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SOLUTIONS

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JP.226. Let a, b, c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq k \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq k \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right) \quad (*)$$

But: $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \stackrel{\text{Nesbit}}{\geq} \frac{3}{2}$; $a^2 + b^2 + c^2 \geq ab + bc + ca \rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 > 0$

So, $k > 0 \rightarrow k_{max} > 0$.

Because: (*) is true for all $a, b, c > 0$

Hence, we let: $a = b = 1$; $c = \frac{1}{n} (\forall n \in \mathbb{N}^*)$ then:

$$k \leq \frac{\frac{3}{1 + \frac{1}{n}} - \frac{3}{2}}{\frac{2 + \frac{1}{n^2}}{1 + \frac{1}{n}} - 1}; (\forall n \in \mathbb{N}^*)$$

Let $n \rightarrow +\infty$ we have: $k \leq \frac{1}{2}$

Now, we show that: $k_{max} = \frac{1}{2}$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq \frac{1}{2} \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right)$$

$$\Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2} \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} \right) + 1 = \frac{(a+b+c)^2}{2(ab + bc + ca)}$$

It is true because:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{ab+ac} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb} \stackrel{\text{Schwarz}}{\geq} \frac{(a+b+c)^2}{2(ab + bc + ca)}$$

Proved. Equality if and only if $a = b = c$.

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JP.227. Prove that in any ABC triangle the following relationship holds:

$$3(r_a r_b^3 + r_b r_c^3 + r_c r_a^3) \geq r(4R + r)^3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

Because $r_a + r_b + r_c = 4R + r$ we must show:

$$3(r_a r_b^3 + r_b r_c^3 + r_c r_a^3) \geq r(r_a + r_b + r_c)^3$$

$$\left. \begin{aligned} 3(r_a r_b^3 + r_b r_c^3 + r_c r_a^3) &\geq r(r_a + r_b + r_c)^3 \\ \text{But } r_a &= \frac{s}{s-a}, r = \frac{s}{s}, s = \frac{a+b+c}{2} \end{aligned} \right\} \Rightarrow \text{we must show:}$$

$$\begin{aligned} 3 \left(\frac{1}{(s-a)(s-b)^3} + \frac{1}{(s-b)(s-c)^3} + \frac{1}{(s-c)(s-a)^3} \right) &\geq \frac{1}{s} \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right)^3 \\ \Leftrightarrow 3s((s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3) &\geq \\ &\geq ((s-a)(s-b) + (s-b)(s-c) + (s-b)(s-c))^3 \quad (1) \\ (s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3 &= \\ = \frac{(s-a)^3(s-c)^3}{s-a} + \frac{(s-b)^3(s-a)^3}{s-b} + \frac{(s-c)^3(s-b)^3}{s-c} &\stackrel{\text{Holder}}{\geq} \\ \geq \frac{((s-a)(s-c) + (s-b)(s-a) + (s-c)(s-b))^3}{3(s-a+s-b+s-c)} &\Leftrightarrow \\ \Leftrightarrow 3s((s-a)^2(s-c)^3 + (s-b)^2(s-a)^3 + (s-c)^2(s-b)^3) &\geq \\ \geq ((s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a))^3 &\Rightarrow (1) \text{ it is true.} \end{aligned}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $r_a + r_b + r_c = 4R + r$. And: $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$

$$\text{Now, LHS} = 3 \left(\frac{r_b^3}{\left(\frac{1}{r_a}\right)} + \frac{r_c^3}{\left(\frac{1}{r_b}\right)} + \frac{r_a^3}{\left(\frac{1}{r_c}\right)} \right) \stackrel{\text{Holder}}{\geq} 3 \cdot \frac{(r_b+r_c+r_a)^3}{3\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)} = \frac{(4R+r)^3}{\frac{1}{r}} = r(4r+r)^3$$

Proved. Equality $\Leftrightarrow r_a = r_b = r_c \Leftrightarrow a = b = c$.

Solution 3 by Adrian Popa-Romania

$$\text{LHS} = 3 \left(\frac{r_a r_b^3 r_c}{r_c} \right) + \left(\frac{r_b r_c^3 r_a}{r_a} \right) + \left(\frac{r_c r_a^3 r_b}{r_b} \right) = 3\Delta s \left(\frac{(r_a^2 + r_b^2 + r_c^2)^2}{r_a + r_b + r_c} \right) =$$

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$$= 3\Delta s(r_a + r_b + r_c) = 3\Delta s(4R + r) \stackrel{?}{\geq} r(4R + r)^2$$

$$?: 3\Delta s \geq r(4R + r)^2 \leftrightarrow 3s^2 \geq (4R + r)^2$$

$$3s^2 \geq 3(16rR - 4r^2) \leftrightarrow 3s^2 - (4R + r)^2 \geq 3(16rR - 5r^2) - (4R + r)^2 \leftrightarrow$$

$$\leftrightarrow 40Rr - 16R^2 - 16r^2 \geq 0$$

$$R \geq 2r; r = \frac{\Delta}{s}; r_a r_b r_c = \Delta s$$

$$r_a + r_b + r_c = 4R + r$$

$$s^2 \geq 16rR - 5r^2$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\therefore m^2 + n^2 + p^2 \geq mn + np + pm, \forall m, n, p \in \mathbb{R}$$

$$\therefore x^2 y^4 + y^2 z^4 + z^2 x^4 \geq xy^2 \cdot yz^2 + yz^2 \cdot zx^2 + zx^2 \cdot xy^2 = xyz \left(\sum x^2 y \right)$$

$$\Rightarrow \frac{\sum x^2 y^4}{xyz} \geq \sum x^2 y \Rightarrow \sum \frac{xy^3}{z} \stackrel{(1)}{\geq} \sum x^2 y$$

$$\text{Now, } 3 \sum xy \left(\sum \frac{x^2}{y} \right) = 3 \sum x^3 + 3 \sum xy^2 + 2 \sum \frac{xy^3}{z} \stackrel{\text{by (1)}}{\geq} 3 \sum x^3 + 3 \sum xy^2 + 3 \sum x^2 y$$

$$\stackrel{A-G}{\geq} \sum x^3 + 6xyz + 3 \sum xy^2 + 3 \sum x^2 y = \sum x^3 + 3 \prod (x + y) = \left(\sum x \right)^3$$

$$\therefore 3 \sum xy \left(\sum \frac{x^2}{y} \right) \stackrel{(2)}{\geq} \left(\sum x \right)^3$$

Choosing $x = r_a, y = r_b, z = r_c$ in (2),

$$3 \sum r_a r_b \left(\sum \frac{r_a^2}{r_b} \right) \stackrel{(3)}{\geq} \left(\sum r_a \right)^3$$

$$\text{Now, } 3 \sum r_a r_b^3 = 3s \sum r_a^2 (s - b) \quad (\because r_a r_b = s(s - c), \text{ etc})$$

$$= 3s \sum r_a^2 \frac{rs}{r_b} = 3r(s^2) \left[\sum \frac{r_a^2}{r_b} \right] = 3r(\sum r_a r_b) \left(\sum \frac{r_a^2}{r_b} \right) \stackrel{\text{by (3)}}{\geq} r(4R + r)^3 \quad (\text{proved})$$

JP.228. Prove that if a, b, c are the lengths of the sides of a triangle, then:

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$$\sqrt{\frac{a}{b+c-a}} + \sqrt{\frac{b}{c+a-b}} + \sqrt{\frac{c}{a+b-c}} \geq \frac{(a+b+c)^2}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{a}{b+c-a} \cdot 1} \geq \frac{2}{\frac{b+c-a}{a}+1} = \frac{2a}{b+c} \Rightarrow \text{we must show:}$$

$$2 \left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \right) \geq \frac{(a+b+c)^2}{ab+bc+ca} \quad (1)$$

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = \frac{a^2}{ab+ac} + \frac{b^2}{ab+bc} + \frac{c^2}{ac+bc} \stackrel{\text{Bergstrom}}{\geq} \frac{(a+b+c)^2}{2(ab+ac+bc)} \Rightarrow (1) \text{ it is true.}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \text{LHS} &= \frac{a^2}{a\sqrt{ba+ca-a^2}} + \frac{b^2}{b\sqrt{cb+ab-b^2}} + \frac{c^2}{c\sqrt{ac+bc-a^2}} \stackrel{\text{Schwarz}}{\geq} \\ &\geq \frac{(a+b+c)^2}{a\sqrt{ba+ca-a^2} + b\sqrt{cb+ab-b^2} + c\sqrt{ac+bc-a^2}} = A \end{aligned}$$

$$a\sqrt{bc+ca-a^2} \stackrel{\text{AM-GM}}{\leq} \frac{a^2 + (bc+ca-a^2)}{2} = \frac{bc+ca}{2}$$

$$b\sqrt{cb+ab-b^2} \stackrel{\text{AM-GM}}{\leq} \frac{b^2 + (cb+ab-b^2)}{2} = \frac{cb+ab}{2}$$

$$c\sqrt{ac+bc-c^2} \stackrel{\text{AM-GM}}{\leq} \frac{c^2 + (ac+bc-c^2)}{2} = \frac{ab+bc}{2}$$

$$\Rightarrow a\sqrt{bc+ca-a^2} + b\sqrt{cb+ab-b^2} + c\sqrt{ac+bc-c^2} \leq ab+bc+ca$$

$$\Rightarrow A \geq \frac{(a+b+c)^2}{ab+bc+ca} = \text{RHS}$$

Proved. Equality $\Leftrightarrow a = b = c$.

Solution 3 by Michael Sterghiou-Greece

$$\sum_{\text{cyc}} \sqrt{\frac{a}{b+c-a}} \geq \frac{(\sum_{\text{cyc}} a)^2}{\sum_{\text{cyc}} ab} \quad (1)$$

(1) homogeneous so, WLOG $\sum_{\text{cyc}} a = p = 3$. Let $\sum_{\text{cyc}} ab = a$, $abc = r$

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(1) $\rightarrow \sum_{cyc} \frac{a}{\sqrt{(b+c-a)a}} \geq \frac{a}{q}$ (2). The function $f(x) = \frac{1}{\sqrt{x}}$ is convex hence by the

generalized Jensen inequality we have LHS of (2) $\geq (a+b+c) \cdot \frac{1}{\sqrt{\frac{\sum_{cyc} a \cdot a(b+c-a)}{a+b+c}}} \stackrel{?}{\geq} \frac{9}{q}$

(where a, b, c are "weights"). The last inequality reduces to $\frac{3}{\sqrt{\frac{\sum_{cyc} a^2(3-2a)}{3}}} \geq \frac{9}{q}$ or

$$q^2 - 36q + 18r + 81 \geq 0 \quad (3)$$

as $\sum_{cyc} a^2 = 9 - 2q$, $\sum_{cyc} a^3 = 27 - 9q + 3r$. As $q \leq \frac{p^3+9r}{4p}$ from Schur inequality 3rd

degree we have $3r \geq 4q - 9$ so (3) $\rightarrow q^2 - 12q + 27 \geq 0$ or $(3-q)(9-q) \geq 0$ which

$$\text{holds as } q \leq \frac{p^2}{3} = 3$$

Solution 4 by Jalil Hajimir-Canada

$$\sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{1}{\frac{b+c}{a}-1}} \stackrel{(*)}{\geq} \frac{1}{1 + \frac{1}{2}\left(\frac{b+c}{a}-1\right)} = \frac{2a}{b+c}$$

$$\left(\frac{b+c}{a}-1\right)^{\frac{1}{2}} = \left(1 + \frac{b+c}{a}-2\right)^{\frac{1}{2}} \geq 1 + \frac{1}{2}\left(\frac{b+c}{a}-2\right) \quad (*) \text{ Bernoulli}$$

$$\sum_{cyc} \sqrt{\frac{a}{b+c-a}} \geq \sum_{cyc} \frac{2a}{b+c} \quad (1)$$

$$\sum_{cyc} \frac{2a}{b+c} = 2 \sum_{cyc} \frac{a^2}{ab+ac} \geq \frac{2(a+b+c)^2}{2(ab+bc+ca)} \quad (2)$$

$$\text{From (1) and (2): } \sum_{cyc} \sqrt{\frac{a}{b+c-a}} \geq \frac{(a+b+c)}{(a^2+b^2+c^2)}$$

