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UP.240. Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of:

$$T = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc}$$

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Solution by Michael Sterghiou-Greece

$$T = \left(\sum_{cyc}\frac{a}{b}\right) + \sum_{cyc}\frac{1}{a^3+b^3+abc} \quad \textit{(1)}$$

We will prove that $T \geq 4$. We use the following lemma: With the problem's conditions it holds that: $G\left(\sum_{cyc}\frac{a}{b}\right) + 3 \geq 7 \cdot \sum_{cyc} a^2$ (L). To prove this, we homogenize the inequality by multiplying by $3abc$ (and taking into account that $\sum_{cyc} a = 3$) as follows:

(L) $\rightarrow 2\left(\sum_{cyc} a\right)^2 \cdot \left(\sum ab^2\right) + abc \cdot \left(\sum_{cyc} a\right)^2 - 21abc\sum_{cyc} a^2 \geq 0$ or the equivalent $\sum_{cyc} a^4c + \sum_{cyc} a^3b^2 + 2\sum_{cyc} a^2b^3 + 4abc \cdot \sum_{cyc} ab - 8abc\sum_{cyc} a^2 \geq 0$. The last reduces to: $\sum_{cyc} a\,(a-b)^2(b-2c)^2 \geq 0$ which is true. Now, as $\sum_{cyc} a^2 \geq 3$ be adding this to (L) we get $\left(\sum_{cyc}\frac{a}{b}\right) \geq \sum_{cyc} a^2$ (2). Let $(p, q, r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$ with

$p = 3$ ***(1) becomes the stronger inequality*** $\left(\sum_{cyc} a^2\right) + \frac{9}{2\left(\sum_{cyc} a^3\right)+3abc} \geq 4$ ***[by (2) and BCS] or*** $9 - 2q + \frac{1}{6-2q+r} - 4 \geq 0$ ***[Note that*** $\sum_{cyc} a^3 = p^3 - 3pq + 3r$***] which reduces to*** $4q^2 - 22q + (5-2q)r + 31 \geq 0$ ***(3). This is either an increasing or decreasing function of*** r ***(depending on the sign of*** $5 - 2q$***). In either case it suffices to hold when*** $r = \max$ ***or*** $r = \min$ ***which according to V. Cîrtoaje theorem happens (for any fixed*** q***) when any two of*** a, b, c ***are equal. Let WLOG*** $b = c\left(< \frac{3}{2}\right) \Rightarrow a = 3 - 2b$***. Now (3) becomes*** $-(b-1)^2(12b^3 - 54b^2 + 70b - 31) \geq 0$ ***for*** $0 < b < \frac{3}{2}$***. This can be easily show to be true because*** $12b^3 - 54b^2 + 70b - 31 < 0$ ***for*** $b \in \left(0, \frac{3}{2}\right)$***. Done!***