PROPOSED PROBLEM

JOŚE LUIS DIAZ - BARRERO
SCHOOL OF CIVIL ENGINEERING, BERCELONA TECH - UPC, BARCELONA, SPAIN

Show that the equation $x^6 - 5x^5 - 6x^4 + 2x^3 + 9x^2 - 17x + 1 = 0$ has no negative roots.

Solution 1 by Marian Ursărescu - Romania.

Let's assume that the equation has a negative root, let $\alpha < 0$ such that:

$$\alpha^6 - 5\alpha^5 - 6\alpha^4 + 2\alpha^3 + 9\alpha^2 - 17\alpha + 1 = 0 \Rightarrow$$
$$\Rightarrow \alpha^6 - 6\alpha^4 + 9\alpha^2 - 5\alpha^5 + 2\alpha^3 - 17\alpha + 1 = 0 \Leftrightarrow$$
$$\alpha^2(\alpha^4 - 6\alpha^2 + 9) - \alpha(5\alpha^4 - 2\alpha^2 + 17) + 1 = 0 \Leftrightarrow$$

$$\alpha^2(\alpha - 3)^2 - \alpha(5\alpha^4 - 2\alpha^2 + 17) + 1 = 0 \tag{1}$$

$$\text{But } \alpha^2(\alpha - 3)^2 \geq 0 \tag{2}$$

$$5\alpha^4 - 2\alpha^2 + 17 > 0, \text{ because if } \alpha^2 = x \Rightarrow 5x^2 - 2x + 17 > 0 \text{ because } \Delta < 0 \tag{3}$$

$$-\alpha > 0 \tag{4}$$

$$\text{From } (1) + (2) + (3) \Rightarrow \text{ the equation has no negative roots.}$$

□

Solution 2 by Michael Sterghiou - Greece.

$$x^6 - 5x^5 - 6x^4 + 2x^3 + 9x^2 - 17x + 1 = 0 \tag{1}$$

Let $x_0 < 0$ be a root of (1). Then:

$$\frac{x_0^6 + 2x_0^3 + 9x_0^2 + 1}{5x_0^4 + 6x_0^3 + 17} = x_0 < 0 \tag{2}$$

$$\text{The numerator of LHS of (2) is } = (x_0 + 1)^2(x_0^2 - x_0 + 1)^2 + 9x_0^2 > 0$$

$$\text{Let } f(x_0) = 5x_0^4 + 6x_0^3 + 17. f'(x_0) = 2x_0^2(10x_0 + 9) \text{ with roots to } t_0 \in \left\{ -\frac{9}{10}, 0 \right\}$$

$$\text{which are critical points of } f(x). \text{ But } f(0) = 17, f\left(-\frac{9}{10}\right) > 0 \text{ and}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = \infty \text{ therefore } f_{\min} \in \left\{ f(0), f\left(-\frac{9}{10}\right) \right\} > 0$$

Thus LHS of (2) > 0 while RHS of (2) < 0 by assumption which is a contradiction.

Therefore (1) does not have negative roots. Done!

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Solution 3 by Ravi Prakash - New Delhi - India.

$$\textit{Let } x = -t \textit{ where } t > 0$$

$$f = x^6 - 5x^5 - 6x^4 + 2x^3 + 9x^2 - 17x + 1 = t^6 + 5t^5 - 6t^4 - 2t^3 + 9t^2 + 17t + 1$$

$$\textit{If } 0 < t \leq 1, \textit{ then: } f = t^6 + 5t^5 + 6t^2(1 - t^2) + 3t^2(1 - t) + 1 > 0$$

$$\textit{If } t > 1, \textit{ then: } f = t^2(t^2 - 3)^2 + 3t^5 + 2t^3(t^2 - 1) + 17t + 1 > 0$$

$$\textit{Thus, } f > 0, \forall x < 0 \Rightarrow f = 0 \textit{ cannot have negative roots.}$$

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Mehedinti, Romania
Email address: dansitaru63@yahoo.com