Solution 5 by Marian Dinca-Romania

$$\begin{aligned} \sum_{cyc} \sqrt{\frac{a}{b+c-a}} &= \sum_{cyc} \sqrt{\frac{a \cdot a}{(b+c-a)a}} = \sum_{cyc} \frac{a}{\sqrt{(b+c-a)a}} \geq \\ &\geq \sum_{cyc} \frac{a}{\frac{(b+c-a)+a}{2}} = \sum_{cyc} \frac{2a}{b+c} \end{aligned}$$

$$\sum_{cyc} \frac{2a}{b+c} = 2 \cdot \sum_{cyc} \frac{a^2}{(b+c)a} \geq 2 \sum_{cyc} \frac{(a+b+c)^2}{(b+c)a} = \frac{(a+b+c)^2}{ab+bc+ca}$$

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Solution 6 by Anant Bansal-India

We know that the minimum value of $\sum \sin A \cdot \sin B$ is at $A = B = C$

$$\begin{aligned} \Rightarrow \sum \sin A \sin B &\geq \frac{9}{4} > \frac{27}{1024} = \left(\frac{\left(\frac{3\sqrt{3}}{2} \right)^2}{24\sqrt{3}} \right)^2 > \left(\frac{(\sum \sin A)^2}{24\sqrt{3}} \right)^2 \\ &\Rightarrow \frac{(24\sqrt{3})^2}{\sum \sin A \cdot \sin B} > \frac{(\sum \sin A)^4}{(\sum \sin A \cdot \sin B)^2} \\ &\Rightarrow 8 \sqrt{\frac{3}{\sin A \cdot \sin B} + \frac{3}{\sin B \cdot \sin C} + \frac{3}{\sin A \cdot \sin C}} > \frac{(\sum \sin A)^2}{\sum \sin A \cdot \sin B} \end{aligned}$$

Putting $\sin A = \frac{a}{2R}$

$$\frac{16\sqrt{3}R}{\sqrt{abc}} > \frac{(a+b+c)^2}{ab+bc+ca} = k$$

$$R = \frac{abc}{4\Delta}$$

$$\frac{4\sqrt{3abc}}{\Delta} > k$$

Using Heron's formula and $AM \geq GM$ we get:

$$\sqrt{\frac{a}{b+c-a}} + \sqrt{\frac{b}{a+c-b}} + \sqrt{\frac{c}{a+b-c}} > \frac{(a+b+c)^2}{ab+bc+ca}$$

Equality holds for $a = b = c$

Solution 7 by Soumava Chakraborty-Kolkata-India

$$b+c-a = 4R \cos \frac{A}{2} \cos \frac{B-C}{2} - 4R \sin \frac{A}{2} \cos \frac{A}{2} = 4R \cos \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right)$$

$$= 8R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Rightarrow \sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{4R \sin^2 \frac{A}{2} \cos \frac{A}{2}}{8R \cos \frac{A}{2} \left(\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)}} =$$

$$= \sin \frac{A}{2} \sqrt{\frac{1}{2 \left(\frac{r}{4R} \right)}} \stackrel{(1)}{=} \sqrt{\frac{2R}{r}} \sin \frac{A}{2}$$

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Similarly, $\sqrt{\frac{b}{c+a-b}} \stackrel{(2)}{=} \sqrt{\frac{2R}{r}} \sin \frac{B}{2}$ and $\sqrt{\frac{c}{a+b-c}} \stackrel{(3)}{=} \sqrt{\frac{2R}{r}} \sin \frac{C}{2}$

$$(1)+(2)+(3) \Rightarrow \sum \sqrt{\frac{a}{b+c-a}} = \sqrt{\frac{R}{2r}} \sum \frac{2 \sin \frac{A}{2} \cos \frac{B-C}{2}}{\cos \frac{B-C}{2}} \stackrel{(4)}{\geq} \sqrt{\frac{R}{2r}} \sum \left(2 \cos \frac{B+C}{2} \cos \frac{B-C}{2} \right)$$

$$\left(\because 0 < \cos \frac{B-C}{2} \leq 1 \right)$$

$$= \sqrt{\frac{R}{2r}} \sum (\cos B + \cos C) = \sqrt{\frac{2R}{r}} \left(1 + \frac{r}{R} \right)$$

We shall now prove $\frac{\sum a^2}{\sum ab} \stackrel{(a)}{\leq} \sqrt{\frac{R}{2r}}$

$$(a) \Leftrightarrow R(s^2 + 4Rr + r^2)^2 \geq 8r(s^2 - 4Rr - r^2)^2$$

$$\Leftrightarrow R\{s^4 + (4Rr + r^2)^2 + 2(4Rr + r^2)s^2\} \geq 8r\{s^4 + (4Rr + r^2)^2 - 2(4Rr + r^2)s^2\}$$

$$\Leftrightarrow (R - 2r)s^4 + (R - 8r)(4Rr + r^2)^2 + (2R + 16r)(4Rr + r^2)s^2 \stackrel{(b)}{\geq} 6rs^4$$

Now, LHS of (b) $\stackrel{\text{Gerretsen}}{\underset{(i)}{\geq}} (R - 2r)(16Rr - 5r^2)s^2 + (R - 8r)(4Rr + r^2)^2 +$

$$+ (2R + 16r)(4Rr + r^2)s^2 \text{ and RHS of (b) } \stackrel{\text{Gerretsen}}{\underset{(ii)}{\leq}} 6rs^2(4R^2 + 4Rr + 3r^2)$$

(i)(ii) \Rightarrow in order to prove (b), it suffices to prove:

$$s^2\{(R - 2r)(16Rr - 5r^2) + (2R + 16r)(4Rr + r^2) - 6r(4R^2 + 4Rr + 3r^2)\} + (R - 8r)(4Rr + r^2)^2 \geq 0$$

$$\Leftrightarrow s^2(5R + 8r) + (R - 8r)(4R + r)^2 \stackrel{(c)}{\geq} 0$$

Now, LHS of (c) $\stackrel{\text{Gerretsen}}{\geq} (5R + 8r)(16Rr - 5r^2) + (R - 8r)(4R + r)^2 \stackrel{?}{\geq} 0$

$$\Leftrightarrow 2t^3 - 5r^2 + 5t - 6 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{2t(t - 2) + 3t + 3\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow (c) \Rightarrow (b) \Rightarrow (a) \text{ is true. Now, } \frac{(\sum a)^2}{\sum ab} = \frac{\sum a^2 + 2\sum ab}{\sum ab} = 2 + \frac{\sum a^2}{\sum ab} \stackrel{(5)}{\leq} 2 + \sqrt{\frac{R}{2r}}$$

$$(4), (5) \Rightarrow \text{it suffices to prove: } \sqrt{\frac{2R}{r}} \left(\frac{R+r}{R} \right) \geq 2 + \frac{1}{2} \sqrt{\frac{2R}{r}} \Leftrightarrow \sqrt{\frac{2R}{r}} \left(\frac{2R+2r-R}{2R} \right) \geq 2 \Leftrightarrow$$

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$$\Leftrightarrow \frac{2R}{r} \cdot \frac{(R+2r)^2}{4R^2} \geq 4 \Leftrightarrow (R+2r)^2 \geq 8Rr \Leftrightarrow R^2 - 4Rr + 4r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)^2 \geq 0 \rightarrow \text{true (Proved)}$$

JP.229. Let a, b, c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2} - 1 \geq k \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2} - 1 \geq k \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right) \quad (*)$$

Using Inequality: $x^2 + y^2 + z^2 \geq xy + xz + yz$ we have:

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \rightarrow \frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2} - 1 > 0$$

$$a^2 + b^2 + c^2 \geq ab + bc + ca \rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 > 0$$

So, we only check case: $k > 0 \rightarrow k_{max} > 0$. Because: () is true for all $a, b, c > 0$*

Hence, we let: $a = b = 1; c = \frac{1}{n} (\forall n \in \mathbb{N}^*)$ then: $k \leq \frac{\frac{2+\frac{1}{n^4}-1}{\frac{2}{n^2}+1}}{\frac{2+\frac{1}{n^2}-1}{1+\frac{2}{n}}}$; $(\forall n \in \mathbb{N}^*)$

Let $n \rightarrow +\infty$ we have: $k \leq 1$. Now, we show that: $k_{max} = 1$

$$\frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2} - 1 \geq 1 \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \right)$$

$$\Leftrightarrow \frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \Leftrightarrow abc(a^3 + b^3 + c^3) + \sum_{cyc} (a^5b + ab^5)$$

$\geq 3a^2b^2c^2 + \sum_{cyc} (a^2b^4 + a^4b^2)$. Because:

$$a^3 + b^3 + c^3 \stackrel{AM-GM}{\geq} 3abc \rightarrow abc(a^3 + b^3 + c^3) \geq 3a^2b^2c^2 \quad (1)$$

$$a^5b + ab^5 = ab(a^4 + b^4) \stackrel{(a^4+b^4) \geq (ab^3+ba^3)}{\geq} ab(ab^3 + ba^3) = (a^2b^4 + a^4b^2)$$

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$$\rightarrow \sum_{cyc}(a^5b + ab^5) \geq \sum_{cyc}(a^2b^4 + a^4b^2) \quad (2)$$

From (1)+(2) we have: $abc(a^3 + b^3 + c^3) + \sum_{cyc}(a^5b + ab^5) \geq 3a^2b^2c^2 + \sum_{cyc}(a^2b^4 + a^4b^2)$. Proved. Equality holds if and only if $a = b = c$

JP.230. Prove that for any ABC triangle the following relationship holds:

$$\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} + \frac{3\sqrt[4]{3}}{\sqrt{2}} \leq 2 \left(\sqrt{\cos \frac{A}{2}} + \sqrt{\cos \frac{B}{2}} + \sqrt{\cos \frac{C}{2}} \right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Marian Ursărescu – Romania

Let $f: (0, +\infty) \rightarrow \mathbb{R}$, $f(x) = \sqrt{x}$, because $f''(x) < 0 \Rightarrow$ we can use Tiberiu Popoviciu's inequality:

$$\frac{1}{3}(f(x) + f(y) + f(z)) + f\left(\frac{x+y+z}{3}\right) \leq \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+z}{2}\right)\right)$$

$$x = \sin A, y = \sin B, z = \sin C \Rightarrow$$

$$\Rightarrow \frac{1}{3}(\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}) + \sqrt{\sin\left(\frac{\pi}{3}\right)} \leq \frac{2}{3} \sum \sqrt{\frac{\sin A + \sin B}{2}} \Rightarrow$$

$$\Rightarrow \sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} + \frac{3\sqrt[4]{3}}{\sqrt{2}} \leq \frac{2}{3} \sum \sqrt{\frac{\sin A + \sin B}{2}} \quad (1)$$

$$\frac{\sin A + \sin B}{2} = \frac{2 \sin\left(\frac{A+B}{2}\right) \cos\frac{A-B}{2}}{2} = \cos\frac{C}{2} \cdot \frac{A-B}{2} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow \sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} + \frac{3\sqrt[4]{3}}{\sqrt{2}} \leq 2 \sum \sqrt{\cos\frac{C}{2} \cdot \cos\frac{A-B}{2}} \quad (3)$$

$$\text{But } \cos\frac{A-B}{2} \leq 1, \text{ with equality for } A = B \quad (4)$$

$$\text{From (3)+(4)} \Rightarrow \sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} + \frac{3\sqrt[4]{3}}{\sqrt{2}} \leq 2 \left(\sqrt{\cos\frac{A}{2}} + \sqrt{\cos\frac{B}{2}} + \sqrt{\cos\frac{C}{2}} \right)$$

$$\text{Equality for } A = B = C = \frac{\pi}{3}$$

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JP.231. Prove that for any positive real numbers a, b, c the following relationship holds:

$$\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 + ab} \geq \frac{a + b + c}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu-Romania

$$\begin{aligned} & \frac{a^3 + abc - abc}{a^2 + bc} + \frac{b^3 + abc - abc}{b^2 + ca} + \frac{c^3 + abc - abc}{c^2 + ab} \geq \frac{a + b + c}{2} \Leftrightarrow \\ \Leftrightarrow & \frac{a(a^2 + bc)}{a^2 + bc} - \frac{abc}{a^2 + bc} + \frac{b(b^2 + ca)}{b^2 + ca} - \frac{abc}{b^2 + ca} + \frac{c(c^2 + ab)}{c^2 + ab} - \frac{abc}{c^2 + ab} \geq \frac{a + b + c}{2} \Leftrightarrow \\ & \Leftrightarrow \frac{a+b+c}{2} \geq \frac{abc}{a^2+bc} + \frac{abc}{b^2+ca} + \frac{abc}{c^2+ab} \quad (1) \end{aligned}$$

$$a^2 + bc \geq 2a\sqrt{bc} \Leftrightarrow \frac{1}{a^2 + bc} \leq \frac{1}{2a\sqrt{bc}} \Leftrightarrow \frac{abc}{a^2 + bc} \leq \frac{\sqrt{bc}}{2} \Rightarrow$$

$$\frac{abc}{a^2+bc} + \frac{abc}{b^2+ca} + \frac{abc}{c^2+ab} \leq \frac{\sqrt{ab} + \sqrt{ac} + \sqrt{bc}}{2} \quad (2)$$

