqrt[n]{n!}}{n} \le \frac{\zeta_n}{n} \le \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n}$, where taking to limit with $n\to\infty$, by (1) we have:

$$\frac{1}{e} \le \lim_{n\to\infty} \frac{\zeta_n}{n} \le \frac{1}{e} \Rightarrow \lim_{n\to\infty} \frac{\zeta_n}{n} = \frac{1}{e} \tag{3}$$

By (1); (2); (3) we obtain that:

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\Big(\frac{x}{n}\Big)dx = \lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) \cdot \lim_{n\to\infty} f\Big(\frac{\zeta_n}{n}\Big) =$$

$$= \frac{1}{e} \cdot f\Big(\lim_{n\to\infty} \frac{\zeta_n}{n}\Big) = \frac{1}{e} f\Big(\frac{1}{e}\Big)$$

□

Theorem 5.

Let $(\zeta_n)_{n\ge1}, \gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$ and $f:(0,\infty)\to(0,\infty)$ be a

continue function, then $\lim_{n\to\infty} \sqrt[n]{n!} \int_{\gamma}^{\gamma_n} f(x)dx = \frac{f(\gamma)}{2e}$.

Proof.

$$\text{It is well-known that } \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \tag{1}$$

By the mean value theorem for definite integrals we have that there exist $\zeta_n \in (\gamma, \gamma_n)$ such that:

$$\int_{\gamma}^{\gamma_n} f(x)dx = (\gamma_n - \gamma) f(\zeta_n), \forall n \in \mathbb{N}^*, \text{ so}$$

(2) $$\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx = \sqrt[n]{n!}(\gamma_n-\gamma)f(\zeta_n) = \frac{\sqrt[n]{n!}}{n}\cdot n(\gamma_n-\gamma)f(\zeta_n)$$

Then by (1) and (2):

$$\lim_{n\to\infty}\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx = \frac{1}{e}f\Big(\lim_{n\to\infty}\zeta_n\Big)\lim_{n\to\infty} n(\zeta_n-\gamma) =$$

$$= \frac{1}{e}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}} \overset{\text{Cesaro-Stolz}}{\underset{(\frac{0}{0})}{=}} \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}} =$$

$$= \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}} = \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}} \overset{(\frac{1}{n}=x)}{=}$$

$$= \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2} = \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{(x+1)\ln(1+x)-x}{x^2(1+x)} =$$

$$= \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{(x+1)\ln(1+x)-x}{x^2} = \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)+1-1}{2x} =$$

$$= \frac{f(\gamma)}{2e}\lim_{\substack{x\to 0\\ x>0}}\ln(1+x)^{\frac{1}{x}} = \frac{f(\gamma)}{2e}\ln e = \frac{f(\gamma)}{2e}.$$

□

Theorem 6.

Let $(\zeta_n)_{n\geq 1}, \gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$ with $\lim_{n\to\infty}\gamma_n = \gamma$ and $f:(0,\infty)\to(0,\infty)$be a continue function, then:

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\int_{\gamma}^{\gamma_n} f(x)dx = \frac{f(\gamma)}{e}.$$

Proof. We have that:

(1)

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} = \lim_{n\to\infty}\frac{2n+1}{n+1}\Big(\frac{n}{n+1}\Big)^n = \frac{2}{e}$$

By the mean value theorem for definite integrals we have that there exist $\zeta_n\in(\gamma,\gamma_n)$ such that:

$$\zeta_{\gamma}^{\gamma_n} f(x)dx = (\gamma_n-\gamma)f(\zeta_n), \forall n\in\mathbb{N}^*, \text{ so}$$

(2)

$$\sqrt[n]{(2n-1)!!}\int_{\gamma}^{\gamma_n} f(x)dx = \sqrt[n]{(2n-1)!!}(\gamma_n-\gamma)f(\zeta_n) = \frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n(\gamma_n-\gamma)f(\zeta_n)$$

Then by (1) and (2):

