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JP.232. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{\frac{r_a}{r_b r_c}}+\sqrt{\frac{r_b}{r_c r_a}}+\sqrt{\frac{r_c}{r_a r_b}}\ge\sqrt{\frac{3}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mustafa Tarek-Cairo-Egypt, Solution 3 by Jalil Hajimir-Canada, Solution 4 by Bogdan Fustei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{r_a}{\sqrt{r_a r_b r_c}}=\frac{4R+r}{S\sqrt{r}}\overset{Trucht}{\ge}\frac{\sqrt{3}S}{S\sqrt{r}}=\sqrt{\frac{3}{r}}$$

Solution 2 by Mustafa Tarek-Cairo-Egypt

$$\sum_{cyc}\sqrt{\frac{r_a}{r_b r_c}}\ge\sqrt{\frac{3}{r}}$$

First, we will prove the following inequality:

$\sum_{cyc}\sqrt{\frac{x}{yz}}\ge\sqrt{3\sum_{cyc}\frac{1}{x}}$ ***where*** $x,y,z>0$ ***(1)***

$$(x-y)^2+(y-z)^2+(z-x)^2\ge 0$$

$$x^2+y^2+z^2\ge xy+yz+xz$$

$$(x+y+z)^2 \ge 3(xy+yz+xz)$$

$$\frac{x+y+z}{\sqrt{xyz}} \ge \frac{\sqrt{3(xy+yz+xz)}}{\sqrt{xyz}} \Rightarrow \sum_{cyc}\sqrt{\frac{x}{yz}} \ge \sqrt{3\sum_{cyc}\frac{1}{x}}$$

(1) it is true

In (1) let $x = r_a, y = r_b, z = r_c$ and using the identity

$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$ ***then:***

$$\sum_{cyc}\sqrt{\frac{r_a}{r_b r_c}} \ge \sqrt{\frac{3}{r}}$$

Solution 3 by Jalil Hajimir-Canada

$$\sqrt{\frac{r_a}{r_b r_c}} + \sqrt{\frac{r_b}{r_c r_a}} + \sqrt{\frac{r_c}{r_a r_b}} = \frac{r_a+r_b+r_c}{\sqrt{r_a r_b r_c}} \ge \sqrt{3}\sqrt{\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}} = \sqrt{\frac{3}{r}}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{r}$$

$$\frac{x+y+z}{\sqrt{3xyz}} \ge \sqrt{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}};\ x,y,z > 0$$

Solution 4 by Bogdan Fustei-Romania

$s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$ ***(Blundan Inequality)***

(RMM – Famous Inequalities Marathon 1 – 100)

$$\Rightarrow s^2 \cdot \frac{2(2R-r)}{R} \le (4R+r)^2 \le s^2 \cdot \frac{(4R-2r)}{R} \le (4R+r)^2$$

$$s^2\left(4-\frac{2r}{R}\right) \le (4R+r)^2 \Rightarrow s\sqrt{4-\frac{2r}{R}} \le 4R+r$$

$$\sum\sqrt{\frac{r_a}{r_b r_c}} = \sum\sqrt{\frac{r_a^2}{r_a r_b r_c}};\ r_a r_b r_c = Ss = s\cdot r\cdot s = s^2 r$$

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$$\sum\sqrt{\frac{r_a}{r_b r_c}}=\sum\frac{r_a}{s\sqrt{r}}=\frac{r_a+r_b+r_c}{s\sqrt{r}}\geq\frac{s\sqrt{4-\frac{2r}{R}}}{s\sqrt{r}}=\sqrt{\frac{4}{R}-\frac{2}{R}}$$

Now, we will prove that $\frac{4}{r}-\frac{2}{R}\geq\frac{3}{r}\Rightarrow\frac{4}{r}-\frac{3}{r}\geq\frac{2}{R}\Rightarrow\frac{1}{r}\geq\frac{2}{R}\Rightarrow R\geq 2r$ ***(Euler)***

So, finally we have the following: $\sum\sqrt{\frac{r_a}{r_b r_c}}\geq\sqrt{\frac{4}{r}-\frac{2}{R}}\geq\sqrt{\frac{3}{r}}$ ***Q.E.D.***