From (1)+(2) we must show:

$$\frac{a+b+c}{2} \geq \frac{\sqrt{ab} + \sqrt{ac} + \sqrt{bc}}{2} \Leftrightarrow a + b + c \geq \sqrt{ab} + \sqrt{ac} + \sqrt{bc} \text{ true.}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\begin{aligned} & \frac{a^3}{a^2 + b \cdot c} + \frac{b^3}{b^2 + c \cdot a} + \frac{c^3}{c^2 + a \cdot b} \stackrel{?}{\geq} \frac{a + b + c}{2} \\ & \left(\frac{a^3}{a^2 + b \cdot c} - \frac{a}{2} \right) + \left(\frac{b^3}{b^2 + c \cdot a} - \frac{b}{2} \right) + \left(\frac{c^3}{c^2 + a \cdot b} - \frac{c}{2} \right) \stackrel{?}{\geq} 0 \end{aligned}$$

$$\text{Let be the function: } f(x) = \frac{x^3}{x^2 + \alpha} - \frac{x}{2} = \frac{2x^3 - x^3 - \alpha x}{2(x^2 + \alpha)}$$

$$f(x) = \frac{x^3 - \alpha \cdot x}{2(x^2 + \alpha)}, x > 0, \alpha > 0$$

$$\lim_{n \rightarrow 0} f(x) = 0, \lim_{n \rightarrow 0} f(x) = +\infty$$

$$f'(x) = \frac{(3x^2 - \alpha)(2x^2 + 2\alpha) - 4x(x^3 - \alpha x)}{4(x^2 + \alpha)^2} =$$

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$$= \frac{6x^4 + 6\alpha x^2 - 2\alpha x^2 - 2\alpha^2 - 4x^4 + 4\alpha x^2}{4(x^2 + \alpha)^2}$$

$$f'(x) = \frac{2x^4 + 8\alpha x^2 - 2\alpha^2}{4(x^2 + \alpha)^2}, f'(x) = 0 \Rightarrow 2x^4 - 8\alpha x^2 - 2\alpha^2 = 0 \div (2)$$

$$x^4 - 4\alpha x^2 - \alpha^2 = 0$$

$$\Delta = 16\alpha^2 - 4(1)(-\alpha^2) = 16\alpha^2 + 4\alpha^2 = 20\alpha^2 > 0$$

| | | |
|---------|-------|-----------|
| x | 0 | $+\infty$ |
| $f'(x)$ | +++++ | |
| $f(x)$ | 0 | $+\infty$ |

So: $\forall x \in]0, +\infty[: f(x) > 0$

$$\frac{a^3}{a^2 + bc} - \frac{a}{2} > 0, \frac{b^3}{b^2 + ca} - \frac{b}{2} > 0, \frac{c^3}{c^2 + ab} - \frac{c}{2} > 0$$

$$\text{So: } \frac{a^3}{a^2 + bc} - \frac{a}{2} + \frac{b^3}{b^2 + ca} - \frac{b}{2} + \frac{c^3}{c^2 + ab} - \frac{c}{2} > 0$$

$$\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 + ab} > \frac{a + b + c}{2}$$

and the inequality is true when $a = b = c$.

Solution 3 by Jalil Hajimir-Canada

Lemma x, y, z, a, b and c are positive real numbers. Prove:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a + b + c)^3}{3(x + y + z)}$$

Proof:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{\left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}}\right)^2}{x + y + z} \geq \frac{(a + b + c)^3}{3(x + y + z)}$$

$$* \left(\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}} \geq \frac{a + b + c}{3} \leftrightarrow \left(a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}}\right)^2 \geq \frac{(a + b + c)^3}{3}$$

* Power Mean

Proof based on the lemma:

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$$\begin{aligned} \frac{a^3}{a^2+bc} + \frac{b^3}{b^2+ac} + \frac{c^3}{c^2+ba} &\geq \frac{(a+b+c)^3}{a^2+bc+b^2+ac+c^2+ab} \geq \\ &\geq \frac{(a+b+c)^3}{3(a+b+c)^2} = \frac{a+b+c}{3} \end{aligned}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{a^3}{a^2+bc} &= \sum \frac{a^4}{a^3+abc} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a^2)^2}{\sum a^3+3abc} \\ &\stackrel{?}{\geq} \frac{\sum a}{2} \Leftrightarrow 2(\sum a^2)^2 \stackrel{?}{\geq} (\sum a)(\sum a^3+3abc) \\ \Leftrightarrow \sum a^4 + 4\sum a^2b^2 &\stackrel{?}{\geq} \sum a^3b + \sum ab^3 + 3abc(\sum a) \end{aligned}$$

$$\text{Now, } \sum a^2b^2 \geq ab \cdot bc + bc \cdot ca + ca \cdot ab = abc(\sum a)$$

$$\Rightarrow 3\sum a^2b^2 \stackrel{(a)}{\geq} 3abc(\sum a)$$

$$\text{Also, } \frac{a^4+a^2b^2}{2} \stackrel{A-G}{\geq} \frac{a^4}{(b)}, \frac{b^4+a^2b^2}{2} \stackrel{A-G}{\geq} \frac{ab^3}{(c)}, \frac{b^4+b^2c^2}{2} \stackrel{A-G}{\geq} \frac{c^4+b^2c^2}{(d)}, \frac{c^4+b^2c^2}{2} \stackrel{A-G}{\geq} \frac{bc^3}{(e)},$$

$$\frac{c^4+c^2a^2}{2} \stackrel{A-G}{\geq} \frac{c^3a}{(f)}, \frac{a^4+c^2a^2}{2} \stackrel{A-G}{\geq} \frac{ca^3}{(g)}$$

$$(a) + (b) + (c) + (d) + (e) + (f) + (g) \Rightarrow (1) \text{ is true (proved)}$$

Solution 5 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^3}{a^2+bc} + \frac{b^3}{b^2+ca} + \frac{c^3}{c^2+ab} \geq \frac{a+b+c}{2}$$

$$\Leftrightarrow 2(a^3(b^2+ca)(c^2+ab) + b^3(a^2+bc)(c^2+ab) + c^3(a^2+bc)(b^2+ca)) \geq (a+b+c)(a^2+bc)(b^2+ca)(c^2+ab)$$

$$\Leftrightarrow 2[abc(a^4+b^4+c^4) + a^2b^2c^2(a+b+c) + a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4 + b^3c^4 + c^3a^4]$$

$$\geq abc(a^4+b^4+c^4) + a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4$$

$$+ b^3c^4 + c^3a^4 + 2a^2b^2c^2(a+b+c)$$

$$+ abc(ba^3 + ac^3 + cb^3) + abc(ab^3 + bc^3 + ca^3)$$

$$+ abc(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow abc(a^4+b^4+c^4) + a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4 + b^3c^4 + c^3a^4$$

$$\geq abc(ba^3 + ac^3 + cb^3 + ab^3 + bc^3 + ca^3) + abc(a^2b^2 + b^2c^2 + c^2a^2)$$

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Because:

$$\begin{aligned} a^4 + b^4 + c^4 &\geq a^2b^2 + b^2c^2 + c^2a^2 \rightarrow abc(a^4 + b^4 + c^4) \geq \\ &\geq abc(a^2b^2 + b^2c^2 + c^2a^2) \quad (*) \end{aligned}$$

$$a^4b^3 + a^4b^3 + c^3a^4 \stackrel{AM-GM}{\geq} 3\sqrt[3]{(a^4)^3(b^3)^2c^3} = 3a^4b^2c = 3abc \cdot ba^3 \quad (1)$$

$$b^4c^3 + b^4c^3 + a^3b^4 \stackrel{AM-GM}{\geq} 3abc \cdot cb^3 \quad (2)$$

$$c^4a^3 + c^4a^3 + b^3c^4 \stackrel{AM-GM}{\geq} 3abc \cdot ac^3 \quad (3)$$

$$a^3b^4 + a^3b^4 + c^3b^4 \stackrel{AM-GM}{\geq} 3abc \cdot ab^3 \quad (4)$$

$$b^3c^4 + b^3c^4 + c^4a^3 \stackrel{AM-GM}{\geq} 3abc \cdot bc^3 \quad (5)$$

$$c^3a^4 + c^3a^4 + a^4b^3 \stackrel{AM-GM}{\geq} 3abc \cdot ca^3 \quad (6)$$

$$(1)+(2)+(3)+(4)+(5)+(6) \rightarrow$$

$$\begin{aligned} 3(a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4 + b^3c^4 + c^3a^4) &\geq \\ &\geq 3abc(ba^3 + ac^3 + cb^3 + ab^3 + bc^3 + ca^3) \end{aligned}$$

$$\Leftrightarrow a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4 + b^3c^4 + c^3a^4 \geq$$

$$\geq abc(ba^3 + ac^3 + cb^3 + ab^3 + bc^3 + ca^3) \quad (**)$$

$$\begin{aligned} (*)+(**) \rightarrow abc(a^4 + b^4 + c^4) + a^4b^3 + b^4c^3 + c^4a^3 + a^3b^4 + b^3c^4 + c^3a^4 &\geq \\ &\geq abc(ba^3 + ac^3 + cb^3 + ab^3 + bc^3 + ca^3) + abc(a^2b^2 + b^2c^2 + c^2a^2) \end{aligned}$$

Proved. Equality if and only if $a = b = c$.