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\int_{\gamma}^{\gamma_n} f(x)dx = \frac{2}{e}f\Big(\lim_{n\to\infty}\zeta_n\Big)\lim_{n\to\infty} n(\gamma_n-\gamma) =$$

$$= \frac{2}{e}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}} \overset{\text{Cesaro-Stolz}}{\underset{(\frac{0}{0})}{=}} \frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}} = \frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}} =$$

$$= \frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}} \overset{(\frac{1}{n}=x)}{=} \frac{2f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2} =$$

$$= \frac{2f(\gamma)}{e} \lim_{\substack{x\to 0\\ x>0}} \frac{(x+1)\ln(1+x)-x}{x^2(1+x)} = \frac{2f(\gamma)}{e} \lim_{\substack{x\to 0\\ x>0}} \frac{(x+1)\ln(1+x)-x}{x^2} =$$

$$= \frac{2f(\gamma)}{e} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x)+1-1}{2x} = \frac{2f(\gamma)}{2e} \lim_{\substack{x\to 0\\ x>0}} \ln(1+x)^{\frac{1}{x}} = \frac{f(\gamma)}{e} \ln e = \frac{f(\gamma)}{e}.$$

□

Theorem 7.

Let $(\gamma_n)_{n\geq 1}, \gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$ and $f : (0,\infty) \to (0,\infty)$be a continue function, then: $\lim_{n\to\infty} \sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}} \int_{\gamma}^{\gamma_n} f(x)dx = \frac{f(\gamma)}{e^2}$.

Proof. We have that:

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{3!!}\cdot\sqrt[n]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}\cdot\sqrt[n+1]{(2n+1)!!}}{(n+1)^{n+1}} \cdot \frac{n^n}{\sqrt[n]{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}\sqrt[n]{\frac{(2n-1)!!}{n^n}} = \frac{1}{e}\lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} =$$

$$(1) \qquad = \frac{1}{e}\lim_{n\to\infty} \frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e^2}$$

By the mean value theorem for definite integrals we have that there exist $\zeta_n \in (\gamma, \gamma_n)$ such that:

$$\int_{\gamma}^{\gamma_n} f(x)dx = (\gamma_n - \gamma) f(\zeta_n), \forall n \in \mathbb{N}^*, \text{ so}$$

(2)

$$\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}} \int_{\gamma}^{\gamma_n} f(x)dx = \frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n} \cdot n(\gamma_n - \gamma) f(\zeta_n)$$

Then by (1) and (2):

$$\lim_{n\to\infty} \sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}} \int_{\gamma}^{\gamma_n} f(x)dx =$$

$$= \frac{2}{e^2} f(\gamma) \lim_{n\to\infty} \frac{\gamma_n - \gamma}{\frac{1}{n}} \underset{(\frac{0}{0})}{\overset{\text{Cesaro-Stolz}}{=}} \frac{2f(\gamma)}{e^2} \lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\frac{1}{n+1} - \frac{1}{n}} = \frac{2f(\gamma)}{e^2} \lim_{n\to\infty} \frac{-\gamma_{n+1} + \gamma_n}{\frac{1}{n(n+1)}} =$$

$$= \frac{2f(\gamma)}{e^2} \lim_{n\to\infty} \frac{\ln\frac{n+1}{n} - \frac{1}{n+1}}{\frac{1}{n^2}} \overset{(\frac{1}{n}=x)}{=} \frac{2f(\gamma)}{e^2} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x) - \frac{x}{1+x}}{x^2} =$$

$$= \frac{2f(\gamma)}{e^2} \lim_{\substack{x\to 0\\ x>0}} \frac{(x+1)\ln(1+x)-x}{x^2(1+x)} = \frac{2f(\gamma)}{e^2} \lim_{\substack{x\to 0\\ x>0}} \frac{(x+1)\ln(1+x)-x}{x^2} =$$

$$= \frac{2f(\gamma)}{e^2} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x)+1-1}{2x} = \frac{2f(\gamma)}{2e^2} \lim_{\substack{x\to 0\\ x>0}} \ln(1+x)^{\frac{1}{x}} = \frac{f(\gamma)}{e^2} \ln e = \frac{f(\gamma)}{e^2}.$$

□

Theorem 8.

Let Γ be the gamma function, then $\lim\limits_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)dx = e\Gamma(e)$.

