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SP.233. If $A \in M_3(\mathbb{R}); Tr(A^2) = 0; \det = 1$ then:

$$\det(A^2 + A + I_3) \geq (Tr\ A)^3$$

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Solution by Florentin Vișescu – Romania

$$\textit{If } Tr(A^2) = 0 \textit{ and } \det A = 1$$

$$P_A(x) = x^3 - Tr\ A\, x^2 + \frac{(Tr\ A)^2}{2} x - 1$$

$$\det(A^2 + A + I_3) = \det(A - \varepsilon I_3)(A - \bar{\varepsilon} I_3) =$$

$$= P_A(\varepsilon) \cdot P_A(\bar{\varepsilon}) = \left(\varepsilon^3 - Tr\ A\, \varepsilon^2 + \frac{(Tr\ A)^2}{2}\varepsilon - 1\right) \cdot \left(\bar{\varepsilon}^3 - Tr\ A \bar{\varepsilon}^2 + \frac{(Tr\ A)^2}{2}\bar{\varepsilon} - 1\right) =$$

$$= (Tr\ A)^2 \varepsilon \cdot \bar{\varepsilon}\left(\varepsilon - \frac{Tr\ A}{2}\right)\left(\bar{\varepsilon} - \frac{Tr\ A}{2}\right) = (Tr\ A)^2 \cdot \left(\varepsilon\bar{\varepsilon} - \varepsilon\frac{Tr\ A}{2} - \bar{\varepsilon}\frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4}\right)$$

$$= (Tr\ A)^2 \cdot \left(1 - \frac{Tr\ A}{2}(\varepsilon + \bar{\varepsilon}) + \frac{(Tr\ A)^2}{4}\right) = (Tr\ A)^2 \cdot \left(1 + \frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4}\right) \geq (Tr\ A)^3$$

$$1 + \frac{Tr\ A}{2} + \frac{(Tr\ A)^2}{4} \geq Tr\ A$$

$$4 + 2\,Tr\ A + (Tr\ A)^2 \geq 4 Tr\ A$$

$$(Tr\ A)^2 - 2Tr\ A + 4 \geq 0$$

$$(Tr\ A - 1)^2 + 3 \geq 0$$