JP.232. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{\frac{r_a}{r_b r_c}} + \sqrt{\frac{r_b}{r_c r_a}} + \sqrt{\frac{r_c}{r_a r_b}} \geq \sqrt{\frac{3}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{r_a}{\sqrt{r_a r_b r_c}} = \frac{4R + r}{S\sqrt{r}} \stackrel{Trucht}{\geq} \frac{\sqrt{3}S}{S\sqrt{r}} = \sqrt{\frac{3}{r}}$$

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Solution 2 by Mustafa Tarek-Cairo-Egypt

$$\sum_{cyc} \sqrt{\frac{r_a}{r_b r_c}} \geq \sqrt{\frac{3}{r}}$$

First, we will prove the following inequality:

$$\sum_{cyc} \sqrt{\frac{x}{yz}} \geq \sqrt{3 \sum_{cyc} \frac{1}{x}} \text{ where } x, y, z > 0 \quad (1)$$

$$(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$$

$$x^2 + y^2 + z^2 \geq xy + yz + xz$$

$$(x + y + z)^2 \geq 3(xy + yz + xz)$$

$$\frac{x + y + z}{\sqrt{xyz}} \geq \frac{\sqrt{3(xy + yz + xz)}}{\sqrt{xyz}} \Rightarrow \sum_{cyc} \sqrt{\frac{x}{yz}} \geq \sqrt{3 \sum_{cyc} \frac{1}{x}}$$

(1) it is true

In (1) let $x = r_a, y = r_b, z = r_c$ and using the identity

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \text{ then:}$$

$$\sum_{cyc} \sqrt{\frac{r_a}{r_b r_c}} \geq \sqrt{\frac{3}{r}}$$

Solution 3 by Jalil Hajimir-Canada

$$\sqrt{\frac{r_a}{r_b r_c}} + \sqrt{\frac{r_b}{r_c r_a}} + \sqrt{\frac{r_c}{r_a r_b}} = \frac{r_a + r_b + r_c}{\sqrt{r_a r_b r_c}} \geq \sqrt{3} \sqrt{\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}} = \sqrt{\frac{3}{r}}$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

$$\frac{x + y + z}{\sqrt{3xyz}} \geq \sqrt{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}; x, y, z > 0$$

Solution 4 by Bogdan Fustei-Romania

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \quad (\text{Blundan Inequality})$$

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$$\Rightarrow s^2 \cdot \frac{2(2R - r)}{R} \leq (4R + r)^2 \leq s^2 \cdot \frac{(4R - 2r)}{R} \leq (4R + r)^2$$

$$s^2 \left(4 - \frac{2r}{R}\right) \leq (4R + r)^2 \Rightarrow s \sqrt{4 - \frac{2r}{R}} \leq 4R + r$$

$$\sum \sqrt{\frac{r_a}{r_b r_c}} = \sum \sqrt{\frac{r_a^2}{r_a r_b r_c}}; r_a r_b r_c = Ss = s \cdot r \cdot s = s^2 r$$

$$\sum \sqrt{\frac{r_a}{r_b r_c}} = \sum \frac{r_a}{s\sqrt{r}} = \frac{r_a + r_b + r_c}{s\sqrt{r}} \geq \frac{s\sqrt{4 - \frac{2r}{R}}}{s\sqrt{r}} = \sqrt{\frac{4 - \frac{2r}{R}}{R - \frac{2r}{R}}}$$

Now, we will prove that $\frac{4}{r} - \frac{2}{R} \geq \frac{3}{r} \Rightarrow \frac{4}{r} - \frac{3}{r} \geq \frac{2}{R} \Rightarrow \frac{1}{r} \geq \frac{2}{R} \Rightarrow R \geq 2r$ (Euler)

So, finally we have the following: $\sum \sqrt{\frac{r_a}{r_b r_c}} \geq \sqrt{\frac{4 - \frac{2r}{R}}{R - \frac{2r}{R}}} \geq \sqrt{\frac{3}{r}}$ Q.E.D.

JP.233. Find the maximum and minimum possible value of:

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^2 x}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Sohini Mondal-India

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^2 x}$$

We know, $\sin^4 x + \cos^2 x = \sin^4 x + 1 - \sin^2 x = 1 - \sin^2 x (1 - \sin^2 x)$

$$= 1 - \sin^2 x \cos^2 x = 1 - \frac{1}{4} \sin^2 2x$$

and $\cos^4 x + \sin^2 x = 1 - \cos^2 x + \cos^4 x = 1 - \cos^2 x (1 - \cos^2 x)$

$$= 1 - \cos^2 x \sin^2 x = 1 - \frac{1}{4} \sin^2 2x$$

$$\therefore \frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^2 x} = \frac{2}{1 - \frac{1}{4} \sin^2 2x}$$

$$\therefore \frac{2}{1} \leq \frac{2}{1 - \frac{1}{4} \sin^2 2x} \leq \frac{2}{1 - \frac{1}{4}} \Rightarrow 2 \leq \frac{2}{1 - \frac{1}{4} \sin^2 2x} \leq \frac{8}{3}$$

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$$\therefore \min \text{ value} = 2; \max \text{ value} = \frac{8}{3}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\text{Suppose } f(x) = \frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^4 x}$$

$$f(x) = \frac{1}{\sin^4 x + 1 - \sin^2 x} + \frac{1}{\cos^4 x + 1 - \cos^2 x}$$

$$f(x) = \frac{1}{\sin^4 x - \sin^2 x + 1} + \frac{1}{\cos^4 x - \cos^2 x + 1}$$

$$\text{Suppose: } t = \sin^2 x, \cos^2 x = 1 - t, 0 \leq t \leq 1$$

$$f(x) = \frac{1}{t^2 - t + 1} + \frac{1}{t^2 - t + 1} = \frac{2}{t^2 - t + 1}$$

$$f(0) = 2, f(1) = 2, f'(x) = \frac{-2(2t-1) \cdot 1}{(t^2 - t + 1)^2}, f'(x) = 0 \Rightarrow t = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{2}{\frac{1}{4} - \frac{1}{2} + 1} = \frac{8}{3}$$

| | | | |
|---------|-------|---------------|---|
| x | 0 | $\frac{1}{2}$ | 1 |
| $f'(x)$ | +++++ | ----- | |
| $f(x)$ | 2 | $\frac{8}{3}$ | 2 |

$$\max(f(x)) = \frac{8}{3}, \min(f(x)) = 2$$

Solution 3 by Sudhir Jha-Kolkata-India

$$\Leftrightarrow \frac{1}{\sin^2 x + \cos^4 x} + \frac{1}{\sin^4 x + \cos^2 x} =$$

$$= \frac{1}{1 - \cos^2 x (1 - \cos^2 x)} + \frac{1}{1 - \sin^2 x (1 - \sin^2 x)}$$

$$= \frac{2}{1 - \sin^2 x \cdot \cos^2 x} = \frac{1}{1 - \frac{1}{4}(\sin 2x)^2}$$

$$\Leftrightarrow 0 \leq (\sin 2x)^2 \leq 1 \Rightarrow 0 \leq \frac{1}{4}(\sin 2x)^2 \leq \frac{1}{4} \Rightarrow 1 \geq 1 - \frac{1}{4}(\sin 2x)^2 \geq 1 - \frac{1}{4}$$

$$\Rightarrow 2 \leq \frac{2}{1 - \frac{1}{4}(\sin 2x)^2} \leq \frac{8}{3} \Leftrightarrow \max = \frac{8}{3}, \min = 2 \text{ (Answer)}$$

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Solution 4 by Nani Gopal Saha-India

$$T = \frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^2 x} = \frac{1}{\sin^4 x - \sin^2 x + 1} + \frac{1}{\sin^4 x - \sin^2 x + 1}$$

$$= \frac{2}{\sin^4 x - \sin^2 x + 1} = \frac{2}{\left(\sin^2 x - \frac{1}{2}\right)^2 + 1 - \frac{3}{4}} = \frac{2}{\left(\sin^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Clearly T will be maximum when $\sin^2 x = \frac{1}{2} \Rightarrow$ i.e. $\sin^2 x = \frac{1}{2} \therefore T_{\max} = \frac{2}{\frac{3}{4}} = \frac{8}{3}$

and T will be minimum when $\sin^2 x = 1$

$$T_{\min} = \frac{2}{\left(\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\frac{1}{4} + \frac{3}{4}} = 2$$

JP.234. Let x, y be positive real numbers such that $x + y \leq 1$. Prove that:

$$\left(1 - \frac{1}{x^4}\right)\left(1 - \frac{1}{y^4}\right) \geq 225$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$\because x, y > 0$ and $x + y \leq 1, \therefore x, y < 1$

$$\Rightarrow x^4, y^4 < 1 \Rightarrow (x^4 - 1), (y^4 - 1) < 0 \Rightarrow (x^4 - 1)(y^4 - 1) > 0$$

$$\therefore \text{proposed inequality} \Leftrightarrow (x^4 - 1)(y^4 - 1) \geq 225x^4y^4$$

$$\Leftrightarrow 1 \geq 225x^4y^4 - x^4y^4 + x^4 + y^4 \Leftrightarrow 1 \stackrel{(1)}{\geq} 224x^4y^4 + x^4 + y^4$$

$$\because 1 \geq x + y \therefore \text{LHS of (1)} \geq (x + y)^4 \stackrel{?}{\geq} 224x^4y^4 + x^4 + y^4$$

$$\Leftrightarrow 2x^3y + 2xy^3 + 3x^2y^2 \stackrel{?}{\geq} 112x^4y^4$$

$$\text{Now, } 1 \geq x + y \stackrel{A-G}{\geq} 2\sqrt{xy} \Rightarrow 1 \geq 4xy \Rightarrow 1 \geq 16x^2y^2 \Rightarrow x^2y^2 \leq \frac{1}{16}$$

$$\therefore 112x^4y^4 = 112x^2y^2(x^2y^2) \leq 112x^2y^2\left(\frac{1}{16}\right) = 7x^2y^2 \stackrel{?}{\leq} 2x^3y + 2xy^3 + 3x^2y^2$$

$$\Leftrightarrow x^3y + xy^3 - 2x^2y^2 \stackrel{?}{\geq} 0 \Leftrightarrow xy(x - y)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore (2) \Rightarrow (1) \Rightarrow \text{proposed inequality is true (Proved)}$$

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Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y > 0$ and $x + y \leq 1$, we have: $\frac{1}{x} + \frac{1}{y} \leq \frac{1}{xy}$ and $\frac{1}{x} + \frac{1}{y} \geq 4$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{2}{xy} \leq \frac{1}{(xy)^2} \text{ and } \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \geq 1$$

$$\text{Consider } \left(1 - \frac{1}{x^4}\right) \left(1 - \frac{1}{y^4}\right) = \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \left(1 + \frac{1}{x}\right) \left(1 + \frac{1}{y}\right) \left(1 + \frac{1}{x^2}\right) \left(1 + \frac{1}{y^2}\right)$$

$$= \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \left(1 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{(xy)^2}\right)$$

$$\geq \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{x} + \frac{1}{y}\right) \left(1 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{2}{xy}\right)$$

$$\geq \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) \left(1 + 2\left(\frac{1}{x} + \frac{1}{y}\right)\right) \left(1 + \frac{\left(\frac{1}{x} + \frac{1}{y}\right)^2}{2} + \left(\frac{1}{x} + \frac{1}{y}\right)^2\right)$$

$$\geq \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) (1 + 8)(1 + 8 + 16) = \left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{y}\right) 9 \times 25$$

$\geq 1 \times 225 = 225$ ok. Therefore, it is true.