Proof.

$$\text{We have: } \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

We denote $x_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, and we have $\lim\limits_{n\to\infty} x_n = \frac{1}{e}$, where by the mean value theorem for definite integrals we have that there exist $\zeta_n \in \left[\frac{1}{\sqrt[n+1]{(n+1)!}}, \frac{1}{\sqrt[n]{n!}}\right]$ such that:

$$\int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)dx = \left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\Gamma(n\zeta_n) = \frac{x_n}{\sqrt[n]{n!}\cdot \sqrt[n+1]{(n+1)!}}\Gamma(n\zeta_n)$$

So,

$$\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)dx = \lim_{n\to\infty} x_n \cdot \lim_{x\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n}{n+1}\right)\cdot \lim_{n\to\infty}\Gamma(n\zeta_n) =$$

$$= \frac{1}{e}\cdot e\cdot e\cdot 1\cdot\Gamma\left(\lim_{n\to\infty} n\zeta_n\right)$$

Since $\frac{1}{\sqrt[n+1]{(n+1)!}} \le \zeta_n \le \frac{1}{\sqrt[n]{n!}} \Rightarrow \frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n}{n+1} \le n\zeta_n \le \frac{n}{\sqrt[n]{n!}}$, we obtain that

$$\lim_{n\to\infty} n\zeta_n = e. \text{ Hence } \lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)dx = e\Gamma(e).$$

□

Theorem 9.

Let Γ be the gamma function, then:

$$\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(2n+1)!!}}}^{\frac{1}{\sqrt[n]{(2n-1)!!}}} \Gamma(n^2x^2)dx = \frac{e}{2}\Gamma\left(\frac{e^2}{4}\right)$$

Proof. We have that:

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}.$$

We denote $x_n = \sqrt[n]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}$. We have that:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right) = \lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\lim_{n\to\infty} n(u_n - 1) = \frac{2}{e}\lim_{n\to\infty} n(u_n-1) =$$

$$= \frac{2}{e}\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \text{ where } u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}, \forall n \ge 2.$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{n}\right) = \frac{2}{e}\cdot\frac{e}{2}\cdot 1 = 1, \text{ so } \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1.$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{2n+1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2\cdot\frac{e}{2} = e$$

$$\text{and then } \lim_{n\to\infty} x_n = \lim_{n\to\infty} (\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}) = \frac{2}{e}.$$

By the mean value theorem for definite integrals we have that there exist $\zeta_n \in \left[\frac{1}{\sqrt[n+1]{(2n+1)!!}}, \frac{1}{\sqrt[n]{(2n-1)!!}}\right]$ such that:

$$n^2 \int_{\frac{1}{\sqrt[n+1]{(2n+1)!!}}}^{\frac{1}{\sqrt[n]{(2n-1)!!}}} \Gamma(n^2x^2)dx = n^2\Big(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\Big)\Gamma(n^2\zeta_n^2) =$$

$$= x_n \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} \cdot \frac{n}{n+1} \cdot \Gamma(n^2\zeta_n^2)$$

So, $\displaystyle\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(2n+1)!!}}}^{\frac{1}{\sqrt[n]{(2n-1)!!}}} \Gamma(n^2x^2)dx = \frac{2}{e}\cdot\frac{e}{2}\cdot\frac{e}{2}\cdot 1\cdot\Gamma\Big(\lim_{n\to\infty}(n\zeta_n)^2\Big) = \frac{e}{2}\Gamma\Big(\lim_{n\to\infty}(n\zeta_n^2)\Big)$

Since, $\displaystyle\frac{1}{\sqrt[n+1]{(2n+1)!!}} \le \zeta_n \le \frac{1}{\sqrt[n]{(2n-1)!!}} \Rightarrow \frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{n}{n+1} \le n\zeta_n \le \frac{n}{\sqrt[n]{(2n-1)!!}}$

we obtain that: $\displaystyle\lim_{n\to\infty} n\zeta_n = \frac{e}{2}$. Hence $\displaystyle\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(2n+1)!!}}}^{\frac{1}{\sqrt[n]{(2n-1)!!}}} \Gamma(n^2x^2)dx = \frac{e}{2}\Gamma\Big(\frac{e^2}{4}\Big)$

□

Theorem 10.

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function and $\gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma =$ Euler - Mascheroni's constant. Then:

$$\lim_{n\to\infty} n\int_{\gamma}^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n - x) + f(x-\gamma)}dx = \frac{1}{4}$$

Proof.

