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In ΔABC, n_a –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\frac{1}{4r}+\frac{1}{8r_ar_br_c}\sum_{cyc}(n_a^2+g_a^2)\geq\sum_{cyc}\frac{r_b+r_c}{(r_a+r_b)(r_a+r_c)}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{4r}+\frac{1}{8r_ar_br_c}\sum(n_a^2+g_a^2)\geq\sum\frac{r_b+r_c}{(r_a+r_b)(r_a+r_c)}$$

Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)$$

$$\Rightarrow s(b^2+c^2)-2sbc=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_2^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$=as^2-2ah_ar_a\Rightarrow n_a^2\overset{(1)}{=}s^2-2h_ar_a$$

Again, Stewart's theorem $\Rightarrow b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-(b^3+c^3)=ag_a^2+a(s^2-s(2s-a)+bc)$$

$$=ag_a^2+a(-s^2+as+bc)$$

$$\Rightarrow ag_a^2 = as^2 + \frac{(b^2+c^2)(\sum a) - 2(b^3+c^3) - a^2(\sum a) - 2abc}{2}$$

$$= as^2 + \frac{ab^2 + ac^2 + b^3 + bc^2 + b^2c + c^3 - 2(b^3+c^3) - a^3 - a^2b - a^2c - 2abc}{2}$$

$$= as^2 + \frac{a(b-c)^2 - (a^3+b^3+c^3) + b^2c + bc^2 - a^2b - a^2c}{2}$$

$$= as^2 + \frac{a(b-c)^2 - a^2(\sum a) - (b+c)(b^2 - bc + c^2) + bc(b+c)}{2}$$

$$= as^2 + \frac{a(b-c)^2 - 2sa^2 - (2s-a)(b-c)^2}{2}$$

$$= as^2 + \frac{2a(b-c)^2 - 2sa^2 - 2s(b^2+c^2-2bc)}{2}$$

$$= as^2 + a(b-c)^2 - s\sum a^2 + 2sbc$$

$$\Rightarrow g_a^2 \overset{(2)}{=} (b-c)^2 + s^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a}$$

$$\textbf{(1)+(2)} \Rightarrow n_a^2 + g_a^2 + 2r_br_c$$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - 2\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) + 2\frac{\Delta^2}{(s-b)(s-c)}$$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - \frac{4s(s-a)(s-b)(s-c)}{a(s-a)} +$$

$$+ \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{\sum a^2 + 4(s-b)(s-c) - 2bc}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{a^2 - (b-c)^2 + a^2 + (b^2+c^2-2bc)}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left(\frac{2a^2}{a}\right) = (b-c)^2 + 4s(s-a)$$

$$= (b-c)^2 + (b+c+a)(b+c-a)$$

$$= (b-c)^2 + (b+c)^2 - a^2 = 2b^2 + 2c^2 - a^2 = 4m_a^2$$

$$\therefore n_a^2 + g_a^2 + 2r_br_c = 4m_a^2 \Rightarrow n_a^2 + g_a^2 \overset{(a)}{=} 4m_a^2 - 2r_br_c$$

Similarly, $n_b^2+g_b^2\overset{(b)}{=}4m_b^2-2r_cr_a$ ***and*** $n_c^2+g_c^2\overset{(c)}{=}4m_c^2-2r_ar_b$

(a)+(b)+(c)$\Rightarrow \frac{1}{4r}+\frac{1}{8r_ar_br_c}\sum(n_a^2+g_a^2)$

$$=\frac{1}{4r}+\frac{1}{8r_ar_br_c}\sum(4m_a^2-2r_br_c)=\frac{1}{4r}+\frac{\frac{3}{4}\sum a^2}{2rs^2}-\frac{1}{4}\left(\sum\frac{1}{r_a}\right)$$

$$=\frac{1}{4r}+\frac{3(s^2-4Rr-r^2)}{4rs^2}-\frac{1}{4r}=\frac{3(s^2-4Rr-r^2)}{4rs^2}$$

$$\Rightarrow LHS\overset{(i)}{=}\frac{3(s^2-4Rr-r^2)}{4rs^2}$$

Now, $r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=s\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=\frac{s\cos^2\frac{A}{2}}{\prod\cos\frac{A}{2}}\overset{(3)}{=}4R\cos^2\frac{A}{2}$

Similarly, $r_c+r_a\overset{(5)}{=}4R\cos^2\frac{B}{2}$ ***and*** $r_a+r_b\overset{(6)}{=}4R\cos^2\frac{C}{2}$

(4),(5),(6)$\Rightarrow \sum\frac{r_b+r_c}{(r_a+r_b)(r_a+r_c)}=\sum\frac{(r_b+r_c)^2}{\prod(r_b+r_c)}=\frac{\sum(r_b^2+r_c^2+2r_br_c)}{64R^3\left(\frac{s}{4R}\right)^2}$

$$=\frac{\sum r_a^2+\sum r_br_c}{2Rs^2}=\frac{(4R+r)^2-s^2}{2Rs^2}\Rightarrow RHS\overset{(ii)}{=}\frac{(4R+r)^2-s^2}{2Rs^2}$$

(i), (ii)$\Rightarrow$ ***proposed inequality*** $\Leftrightarrow \frac{3(s^2-4Rr-r^2)}{4rs^2}\geq\frac{(4R+r)^2-s^2}{2Rs^2}$

$$\Leftrightarrow 3R(s^2-4Rr-r^2)\geq 2r(4R+r)^2-2rs^2$$

$$\Leftrightarrow (3R+2r)s^2\overset{(iii)}{\geq}3Rr(4R+r)+2r(4R+r)^2=r(4R+r)(11R+2r)$$

Now, LHS of (iii) $\underset{?}{\overset{Gerretsen}{\geq}}(3R+2r)(16Rr-5r^2)\overset{?}{\geq}r(4R+r)(11R+2r)$

$$\Leftrightarrow 2R^2-Rr-6r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(2R+3r)\overset{?}{\geq}0\rightarrow \textbf{true}\because R\overset{Euler}{\geq}2r$$

$\Rightarrow$ ***(iii)*** $\Rightarrow$ ***proposed inequality is true (Proved)***