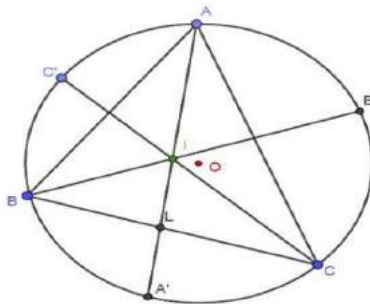
JP.235. In $\triangle ABC$; I – incenter; A', B', C' - lies on circumcircle such that:

(A, I, A') ; (B, I, B') ; (C, I, C') are collinear. Prove that:

$$\frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} \geq \frac{a + b + c}{R}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam



Using Ptolemy's theorem with Cyclic quadrilateral $ABA'C$:

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$$AA' \cdot BC = AB \cdot CA' + AC \cdot BA' \leftrightarrow AA' \cdot a = c \cdot CA' + b \cdot BA'$$

$$\text{Because } \widehat{BAA'} = \widehat{CAA'} \rightarrow A'B = A'C \rightarrow AA' \cdot a = (b+c)A'B$$

$$\text{But: } \frac{A'B}{\sin \widehat{BAA'}} = 2R \rightarrow A'B = 2R \sin \widehat{BAA'} = 2R \sin \frac{A}{2}. \text{ So, } AA' = \frac{b+c}{a} \cdot 2R \sin \frac{A}{2} = \frac{b+c}{2 \cos \frac{A}{2}}$$

$$\rightarrow IA' = AA' - AI = \frac{b+c}{2 \cos \frac{A}{2}} - \frac{b+c}{2s} \cdot \frac{2bc}{b+c} \cdot \cos \frac{A}{2} = \frac{b+c}{2 \cos \frac{A}{2}} - \frac{bc}{s} \cos \frac{A}{2}$$

$$= \frac{s(b+c) - 2bc \cos^2 \frac{A}{2}}{2s \cos \frac{A}{2}} = \frac{s(b+c) - 2bc \cdot \frac{s(s-a)}{bc}}{2s \cos \frac{A}{2}}$$

$$= \frac{b+c - 2(s-a)}{2 \cos \frac{A}{2}} = \frac{b+c - (b+c-a)}{2 \cos \frac{A}{2}} = \frac{a}{2 \cos \frac{A}{2}} \rightarrow \frac{a}{IA'} = 2 \cos \frac{A}{2}$$

$$\text{Similarly: } \frac{b}{IB'} = 2 \cos \frac{B}{2}; \frac{c}{IC'} = 2 \cos \frac{C}{2}$$

$$\rightarrow \frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} = 2 \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \right) \stackrel{AM-GM}{\geq} 6 \sqrt[3]{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = 6 \sqrt[3]{\frac{s}{4R}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Let's us join AIA' and BA' . \therefore angles $AA'B$ and ACB are angles on the same arc $AC'B$

and the same side of it, $\therefore \angle AA'B = \angle ACB = C$

$$\begin{aligned} \therefore \text{in } \triangle AA'B, \angle ABA' &= 180^\circ - \left(\frac{A}{2} + C \right) = A + B - \frac{A}{2} = \frac{A+2B}{2} = \frac{B+(180^\circ-C)}{2} \\ &= 90^\circ + \frac{B-C}{2} \therefore \angle ABA' \stackrel{(1)}{=} 90^\circ + \frac{B-C}{2} \end{aligned}$$

\therefore the circumcircle of $\triangle ABC$ coincides with the circumcircle of $\triangle AA'B$

$$\therefore \text{circumradius of } \triangle AA'B \stackrel{(2)}{=} R$$

$$(1), (2) \Rightarrow AA' \stackrel{\text{sine rule}}{=} 2R \sin \left(90^\circ + \frac{B-C}{2} \right) = 2R \cos \frac{B-C}{2}$$

$$\Rightarrow IA' = AA' - IA = 2R \cos \frac{B-C}{2} - \frac{r}{\sin \frac{A}{2}} = 2R \cos \frac{B-C}{2} - \frac{r}{\left(\frac{r}{4R} \right)} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 2R \cos \frac{B-C}{2} - 2R \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) = 2R \sin \frac{A}{2}$$

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$$\begin{aligned} \Rightarrow \frac{a}{IA'} &= \frac{4R \sin \frac{A}{2} \cos \frac{A}{2}}{2R \sin \frac{A}{2}} = 2 \cos \frac{A}{2} = \frac{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}{\cos \frac{B-C}{2}} \\ &\geq 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} \left(\because 0 < \cos \frac{B-C}{2} \leq 1 \because -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2} \right) \\ &= \sin B + \sin C = \frac{b+c}{2R} \Rightarrow \frac{a}{IA'} \stackrel{(a)}{\geq} \frac{b+c}{2R}. \text{ Similarly, } \frac{b}{IB'} \stackrel{(b)}{\geq} \frac{c+a}{2R} \text{ and } \frac{c}{IC'} \stackrel{(c)}{\geq} \frac{a+b}{2R} \\ (a)+(b)+(c) &\Rightarrow \frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} \geq \frac{2(a+b+c)}{2R} = \frac{a+b+c}{R} \quad (\text{Proved}) \end{aligned}$$

Solution 3 by Marian Dinca-Romania

$$IA \cdot IA' = IB \cdot IB' = IC \cdot IC' = R^2 - OI^2 = 2Rr$$

$$\frac{r}{IA} = \sin \frac{A}{2} \Rightarrow IA' = \frac{2Rr}{IA} = 2R \sin \frac{A}{2}$$

$$\frac{r}{IB} = \sin \frac{B}{2} \Rightarrow IB' = \frac{2Rr}{IB} = 2R \sin \frac{B}{2}$$

$$\frac{r}{IC} = \sin \frac{C}{2} \Rightarrow IC' = \frac{2Rr}{IC} = 2R \sin \frac{C}{2}$$

$$\frac{a}{IA'} + \frac{b}{IB'} + \frac{c}{IC'} = \frac{2R \sin A}{2R \sin \frac{A}{2}} + \frac{2R \sin B}{2R \sin \frac{B}{2}} + \frac{2R \sin C}{2R \sin \frac{C}{2}} = 2 \cos \frac{A}{2} + 2 \cos \frac{B}{2} + 2 \cos \frac{C}{2}$$

$$\text{and: } \frac{a+b+c}{R} = 2(\sin A + \sin B + \sin C) = 4 \left(\sin \frac{A}{2} \cdot \cos \frac{A}{2} + \sin \frac{B}{2} \cdot \cos \frac{B}{2} + \sin \frac{C}{2} \cdot \cos \frac{C}{2} \right) \leq$$

$$\leq 4 \cdot \frac{1}{3} \left(\sin \frac{A}{2} + \sin \frac{A}{2} + \sin \frac{A}{2} \right) \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \right) \leq$$

$$\leq 4 \sin \left(\frac{A+B+C}{6} \right) \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \right) =$$

$$= 2 \cos \frac{A}{2} + 2 \cos \frac{B}{2} + 2 \cos \frac{C}{2}. \text{ Using Chebyshev and Jensen's inequality. We must show}$$

$$\text{that: } 6 \sqrt[3]{\frac{s}{4R}} \geq \frac{a+b+c}{R} = \frac{2s}{R} \Leftrightarrow 3 \sqrt[3]{\frac{s}{4R}} \geq \frac{s}{R} \Leftrightarrow 27 \cdot \frac{s}{4R} \geq \left(\frac{s}{R} \right)^3 \Leftrightarrow \frac{27}{4} \geq \left(\frac{s}{R} \right)^2. \text{ It is true because:}$$

$$s \leq \frac{3\sqrt{3}}{2} R \rightarrow \left(\frac{s}{R} \right)^2 \leq \left(\frac{3\sqrt{3}}{2} \right)^2 = \frac{27}{4}. \text{ Proved.}$$

JP.236. If $a, b, c, x, y, z > 0$; $a + b + c = 3$ then:

$$a^a \cdot b^b \cdot c^c \cdot (x + y + z)^3 \geq 27x^a y^b z^c$$

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When does the equality holds?

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $(x + y + z)^3 \geq 27 \left(\frac{x}{a}\right)^a \cdot \left(\frac{y}{b}\right)^b \cdot \left(\frac{z}{c}\right)^c \Leftrightarrow$

$$\frac{x+y+z}{3} \geq \left(\frac{x}{a}\right)^{\frac{a}{3}} \left(\frac{y}{b}\right)^{\frac{b}{3}} \left(\frac{z}{c}\right)^{\frac{c}{3}} \quad (1)$$

From inequality of generalized environments \Rightarrow

$$\left(\frac{x}{a}\right)^{\frac{a}{3}} \left(\frac{y}{b}\right)^{\frac{b}{3}} \left(\frac{z}{c}\right)^{\frac{c}{3}} \leq \frac{a}{3} \cdot \frac{x}{a} + \frac{b}{3} \cdot \frac{y}{b} + \frac{c}{3} \cdot \frac{z}{c} = \frac{x+y+z}{3} \Rightarrow (1) \text{ it is true, equality } a = b = c = 1 \text{ and}$$

$$x = y = z.$$

Solution 2 by Sudhir Jha-Kolkata-India

Considering $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ with associated weights a, b, c respectively, we get, by $AM \geq GM$

$$\begin{aligned} \frac{a \cdot \left(\frac{x}{a}\right) + b \left(\frac{y}{b}\right) + c \left(\frac{z}{c}\right)}{a + b + c} &\geq \left[\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c\right]^{\frac{1}{a+b+c}} \\ \Rightarrow \frac{x + y + z}{3} &\geq \left[\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c\right]^{\frac{1}{3}} \quad (\because a + b + c = 3) \\ \Rightarrow \frac{(x + y + z)^3}{27} &\geq \frac{x^a y^b z^c}{a^a b^b c^c} \Rightarrow a^a b^b c^c (x + y + z)^3 \geq 27 x^a y^b z^c \end{aligned}$$

(Proved)

Solution 3 by Daniel Văcaru – Romania

We have:

$$\begin{aligned} \left(\frac{x}{a}\right)^{\frac{a}{a+b+c}} \cdot \left(\frac{y}{b}\right)^{\frac{b}{a+b+c}} \cdot \left(\frac{z}{c}\right)^{\frac{c}{a+b+c}} &\leq \left(\frac{a}{a+b+c}\right) \cdot \left(\frac{x}{a}\right) + \left(\frac{b}{a+b+c}\right) \cdot \left(\frac{y}{b}\right) + \\ &+ \left(\frac{c}{a+b+c}\right) \cdot \left(\frac{z}{c}\right) = \frac{x + y + z}{3} \Rightarrow \\ \left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c &\leq \left(\frac{x + y + z}{3}\right)^3 \Rightarrow 27 x^a y^b z^c \leq a^a b^b c^c (x + y + z)^3 \end{aligned}$$

JP.237. If $a, b, c, x, y, z > 0; a + b + c \geq x + y + z$ then:

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$$a^a b^b c^c \geq x^a y^b z^c$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$\begin{aligned} a^a b^b c^c \geq x^a y^b z^c &\Leftrightarrow a \ln a + b \ln b + c \ln c \geq a \ln x + b \ln y + c \ln z \Leftrightarrow \\ &\Leftrightarrow a \ln \frac{a}{x} + b \ln \frac{b}{y} + c \ln \frac{c}{z} \geq 0 \Leftrightarrow \ln \left(\frac{a}{x}\right)^a + \ln \left(\frac{b}{y}\right)^b + \ln \left(\frac{c}{z}\right)^c \geq 0 \Leftrightarrow \\ &\Leftrightarrow \ln \left(\frac{a}{x}\right)^a \left(\frac{b}{y}\right)^b \left(\frac{c}{z}\right)^c \geq 0 \Leftrightarrow \left(\frac{a}{x}\right)^a \left(\frac{b}{y}\right)^b \left(\frac{c}{z}\right)^c \geq 1 \text{ true, because: by weighted GM-HM} \Rightarrow \\ &\left(\frac{a}{x}\right)^a \left(\frac{b}{y}\right)^b \left(\frac{c}{z}\right)^c \geq \left(\frac{a+b+c}{\frac{a}{x} + \frac{b}{y} + \frac{c}{z}}\right)^{a+b+c} = \left(\frac{a+b+c}{x+y+z}\right)^{a+b+c} \geq 1^{a+b+c} = 1 \end{aligned}$$

Solution 2 by Sudhir Jha-Kolkata-India

Considering $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ with associated weights a, b, c and applying weighted GM \leq

weighted AM, we get

$$\begin{aligned} \left(\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c\right)^{\frac{1}{a+b+c}} &\leq \frac{a\left(\frac{x}{a}\right) + b\left(\frac{y}{b}\right) + c\left(\frac{z}{c}\right)}{a+b+c} \Rightarrow \\ &\Rightarrow \left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c \leq \left(\frac{x+y+z}{a+b+c}\right)^{a+b+c} \\ &\Rightarrow \left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b \left(\frac{z}{c}\right)^c \leq \dots (\because a+b+c \geq x+y+z) \\ &\Rightarrow a^a b^b c^c \geq x^a y^b z^c \text{ (proved)} \end{aligned}$$

Equality holds for $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 1$

Solution 3 by Michael Sterghiou-Greece

$$a^a b^b c^c \geq x^a y^b z^c \quad (1)$$

(1) $\rightarrow \left(\frac{x}{a}\right)^a \cdot \left(\frac{y}{b}\right)^b \cdot \left(\frac{z}{c}\right)^c \leq 1$ or $\sum_{cyc} a \ln \frac{x}{a} \leq 0$. As $f(t) = \ln t$ is concave

$(f''(t) = -\frac{1}{t^2} < 0)$ using generalized Jensen we get:

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$$\sum_{cyc} a \ln \frac{x}{a} \leq (\sum_{cyc} a) \cdot \ln \left(\frac{\sum_{cyc} a \cdot \frac{x}{a}}{\sum_{cyc} a} \right) = (\sum_{cyc} a) \cdot \ln \frac{x+y+z}{a+b+c} \leq 0 \text{ as } x + y + z \leq a + b + c.$$

Equality when $x = a, y = b, z = c$.