(1) Let $a, b \in \mathbb{R}_+^*, a < b$, then $\displaystyle I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx = \frac{b-a}{2}$

Indeed, we make $t = u(x) = a+b-x, u'(x) = -1, u(a) = b, u(b) = a$ and we obtain:

$$I = \int_a^b \frac{f(a+b-t-a)}{f(b-a-b+t)+f(a+b-t-a)}(-1)dt = \int_a^b \frac{f(b-t)}{f(t-a)+f(b-t)}dt$$

Therefore, $\displaystyle 2I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx + \int_a^b \frac{f(b-x)}{f(x-a)+f(b-x)}dx = \int_a^b dx = b-a$, i.e. (1)

$$\text{Hence, } I_n = \int_{\gamma}^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx = \frac{y_n-\gamma}{2} \text{ and then:}$$

$$\lim_{n\to\infty} nI_n = \frac{1}{2}\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}} \overset{\text{Cesaro-Stolz}}{\underset{(\frac{0}{0})}{=}} \frac{1}{2}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty}\frac{\gamma_n-\gamma_{n+1}}{\frac{1}{n(n+1)}} =$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{-\frac{1}{n+1}+\ln\frac{n+1}{n}}{\frac{1}{n(n+1)}} = \frac{1}{2}\lim_{n\to\infty}\frac{-1+(n+1)\ln(1+\frac{1}{n})}{\frac{1}{n}} \overset{(x=\frac{1}{n})}{=}$$

$$= \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{-x+(1+x)\ln(1+x)}{x^2} \underset{\left(\frac{0}{0}\right)}{\overset{\text{L'H}}{=}} \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{-1+\ln(1+x)+(1+x)\cdot\frac{1}{1+x}}{2x} =$$

$$= \frac{1}{4}\lim_{\substack{x\to 0\\ x>0}} \ln(1+x)^{\frac{1}{x}} = \frac{1}{4}\ln e = \frac{1}{4}$$

□

Theorem 11.

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function and $(x_n)_{n\geq 1}, x_n = \sum_{k=1}^{n} \frac{1}{k}$. Then:

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})} dx = \frac{1}{2}e^{\gamma}$$

Proof.

(1) Let $a, b \in \mathbb{R}_+^*, a < b$ then $I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)} dx = \frac{b-a}{2}$

Indeed, we make $t = u(x) = a+b-x, u'(x) = -1, u(a) = b, u(b) = a$ and we obtain:

$$I = \int_a^b \frac{f(a+b-t-a)}{f(b-a-b+t)+f(a+b-t-a)}(-1)dt = \int_a^b \frac{f(b-t)}{f(t-a)+f(b-t)} dt.$$

Therefore, $2I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)} dx + \int_a^b \frac{f(b-x)}{f(x-a)+f(b-x)} dx = \int_a^b dx = b-a$, i.e. (1)

$$\text{Hence, } \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})} dx = \frac{e^{x_{n+1}}-e^{x_n}}{2} =$$

$$= \frac{1}{2}\cdot e^{x_n}\cdot(e^{x_{n+1}-x_n}-1) = \frac{1}{2}\cdot e^{x_n}\cdot(x_{n+1}-x_n)\cdot\frac{e^{x_{n+1}-x_n}-1}{x_{n+1}-x_n} =$$

$$= \frac{1}{2}\cdot e^{x_n}\cdot\frac{1}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}} = \frac{1}{2}\cdot\frac{e^{x_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}} = \frac{1}{2}\cdot e^{-\ln n + x_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}} =$$

$$= \frac{1}{2}\cdot e^{\gamma_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}, \text{ so}$$

$$\lim_{n\to\infty} I_n = \frac{1}{2}\cdot e^{\gamma}\cdot 1\cdot 1 = \frac{1}{2}e^{\gamma}$$

where $\gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$ and $\lim_{n\to\infty}\gamma_n = \gamma =$ Euler-Mascheroni's constant.

□

Theorem 12.

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function and $(a_n)_{n\geq 1}, a_1 = a_2 = 1$,

$$a_{n+1} = \sum_{k=1}^{n}\frac{a_k}{k}, \forall n \geq 2, (x_n)_{n\geq 1}, x_n = \sum_{k=1}^{n}\frac{1}{a_k}. \text{ Then:}$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})} dx = e^{2\gamma-1}$$

Proof.

