A Combinatorial Proof of the First Strehl Identity

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Abstract

We give a combinatorial proof of the first Strehl Identity by using double counting argument.

1 Introduction

Let n be a non-negative integer. The first Strehl Identity is the binomial coefficient identity

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n} \tag{1}$$

The numbers $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$ (the numbers on the left side of the Identity (1)) are also known as the Franel numbers. In 1894, Franel [2] found a second-order recurrence for f_n:

$$(n+1)^2f_{n+1}-(7n^2+7n+2)f_n-8n^2f_{n-1}=0\ ; \tag{2}$$

where n is a positive integer.

Franel proved that no first-order recurrence of this type exists for f_n. Therefore, there is no simple closed form for f_n. Note that there are some interesting congruences and other formulas for f_n (see [3, 4, 7]).

In 1993, Strehl [6] gave a simple proof of the Identity (1) by using the Chu-Vandermonde convolution formula. A combinatorial proof of the Chu-Vandermonde formula can be found in [1]. Furthermore, Strehl proved the Identity (1) by using the Equation (2) and the Zeilberger method.

The Identity (1) is connected with a deeper binomial coefficient identity:

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{l=0}^{k}\binom{k}{l}^3 \tag{3}$$

In the literature, the Identity (3) is known as the second Strehl Identity [1]; and which relates with the famous Apéry numbers and the well-known Franel numbers in an interesting way.

We give a combinatorial proof of the Identity (1) by using double counting argument. Our proof is a part of the master thesis [5].

2 Notation

We let $[n]$ denote the set $\{1, 2, \ldots, n\}$, if n is a positive integer; and let $[0]$ denote the empty set $\emptyset$.

Let $|A|$ denote the cardinality of the set A; and let $A\backslash B$ denote the set difference: $\{x|x \in A, x \notin B\}$.

Finally, we let $A\Delta B$ denote the symmetric difference of sets: $A\backslash B \cup B\backslash A$.

3 The Proof of the Identity (1)

Let us assume that n is a non-negative number.
We define the set X, as follows:

$$X = \{(A, B)|A, B \subset [2n], |A| = |B| = n, A \subset B\Delta[n]\}.$$

Let us count the elements of X in two different ways.
First, we count the elements of the set B. Let $k = |B\backslash[n]|$. Obviously, k goes from 0 to n. So, we can choose these k elements (from the set $[2n]\backslash[n]$) in $\binom{n}{k}$ ways. The remaining $n-k$ elements belong to the set $B \cap [n]$; and we can choose them (from the set $[n]$) in $\binom{n}{n-k}$ ways. Since, $|B\Delta[n]| = 2k$, we can choose the elements of the set A (from the set $B\Delta[n]$) in $\binom{2k}{n}$ ways. Therefore, we get

$$|X| = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}\binom{2k}{n}. \tag{4}$$

On the other hand,

$$X = \{(A, B)|A, B \subset [2n], |A| = |B| = n, A\backslash[n] \subset B, A \cap B \cap [n] = \emptyset\}.$$

Now, first, we count the elements of the set A. Let $k = |A \cap [n]|$. Obviously, k goes from 0 to n; and we can choose these k elements (from the set $[n]$) in $\binom{n}{k}$ ways. The remaining $n-k$ elements belong to the set $A\backslash[n]$; and we can choose them (from the set $[2n]\backslash[n]$) in $\binom{n}{n-k}$ ways. These $n-k$ elements must be already in B. The remaining k elements of the

[1] This Identity (3) should correctly be named, if one wishes, after both *Assmus Schmidt* and *Volker Strehl*. Strehl proved it independently at the same time, after it had been conjectured by A. Schmidt.

set B we can freely choose from the set $[2n]\backslash A$. It can be done in $\binom{n}{k}$ ways. Therefore, we get

$$|X| = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}\binom{n}{k}. \tag{5}$$

From Eqns. (4),(5), and a symmetry of binomial coefficients, the Identity (1) follows, as desired. ☐

Remark 1. *Let m be a positive integer such that $m > n$. Let Y denote the set*

$$\{(A,B)|A,B \subset [m], |A| = |B| = n, A \subset B\Delta[n]\}.$$

If we count the elements of the set Y in two different ways, we obtain that

$$\sum_{k=0}^{\min(m-n,n)} \binom{m-n}{k}\binom{n}{n-k}\binom{2k}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{m-n}{n-k}\binom{m-n}{k}.$$

4 Acknowledgments

I want to thank to my professor Duško Jojić.

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