JP.238. If $a, b, c, d, x, y, z, t > 0$ then:

$$\frac{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d}{(a + b + c + d)^{a+b+c+d}} \geq \left(\frac{xyzt}{xyz + xyt + xzt + yzt} \right)^{a+b+c+d}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{We must show: } (ax)^a (by)^b (cz)^c (dt)^d \geq \left(\frac{a+b+c+d}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}} \right)^{a+b+c+d} \quad (1)$$

$$\text{Let } \frac{1}{x} = m, \frac{1}{y} = n, \frac{1}{z} = p, \frac{1}{t} = q \quad (2)$$

$$\text{From (1)+(2) we must show: } \left(\frac{a}{m} \right)^a \cdot \left(\frac{b}{n} \right)^b \cdot \left(\frac{c}{p} \right)^c \cdot \left(\frac{d}{q} \right)^d \geq \left(\frac{a+b+c+d}{m+n+p+q} \right)^{a+b+c+d} \Leftrightarrow$$

$$\Leftrightarrow a \ln \left(\frac{a}{m} \right) + b \ln \left(\frac{b}{n} \right) + c \ln \left(\frac{c}{p} \right) + d \ln \left(\frac{d}{q} \right) \geq (a + b + c + d) \ln \left(\frac{a+b+c+d}{m+n+p+q} \right), \text{ which it is true, because it is Gibbs inequality.}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Applying Weighted GM \geq Weighted A.M:

$$\begin{aligned} &^{a+b+c+d} \sqrt{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d} \geq \frac{a + b + c + d}{\frac{a}{ax} + \frac{b}{by} + \frac{c}{cz} + \frac{d}{dt}} \\ \Rightarrow &^{a+b+c+d} \sqrt{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d} \geq \frac{a + b + c + d}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}} \\ \Rightarrow & \frac{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d}{(a+b+c+d)^{a+b+c+d}} \geq \left(\frac{xyzt}{xyz + xyt + xzt + yzt} \right)^{a+b+c+d} \quad (\text{proved}) \end{aligned}$$

Equality at $ax = by = cz = dt$.

JP.239. In $\triangle ABC$ the following relationship holds:

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$$4(m_a + m_b + m_c) \leq 3 \left(\frac{r_a}{\cos^2 \frac{A}{2}} + \frac{r_b}{\cos^2 \frac{B}{2}} + \frac{r_c}{\cos^2 \frac{C}{2}} \right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} r_a &= s \tan \frac{A}{2}; r_b = s \tan \frac{B}{2}; r_c = s \tan \frac{C}{2} \\ r_a + r_b + r_c &= 4R + r \geq m_a + m_b + m_c \\ \frac{1}{\cos^2 \frac{A}{2}} + \frac{1}{\cos^2 \frac{B}{2}} + \frac{1}{\cos^2 \frac{C}{2}} &= \frac{s^2 + (4R + r)^2}{s^2} \end{aligned}$$

$$\text{Suppose: } A \geq B \geq C \rightarrow \begin{cases} r_a \geq r_b \geq r_c \\ \cos \frac{A}{2} \leq \cos \frac{B}{2} \leq \cos \frac{C}{2} \end{cases} \rightarrow \begin{cases} \frac{1}{\cos^2 \frac{A}{2}} \geq \frac{1}{\cos^2 \frac{B}{2}} \geq \frac{1}{\cos^2 \frac{C}{2}} \end{cases}$$

$$\begin{aligned} \text{By Chebyshev's inequality we have: } \text{RHS} &\geq 3 \cdot \frac{1}{3} (r_a + r_b + r_c) \left(\frac{1}{\cos^2 \frac{A}{2}} + \frac{1}{\cos^2 \frac{B}{2}} + \frac{1}{\cos^2 \frac{C}{2}} \right) \\ &\geq 3 \cdot \frac{1}{3} (m_a + m_b + m_c) \cdot \frac{s^2 + (4R + r)^2}{s^2} = (m_a + m_b + m_c) \cdot \frac{s^2 + (4R + r)^2}{s^2} \end{aligned}$$

$$\text{We must show that: } (m_a + m_b + m_c) \cdot \frac{s^2 + (4R + r)^2}{s^2} \geq 4(m_a + m_b + m_c)$$

$$\Leftrightarrow s^2 + (4R + r)^2 \geq 4s^2 \Leftrightarrow (4R + r)^2 \geq 3s^2 \Leftrightarrow 4R + r \geq \sqrt{3}s \text{ (true) (Proved)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Firstly, } \sum \sec^2 \frac{A}{2} &= \sum \frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)} = \frac{\sum bc(s^2 - s(2s-a) + bc)}{r^2 s^2} = \frac{-s^2 \sum ab + 3sabc + (\sum ab)^2 - 2abc(2s)}{r^2 s^2} \\ &= \frac{(s^2 + 4Rr + r^2)(4Rr + r^2) - 4Rrs^2}{r^2 s^2} = \frac{s^2 r^2 + r^2(4R + r)^2}{r^2 s^2} \stackrel{(1)}{=} \frac{s^2 + (4R + r)^2}{s^2} \end{aligned}$$

$$\begin{aligned} \text{Now, } 3 \sum \frac{r_a}{\cos^2 \frac{A}{2}} &= 3 \sum \left[\frac{rs}{s-a} \cdot \frac{bc}{s(s-a)} \right] = \frac{3r}{s} \sum \frac{bc(s-a+a)}{(s-a)^2} \\ &= 3r \sum \frac{bc}{s(s-a)} + \frac{3r}{s} \cdot \frac{4Rrs}{s^2} \sum \frac{s^2}{(s-a)} = 3r \sum \sec^2 \frac{A}{2} + \frac{12R}{s^2} \sum \frac{r^2 s^2}{(s-a)^2} \\ &\stackrel{\text{by (1)}}{=} 3r \left[\frac{s^2 + (4R + r)^2}{s^2} \right] + \frac{12R}{s^2} \sum r_a^2 = 3r + \frac{3r(4R + r)^2}{s^2} + \frac{12R}{s^2} [(4R + r)^2 - 2s^2] \end{aligned}$$

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$$= 3r - 24R + \frac{3(4R+r)^3}{s^2} \Rightarrow 3 \sum \frac{r_a}{\cos^2 \frac{A}{2}} \stackrel{(2)}{=} 3r - 24R + \frac{3(4R+r)^3}{s^2}$$

$$\text{Also, } 4 \sum m_a \leq 4 \sum r_a = 4(4R+r) \stackrel{?}{\leq} 3 \sum \frac{r_a}{\cos^2 \frac{A}{2}} \stackrel{\text{by (2)}}{=} 3r - 24R + \frac{3(4R+r)^3}{s^2}$$

$$\Leftrightarrow (40R+r)s^2 \stackrel{?}{\leq} 3(4R+r)^3$$

$$\text{Now, LHS of (3)} \stackrel{\text{Geretsen}}{\leq} (40R+r)(4R^2+4Rr+3r^2) \stackrel{?}{\leq} 3(4R+r)^3$$

$$\Leftrightarrow 8R^2 - 5Rr - 22r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (8R+11r)(R-2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r$$

\Rightarrow (3) \Rightarrow proposed inequality is true (Proved)

JP.240. In $\triangle ABC$ the following relationship holds:

$$3(m_a + m_b + m_c) \leq r_a \cot^2 \frac{A}{2} + r_b \cot^2 \frac{B}{2} + r_c \cot^2 \frac{C}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\left. \begin{aligned} \sum r_a \cot^2 \frac{A}{2} &= \sum \frac{s}{s-a} \cdot \frac{s(s-a)}{(s-b)(s-c)} = \sum \frac{Ss}{(s-b)(s-c)} \\ \text{but } \sum \frac{1}{(s-b)(s-c)} &= \frac{1}{r^2} \end{aligned} \right\} \Rightarrow$$

$$\sum r_a \cot^2 \frac{A}{2} = \frac{s^2}{r} \quad (1) \Rightarrow \text{we must show: } 3(m_a + m_b + m_c) \leq \frac{s^2}{r} \quad (2)$$

$$\text{But } \frac{m_a}{h_a} \leq \frac{R}{2r} \Rightarrow m_a \leq \frac{R}{2r} h_a \quad (3). \text{ From (2)+(3) we must show:}$$

$$\frac{3R}{2r} (h_a + h_b + h_c) \leq \frac{s^2}{r} \Leftrightarrow \frac{3R}{2} (h_a + h_b + h_c) \leq s^2 \quad (4)$$

$$\text{But } h_a + h_b + h_c = \frac{s^2+r^2+4Rr}{2R} \quad (5). \text{ From (4)+(5) we must show:}$$

$$\frac{3}{4} (s^2 + r^2 + 4Rr) \leq s^2 \Leftrightarrow 3s^2 + 3r^2 + 12Rr \leq 4s^2 \Leftrightarrow$$

$$s^2 \geq 12Rr + 3r^2 \quad (6)$$

$$\text{From Gerretsen's inequality: } s^2 \geq 16Rr - 5r^2 \quad (7)$$

From (6)+(7) we must show:

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$$16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r \text{ true (Euler)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$r_a = s \tan \frac{A}{2}; r_b = s \tan \frac{B}{2}; r_c = s \tan \frac{C}{2}$$

$$\rightarrow RHS = r_a \cdot \cot^2 \frac{A}{2}$$

$$+ r_b \cdot \cot^2 \frac{B}{2} + r_c \cdot \cot^2 \frac{C}{2}$$

$$= s \tan \frac{A}{2} \cdot \cot^2 \frac{A}{2} + s \tan \frac{B}{2} \cdot \cot^2 \frac{B}{2} + s \tan \frac{C}{2} \cdot \cot^2 \frac{C}{2}$$

$$= s \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right) = s \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = s \cdot \frac{s}{r} = \frac{s^2}{r}$$

$$\text{But: } m_a + m_b + m_c \stackrel{BCS}{\leq} \sqrt{3(m_a^2 + m_b^2 + m_c^2)} = \sqrt{\frac{9}{4}(a^2 + b^2 + c^2)} =$$

$$\frac{3}{2} \sqrt{a^2 + b^2 + c^2} \stackrel{\text{Leibniz}}{\leq} \frac{3}{2} \cdot \sqrt{9R^2} = \frac{9R}{2} \rightarrow LHS = 3(m_a + m_b + m_c) \leq \frac{27R}{2}$$

$$\text{We must show that: } \frac{27R}{2} \leq \frac{s^2}{r} \Leftrightarrow 2s^2 \geq 27Rr$$

$$\text{But: } s^2 \geq 16Rr - 5r^2. \text{ We need to prove: } 2(16Rr - 5r^2) \geq 27Rr$$

$$\Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r \text{ (Euler} \rightarrow \text{true) Proved.}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum r_a \cot^2 \frac{A}{2} = \sum \left(s \tan \frac{A}{2} \cot^2 \frac{A}{2} \right) = s \sum \cot \frac{A}{2} = s \sum \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= s \sum \sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} = \frac{s^2}{rs} \sum (s-a) \stackrel{(1)}{=} \frac{s^2}{r}$$

$$\text{Now, Chu and Yang} \Rightarrow 3 \sum m_a \leq 3\sqrt{4s^2 - 16Rr + 5r^2} \stackrel{?}{\leq} \sum r_a \cot^2 \frac{A}{2} \stackrel{\text{by (1)}}{=} \frac{s^2}{r}$$

$$\Leftrightarrow s^4 \stackrel{?}{\geq} 9r^2(4s^2 - 16Rr + 5r^2)$$

$$\text{Now, LHS of (2)} \stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)s^2 \stackrel{?}{\geq} 9r^2(4s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow (16R - 41r)s^2 + 9r^2(16R - 5r) \stackrel{?}{\geq} 0$$