(1) Let $a, b \in \mathbb{R}_+^*, a < b$ then $I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx = \frac{b-a}{2}$

Indeed, we make $t = u(x) = a+b-x, u'(x) = -1, u(a) = b, u(b) = a$ and we obtain

$$I = \int_a^b \frac{f(a+b-t-a)}{f(b-a-b+t)+f(a+b-t-a)}(-1)dt = \int_a^b \frac{f(b-t)}{f(t-a)+f(b-t)}dt$$

Therefore, $2I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx + \int_a^b \frac{f(b-x)}{f(x-a)+f(b-x)}dx = \int_a^b dx = b-a$, i.e. (1)

Hence, $I_n = \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x-e^{x_n})}{f(x^{x_{n+1}}-x)+f(x-e^{x_n})}dx = \frac{e^{x_{n+1}}-e^{x_n}}{2} =$

(2) $= \frac{1}{2} \cdot e^{x_n} \cdot (e^{x_{n+1}-x_n} - 1) = \frac{1}{2} \cdot e^{x_n} \cdot \left(e^{\frac{1}{a_{n+1}}} - 1\right) = \frac{1}{2} \cdot \frac{e^{x_n}}{a_{n+1}} \cdot \frac{e^{\frac{1}{e^{a_{n+1}}}} - 1}{\frac{1}{a_{n+1}}}$

By mathematical induction we have $a_n = \frac{n}{2}, \forall n \geq 2$ which yields $x_n = 2 \cdot \sum_{k=1}^{n} \frac{1}{k} - 1$, then by (2)

(3)
$$\frac{1}{n}I_n = \frac{1}{2} \cdot \frac{e^{x_n}}{n^2} \cdot \left(e^{\frac{1}{a_n}} - 1\right) \cdot n = \frac{1}{2} \cdot \frac{e^{x_n}}{n^2} \cdot \left(e^{\frac{2}{n+1}} - 1\right) \cdot n = \frac{1}{2} \cdot e^{x_n - 2\ln n} \cdot \left(e^{\frac{1}{n+1}} + 1\right)\left(e^{\frac{1}{n+1}} - 1\right) \cdot n =$$

$$= \frac{1}{2} \cdot \left(e^{\frac{1}{n+1}} + 1\right) \cdot e^{2\sum_{k=1}^{n} \frac{1}{k} - 2\ln n - 1} \cdot \frac{e^{\frac{1}{n+1}} - 1}{\frac{1}{n+1}} \cdot \frac{n}{n+1} = \frac{1}{2} \cdot \left(e^{\frac{1}{n+1}} + 1\right) \cdot e^{2\gamma_n - 1} \cdot \frac{e^{\frac{1}{n+1}} - 1}{\frac{1}{n+1}} \cdot \frac{n}{n+1}$$

Since, $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma =$ Euler-Mascheroni's constat and

$\lim_{n\to\infty} \frac{e^{\frac{1}{n+1}} - 1}{\frac{1}{n+1}} = 1$, we obtain $\lim_{n\to\infty} \frac{1}{n} I_n = \frac{1}{2} \cdot (1+1) \cdot e^{2\gamma-1} \cdot 1 \cdot 1 = e^{2\gamma-1}$.

□

Theorem 13.

Let $a \in [0,1], (a_n)_{n\geq 1}, a_n = \sum_{k=1}^{n} \frac{1}{k^a}$ and $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function, then:

$$\lim_{n\to\infty} \int_{n\sqrt[n]{a_n}}^{(n+1)\sqrt[n+1]{a_{n+1}}} \frac{f(x - n\sqrt[n]{a_n})}{f((n+1)\sqrt[n+1]{a_{n+1}} - x) + f(x - n\sqrt[n+1]{a_n})}dx = \frac{1}{2}$$

Proof.

(1) Let $a, b \in \mathbb{R}_+^*, a < b$, then $I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx = \frac{b-a}{2}$

Indeed, we make $t = u(x) = a+b-x, u'(x) = -1, u(a) = b, u(b) = a$ and we obtain:

$$I = \int_a^b \frac{f(a+b-t-a)}{f(b-a-b+t)+f(a+b-t-a)}(-1)dt = \int_a^b \frac{f(b-t)}{f(t-a)+f(b-t)}dt.$$