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$$\Leftrightarrow (16R - 32r)s^2 + 9r^2(16R - 5r) \stackrel{?}{\geq} 9rs^2$$

Now, LHS of (3) $\stackrel{\text{Gerretsen}}{\geq} (a)$ $(16R - 32r)(16Rr - 5r^2) + 9r^2(16R - 5r)$

and, RHS of (3) $\stackrel{\text{Gerretsen}}{\leq} (b)$ $9r(4R^2 + 4Rr + 3r^2)$

(a), (b) \Rightarrow in order to prove (3), it suffices to prove:

$$(16R - 32r)(16Rr - 5r^2) + 9r^2(16R - 5r) - 9r(4R^2 + 4Rr + 3r^2) \geq 0$$

$$\Leftrightarrow 55R^2 - 121Rr + 22r^2 \geq 0 \Leftrightarrow (R - 2r)(55R - 11r) \geq 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (3) \Rightarrow (2) \Rightarrow \text{proposed inequality is true (proved)}$$

Solution 4 by Bogdan Fustei-Romania

$$\sin \frac{A}{2} = \sqrt{\frac{r r_a}{bc}} \text{ (and the analogs); } \cos \frac{A}{2} = \sqrt{\frac{r_b r_c}{bc}} \text{ (and the analogs)}$$

$$\cot^2 \frac{A}{2} = \frac{r_b r_c}{bc} \cdot \frac{bc}{r r_a} = \frac{r_b r_c}{r r_a} \text{ (and the analogs)} \Rightarrow r_a \cot^2 \frac{A}{2} = r_a \cdot \frac{r_b r_c}{r r_a} = \frac{r_b r_c}{r} \text{ (and the analogs)}$$

$$r_a r_b + r_b r_c + r_a r_c = s^2 \Rightarrow \sum r_a \cot^2 \frac{A}{2} = \frac{s^2}{r}$$

$$s^2 \geq 16R - 5r^2 \text{ (Gerretsen's Inequality)}$$

$$16Rr - 5r^2 \geq 3r(4R + r) = 12Rr + 3r^2 \Rightarrow 16Rr - 12Rr \geq 5r^2 + 3r^2 = 8r^2$$

$$4Rr \geq 8r^2 \Rightarrow R \geq 2r \text{ (Euler). So, we will have the inequality:}$$

$$s^2 \geq 3r(4R + r) \Rightarrow \frac{s^2}{r} \geq 3(4R + r) \Leftrightarrow \sum r_a \cot^2 \frac{A}{2} \geq 3(4R + r) \quad (1)$$

$$4R + r \geq m_a + m_b + m_c \Rightarrow 3(4R + r) \geq 3(m_a + m_b + m_c) \quad (2)$$

$$\text{From (1) and (2)} \Rightarrow 3(m_a + m_b + m_c) \leq \sum r_a \cot^2 \frac{A}{2}. \text{ Q.E.D.}$$

SP.226. If $a, b > 0$ then:

$$\left(\sqrt{ab} - \frac{a+b}{2}\right) \arctan\left(\frac{2ab}{a+b}\right) + \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) \arctan(\sqrt{ab}) +$$

$$+ \left(\frac{2ab}{a+b} - \sqrt{ab}\right) \arctan\left(\frac{a+b}{2}\right) \geq 0$$

Proposed by Daniel Sitaru - Romania

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Solution 1 by Marian Ursărescu-Romania

Let $\frac{2ab}{a+b} = x, \sqrt{ab} = y, \text{ and } \frac{a+b}{2} = z \Rightarrow 0 < x < y < z$

Let $f: (0, +\infty) \rightarrow \mathbb{R}; f(x) = \arctan x$. We must show:

$$(y-z)f(x) + (z-x)f(y) + (x-y)f(z) \geq 0 \quad (1)$$

Theorem: Let $f: I \rightarrow \mathbb{R}, I \subset \mathbb{R}, I = \text{interval}$

f its convex $\Leftrightarrow \forall x, y, z \in I$ with $x < y < z$ we have: $\frac{f(x)-f(y)}{x-y} \leq \frac{f(y)-f(z)}{y-z}$

$$f \text{ it is concave} \Leftrightarrow \frac{f(x)-f(y)}{x-y} \geq \frac{f(y)-f(z)}{y-z}$$

In our case f it is concave (for $(0, +\infty)$) $\Rightarrow \frac{f(x)-f(y)}{x-y} \geq \frac{f(y)-f(z)}{y-z} \mid \cdot (x-y)(y-z) > 0 \Rightarrow$

$$\Rightarrow (y-z)(f(x) - f(y)) \geq (x-y)(f(y) - f(z)) \Leftrightarrow$$

$$\Leftrightarrow (y-z)f(x) + (z-x)f(y) + (x-y)f(z) \geq 0 \Rightarrow (1) \text{ it's true.}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

According to $AM \geq GM \geq HM$ we have: $\therefore \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$

Let $f(x) = \tan^{-1} x$ for all $x \geq 0$ then $f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2} \leq 0$ for all $x \geq 0$

hence f is a concave function $\therefore \frac{f(\frac{a+b}{2}) - f(\frac{2ab}{a+b})}{\frac{a+b}{2} - \frac{2ab}{a+b}} \leq \frac{f(\sqrt{ab}) - f(\frac{2ab}{a+b})}{\sqrt{ab} - \frac{2ab}{a+b}}$

$$\Rightarrow \frac{\tan^{-1}\left(\frac{a+b}{2}\right) - \tan^{-1}\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2} - \frac{2ab}{a+b}} \leq \frac{\tan^{-1}(\sqrt{ab}) - \tan^{-1}\left(\frac{2ab}{a+b}\right)}{\sqrt{ab} - \frac{2ab}{a+b}}$$

$$\Rightarrow \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) \tan^{-1}(\sqrt{ab}) - \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) \tan^{-1}\left(\frac{2ab}{a+b}\right)$$

$$\geq \left(\sqrt{ab} - \frac{2ab}{a+b}\right) \tan^{-1}\left(\frac{a+b}{2}\right) - \left(\sqrt{ab} - \frac{2ab}{a+b}\right) \tan^{-1}\left(\frac{2ab}{a+b}\right)$$

$$\therefore \left(\sqrt{ab} - \frac{a+b}{2}\right) \tan^{-1}\left(\frac{2ab}{a+b}\right) + \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) \tan^{-1}(\sqrt{ab}) +$$

$$+ \left(\frac{2ab}{a+b} - \sqrt{ab}\right) \tan^{-1}\left(\frac{a+b}{2}\right) \geq 0 \quad (\text{Proved})$$

SP.227. Prove that for any positive real numbers a, b, c :

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$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \frac{8(a + b + c)^3}{3(a + b)(b + c)(c + a)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Mustafa Tarek-Cairo-Egypt

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \frac{8(a+b+c)^3}{3(a+b)(b+a)(c+a)} \quad (1)$$

$$(1) \text{ is equivalent to } \sum \frac{1}{a} \geq \frac{8(\sum a)^2}{3 \prod(a+b)} \quad (2)$$

Let $x = b + c, y = a + c, z = a + b$. It's easy to show that: $x + y > z$ and $y + z > x$ and $x + z > y$ then x, y, z are sides of a triangle, suppose that it's semiperimeter (s) and

it's circumradius (R) and its inradius (r) then $\sum x = 2 \sum a, \sum a = \frac{\sum x}{2} = s$, and

$$a = \frac{y+z-x}{2}, b = \frac{x+z-y}{2}, c = \frac{x+y-z}{2} \therefore (2) \text{ is equivalent to } \sum \frac{1}{s-x} \geq \frac{8s^2}{3xyz}$$

$$\Leftrightarrow \frac{\sum(s-x)(s-y)}{\prod(s-x)} \geq \frac{8s^2}{3 \cdot 4Rrs} \Leftrightarrow \frac{r(4R+r)}{r^2s} \geq \frac{8s^2}{3 \cdot 4Rrs}$$

[where we've used the well-known identity $\sum(s-x)(s-y) = r(4R+r)$]

$$\Leftrightarrow 4R + r \geq \frac{2s^2}{3R}. \text{ But using Doucet's inequality } 4R + r \geq \sqrt{3}s$$

$$4R + r \geq \sqrt{3}s \stackrel{\text{MITRINOVIC}}{\geq} \frac{2s^2}{3R} \Leftrightarrow \frac{3sR}{2} \geq s \Rightarrow \text{true because it is Mitrinovic's inequality.}$$

Equality holds if $a = b = c$.

Solution 2 by Boris Colakovic-Belgrade-Serbie

We use the well-known inequality $(a + b)(b + c)(c + a) \geq$

$$\geq \frac{8}{9}(a + b + c)(ab + bc + ca)$$

$$RHS \leq \frac{3 \cdot 8(a + b + c)^3}{8(a + b + c)(ab + bc + ca)} = \frac{3(a + b + c)^2}{ab + bc + ca} \leq \frac{(a + b + c)(ab + bc + ca)}{abc} \Rightarrow$$

$$\Rightarrow (ab + bc + ca)^2 \geq 3(a + b + c)abc \text{ true because}$$

$$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \geq$$

$$\geq ab \cdot bc + bc \cdot ca + ab \cdot ca + 2abc(a + b + c) = 3abc(a + b + c)$$

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Solution 3 by Michael Sterghiou-Greece

$$\left(\sum_{cyc} a\right) \left(\sum_{cyc} \frac{1}{a}\right) \geq \frac{8(a+b+c)^3}{3 \prod_{cyc}(a+b)} \quad (1)$$

$$\text{Let } \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right) = (p, q, r)$$

Then observe that $9 \prod_{cyc}(a+b) \geq 8p \cdot q$ (equivalent to $\sum_{cyc} a(b-c)^2 \geq 0$)

so (1) can become the stronger inequality.

$$p \cdot \frac{q}{r} \geq \frac{3 \cdot 8p^3}{8pq} \text{ or } q^2 \geq 3pr \text{ which holds. Done!}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\left(\sum a\right) \left(\sum \frac{1}{a}\right) \stackrel{(1)}{\geq} \frac{8(\sum a)^3}{3 \prod(a+b)}$$

$$\text{Let } \sum a^2 b + \sum ab^2 = p$$

$$(1) \Leftrightarrow \frac{(\sum a)(\sum ab)}{abc} \geq \frac{8 \sum a^3 + 24 \prod(a+b)}{3 \prod(a+b)} \Leftrightarrow \frac{p+3abc}{abc} - 8 \geq \frac{8 \sum a^3}{3(2abc+p)}$$

$$\Leftrightarrow \frac{3(p-5abc)(p+2abc)}{abc} \geq 8 \sum a^3 \Leftrightarrow \frac{3p^2}{abc} - 9p - 30abc \stackrel{(2)}{\geq} 8 \sum a^3$$

$$\text{Now, } \frac{3p^2}{abc} = \frac{3(\sum a^2 b + \sum ab^2)^2}{abc} = \frac{3(\sum a^4 b^2 + \sum a^2 b^4 + 2abc p + 2 \sum a^2 b \cdot \sum ab^2)}{abc}$$

$$= 3 \left(\frac{\sum a^4 b^2 + \sum c^2 a^4}{abc} \right) + 6p + \frac{6 \sum a^2 b \cdot \sum ab^2}{abc}$$

$$= 3 \sum \left\{ a^3 \left(\frac{b}{c} \right) + a^3 \left(\frac{c}{b} \right) \right\} + 6p + \frac{6(\sum a^3 b^3 + 3a^2 b^2 c^2 + abc(\sum a^3))}{abc}$$

$$\stackrel{A-G}{\geq} 3 \cdot 2 \sum a^3 + 6p + 6 \sum a^3 + \frac{6}{abc} (\sum a^3 b^3 + 3a^2 b^2 c^2)$$

$$\stackrel{Schur}{\geq} 12 \sum a^3 + 6p + \frac{6}{abc} (\sum a^2 b^2 c^2 \cdot bc + \sum ab \cdot b^2 c^2)$$

$$= 12 \sum a^3 + 6p + \frac{6}{abc} \cdot abc p = 12 \sum a^3 + 12p$$

$$\Rightarrow \frac{3p^2}{abc} - 9p - 30abc \geq 12 \sum a^3 + 3p - 30abc$$

$$= 8 \sum a^3 + \left\{ 4 \sum a^3 + 3 \left(\sum a^2 b + \sum ab^2 \right) \right\}$$

$$\stackrel{A-G}{\geq} 8 \sum a^3 + (12abc + 18abc) - 30abc = 8 \sum a^3$$

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⇒ (2) ⇒ (1) is true (Proved)