Therefore, $2I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx + \int_a^b \frac{f(b-x)}{f(x-a)+f(b-x)}dx = \int_a^b dx = b-a$, i.e, (1) Then, $I_n = \lim\limits_{n\to\infty} \int_{n\sqrt[n]{a_n}}^{(n+1)\sqrt[n+1]{a_{n+1}}} \frac{f(x - n\sqrt[n]{a_n})}{f((n+1)\sqrt[n+1]{a_{n+1}} - x) + f(x - n\sqrt[n]{a_n})} dx =$

$$= \frac{1}{2}((n+1)\sqrt[n+1]{a_{n+1}} - n\sqrt[n]{a_n}) = \frac{1}{2}\cdot n\sqrt[n]{a_n}\cdot(v_n - 1) = \frac{1}{2}\cdot n\sqrt[n]{a}\cdot\frac{v_n - 1}{\ln v_n}\cdot \ln v_n =$$

$$= \frac{1}{2}\cdot\sqrt[n]{a_n}\cdot\frac{v_n - 1}{\ln v_n}\cdot \ln v_n^n \tag{2}$$

Where $v_n = \frac{(n+1)\sqrt[n+1]{a_{n+1}}}{n\sqrt[n]{a_n}}$

Since, $1 < \sqrt[n]{a_n} \leq \sqrt[n]{n}$, yields $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = 1$, so $\lim\limits_{n\to\infty} v_n = 1$ and consequently

$\lim\limits_{n\to\infty}\frac{v_n - 1}{\ln v_n} = 1$. Also, we have $\lim\limits_{n\to\infty} v_n^n = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n \frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} =$

$$= e\cdot\frac{1}{1}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{a_n} \overset{\text{Cesaro-Stolz}}{=} e\cdot\lim_{n\to\infty}\frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} = e\lim_{n\to\infty}\frac{\frac{1}{(n+2)^2}}{\frac{1}{(n+1)^a}} =$$

$= e\lim\limits_{n\to\infty}\left(\frac{n+1}{n+2}\right)^a = e\cdot 1 = e$. Hence by (2) we obtain $\lim\limits_{n\to\infty} I_n = \frac{1}{2}\cdot 1\cdot 1\cdot \ln e = \frac{1}{2}$.

□

Theorem 14.

Let $(x_n)_{n\geq 1}, x_n = \sum\limits_{k=1}^{n}\frac{1}{k}, \gamma_n = x_n - \ln n$. It is well-known that $\lim\limits_{n\to\infty}\gamma_n = \gamma$,

i.e. Euler-Mascheroni constant. Then, $\lim\limits_{n\to\infty}(e^{x_n} - ne^{\gamma}) = \frac{e^{\gamma}}{2}$

Proof.
An Euler-Mascheroni-Cesàro-Stolz-L'Hôspital collaboration.
We denote $a_n = e^{x_n} - ne^{\gamma} = n\cdot\frac{e^{x_n}}{n} - ne^{\gamma} = ne^{x_n - \ln n} - ne^{\gamma} =$
$= n(e^{\gamma_n} - e^{\gamma}) = e^{\gamma}n(e^{\gamma_n - \gamma} - 1) = e^{\gamma}n\cdot\frac{e^{\gamma_n-\gamma}-1}{\gamma_n - \gamma}(\gamma_n - \gamma), \forall n\in\mathbb{N}^*$. Therefore

$$\lim_{n\to\infty} a_n = e^{\gamma}\cdot\lim_{n\to\infty}\frac{e^{\gamma_n-\gamma}-1}{\gamma_n-\gamma}\cdot\lim_{n\to\infty} n(\gamma_n - \gamma) = e^{\gamma}\cdot 1\cdot\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}} \underset{(\frac{0}{0})}{\overset{\text{Cesaro-Stolz}}{=}}$$

$$= e^{\gamma}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1} - \frac{1}{n}} = e^{\gamma}\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n}) - \frac{1}{n+1}}{\frac{1}{n(n+1)}} = e^{\gamma}\lim_{n\to\infty}\frac{(n+1)\ln(1+\frac{1}{n}) - 1}{\frac{1}{n}} =$$

$$= e^{\gamma}\lim_{n\to\infty}\frac{(1+\frac{1}{n})\ln(1+\frac{1}{n}) - \frac{1}{n}}{\frac{1}{n^2}} \overset{x=\frac{1}{n}}{=} e^{\gamma}\lim_{\substack{x\to 0\\ x>0}}\frac{(1+x)\ln(1+x) - x}{x^2} \overset{\text{L'Holspital}}{=}$$

$$= e^{\gamma}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)+1-1}{2x} = \frac{e^{\gamma}}{2}\lim_{\substack{x\to 0\\ x>0}}\ln(1+x)^{\frac{1}{x}} = \frac{e^{\gamma}}{2}\ln e = \frac{e^{\gamma}}{2}.$$

□

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