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$, we have the following: Because $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a + b + c$

$$\Rightarrow 2(a + b + c) + \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq 3(a + b + c)$$

$$\Rightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(ab + bc + ca) \geq 3(a + b + c)$$

$$\Rightarrow 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{8(ab + bc + ca)(a + b + c)}{9}\right) \geq 8(a + b + c)^2$$

$$\Rightarrow 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(a + b)(b + c)(c + a) \geq 8(a + b + c)^2$$

$$\Rightarrow 3(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(a + b)(b + c)(c + a) \geq 8(a + b + c)^3$$

$$\Rightarrow (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)} \text{ ok. Therefore, it is true.}$$

Solution 6 by Anant Bansal-India

$$\text{By GM} \geq \text{HM}: \left(\frac{a+b+c}{a}\right)\left(\frac{a+b+c}{b}\right)\left(\frac{a+b+c}{c}\right) \geq 3^3 = 27 \quad (i)$$

$$\text{By AM} \geq \text{GM}: x = \prod_{\text{cyc}} \left(\frac{\sum_{\text{cyc}} a}{a+b}\right) = \left(1 + \frac{c}{a+b}\right)\left(1 + \frac{b}{a+c}\right)\left(1 + \frac{a}{b+c}\right) \leq \left(\frac{4}{3}\right)^3$$

$$\frac{512}{81} \times \frac{8}{3} = \frac{8}{3}x$$

$$7 > \frac{8}{3}x \quad (ii)$$

$$\text{From (i) and (ii): } (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)}$$

SP.228. Find the positive real numbers (x, y) such that:

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2\sqrt[4]{\frac{x^4 + y^4}{2}} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

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Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2^4 \sqrt{\frac{x^4+y^4}{2}} & (1) \\ x^2 y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} & (2) \end{cases}, x, y > 0$$

From (1), $x \neq 0, y \neq 0$ $D =]0, +\infty[$

$$\frac{x^2}{y} + \frac{y^2}{x} = 2x^4 \sqrt{\frac{1}{2} \left(1 + \frac{y^4}{x^4}\right)} \Rightarrow \frac{x}{y} \cdot x + \frac{y}{x} \cdot y = 2x^4 \sqrt{\frac{1}{2} \left(1 + \frac{y^4}{x^4}\right)}; x > 0$$

$$\frac{x}{y} + \frac{y}{x} \cdot \frac{y}{x} = 2^4 \sqrt{\frac{1}{2} \left(1 + \frac{y^4}{x^4}\right)}. \text{ Suppose } t = \frac{y}{x}, t > 0, \text{ then:}$$

$$\frac{1}{t} + t^2 = 2^4 \sqrt{\frac{1}{2} (1 + t^4)} \quad (*) \quad D =]0, +\infty[$$

Let be the function: $f(t) = \frac{1}{t} + t^2 - 2^4 \sqrt{\frac{1}{2} (1 + t^4)}, D =]0, +\infty[$

f is Derivative in D and:

$$\lim_{t \rightarrow 0} [f(t)] = +\infty, \lim_{x \rightarrow +\infty} (f(x)) = +\infty$$

$$\left(\lim_{t \rightarrow +\infty} [f(t)] = \lim_{t \rightarrow +\infty} \left[\frac{1}{t} + t^2 - 2 \cdot \frac{1}{t} \sqrt{\frac{1}{2} (1 + t^4)} \right] = +\infty \right)$$

$$f'(t) = ?, f(t) = \frac{1}{t} + t^2 - 2 \left(\frac{1}{2} (1 + t^4) \right)^{\frac{1}{4}} \Rightarrow$$

$$f'(t) = -\frac{1}{t^2} + 2t - 2 \cdot \frac{1}{4} \left(\frac{1}{2} (1 + t^4) \right)^{-\frac{3}{4}} (2t^3)$$

$$f'(t) = -\frac{1}{t^2} + 2t - \frac{t^3}{\sqrt[4]{\left(\frac{1}{2}(1+t^4)\right)^3}}, f'(t) = 0 \Rightarrow t = 1 \text{ in the interval }]0, +\infty[$$

| | | | |
|---------|-----------|----|-----------|
| t | 0 | 1 | + |
| $f'(t)$ | ----- | -0 | +++++ |
| $f(t)$ | $+\infty$ | 0 | $+\infty$ |

So: $t = 1$ is solve to the equation (*). So:

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$\frac{y}{x} = 1 \Rightarrow y = x$. *Substituted in equation (2):*

$$x^4 - x^3 + 1 = \sqrt{2x^2 - 2x + 1}$$

$$(x^4 - x^3)^2 + 2(x^4 - x^3) + 1 = 2x^2 - 2x + 1$$

$$(x^3(x-1))^2 + 2(x^3(x-1)) = 2x(x-1)$$

$$(x-1)[x^6(x-1) + 2x^3 - 2x] = 0$$

$$(x-1)[x^6(x-1) + 2x^3 - 2x] = 0$$

$$x(x-1)[x^5(x-1) + 2x^2 - 2] = 0$$

$$x(x-1)[x^5(x-1) + 2(x-1)(x+1)] = 0$$

$$x(x-1)^2[x^2 + 2x + 1] = 0$$

$x = 0$ *impossible*

$$x = 1 \Rightarrow y = 1$$

$$x^2 + 2x + 1 = 0$$
 impossible

$$x^2 + 2x + 1 > 0$$
 impossible

$$\text{So: system } \begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2\sqrt{\frac{x^4+y^4}{2}} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \end{cases}, x, y > 0$$

Have only: $(x, y) = (1, 1)$

$$S = \{(1, 1)\}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x^4 + y^4}{2}} \leq x^2 - xy + y^2 \Leftrightarrow 2(x^2 - xy + y^2)^2 \geq x^4 + y^4$$

$$\Leftrightarrow 2(x^4 + y^4 + 2x^2y^2) - 4xy(x^2 + y^2) + 2x^2y^2 \geq x^4 + y^4$$

$$\Leftrightarrow (x^2 + y^2)^2 - 4xy(x^2 + y^2) + 4x^2y^2 \geq 0 \Leftrightarrow (x^2 + y^2 - 2xy)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore 2\sqrt{\frac{x^4+y^4}{2}} \leq 2\sqrt{x^2 - xy + y^2} \Rightarrow \frac{x^2}{y} + \frac{y^2}{x} \leq 2\sqrt{x^2 - xy + y^2} \quad (\text{by first equation})$$

$$\Rightarrow \frac{(x+y)(x^2 - xy + y^2)}{xy} \leq 2\sqrt{x^2 - xy + y^2} \Rightarrow (x+y)\sqrt{x^2 - xy + y^2} \stackrel{(1)}{\leq} 2xy$$

$$\text{But A-G} \Rightarrow (x+y)\sqrt{x^2 - xy + y^2} \stackrel{(2)}{\geq} 2\sqrt{xy}\sqrt{xy} = 2xy$$

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$$\therefore (1), (2) \Rightarrow (x + y)\sqrt{x^2 - xy + y^2} = 2xy$$

and (2) suggests equality occurs when $x = y$

$$\therefore x = y \quad (\because \text{equality occurs})$$

Putting $x = y$ in $x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1}$, we get:

$$x^4 - x^3 + 1 \stackrel{(3)}{=} \sqrt{2x^2 - 2x + 1}$$

$$\text{Let } x^2 - x = p. \text{ Then (3)} \Rightarrow x^2p + 1 = \sqrt{2p + 1} \Rightarrow x^4p^2 + 1 + 2x^2p = 2p + 1$$

$$\Rightarrow p(x^4p + 2x^2 - 2) = 0$$

If $p = 0$, $x(x - 1) = 0 \Rightarrow x = 0, 1$ and both values satisfy (3), but $x, y \neq 0$

$$\text{If } x^4p + 2x^2 - 2 = 0, \text{ then: } x^4(x^2 - x) + 2(1 + x)(x - 1) = 0$$

$$\Rightarrow x^5(x - 1) + 2(x + 1)(x - 1) = 0 \Rightarrow (x - 1)(x^5 + 2x + 2) = 0$$

$$\Rightarrow x = 1 \quad (\because x^5 + 2x + 2 > 2 \neq 0 \text{ as } x > 0)$$

\therefore combining all cases, all possible pairs of (x, y) satisfying given system is:

$$\begin{pmatrix} x = 1 \\ y = 1 \end{pmatrix} \text{ (answer)}$$

Solution 3 by Orlando Irahola Ortega-Bolivia

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2\sqrt{\frac{x^4 + y^4}{2}} & (1) \quad \{x \wedge y \neq 0\} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} & (2) \end{cases}$$

$$(2) \rightarrow x^3 + y^3 = 2xy^4\sqrt{\frac{x^4 + y^4}{2}} \Rightarrow x^6 + y^6 + 2x^3y^3 = 4x^2y^2\sqrt{\frac{x^4 + y^4}{2}} \Rightarrow$$

$$\Rightarrow \frac{x^3}{y^3} + \frac{y^3}{x^3} + 2 = \sqrt[4]{\frac{1}{2}\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right)}$$

$$\text{Sea: } t = \frac{x}{y} + \frac{y}{x} \Rightarrow t^3 - 3t + 2 = 2\sqrt{2t^2 - 4} \Rightarrow t(t^2 - 3) = 2(\sqrt{2t^2 - 4} - 1)$$

$$\Rightarrow t^2(t^2 - 3)^2 = 4(\sqrt{2t^2 - 4} - 1)^2 \Rightarrow (\sqrt{2t^2 - 4} + 4)(\sqrt{2t^2 - 4} - 2)^2 =$$

$$= 32(\sqrt{2t^2 - 4} - 1)^2; \text{ sea: } m = \sqrt{2t^2 - 4}$$

$$\Rightarrow (m^2 + 4)(m^2 - 2)^2 = 32(m - 1)^2 \Rightarrow m^6 - 44m^2 + 64m - 16 = 0$$

$$(m - 2)(m^5 + 2m^4 + 4m^3 + 8m^2 - 28m + 8) = 0$$

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$$m - 2 = 0 \Rightarrow m_1 = 2 \Rightarrow \sqrt{2t^2 - 4} = 2 \Rightarrow t = \pm 2 \quad \text{pero: } t = \frac{x}{y} + \frac{y}{x} \Rightarrow \frac{t}{2} \geq \sqrt{\frac{x}{y} \cdot \frac{y}{x}} \Rightarrow t \geq 2$$

$$x \wedge y \in \mathbb{R}$$

$$\text{Then: } t = 2 \Rightarrow \frac{x}{y} + \frac{y}{x} = 2 \Rightarrow (x - y)^2 = 0 \Rightarrow x = y \quad (3)$$

Por otro lado: $m^5 + 2m^4 + 4m^3 + 8m^2 - 28m + 8 = 0 = f(m)$: Condition:

$$m \geq 0 \rightarrow \sqrt{2t^2 - 4} = |m|$$

Si: $\left. \begin{array}{l} m = 0 \Rightarrow f(0) = 8 > 0 \\ m = 1 \Rightarrow f(1) = -5 < 0 \end{array} \right\} \exists \text{ variation of signs} \Rightarrow \exists \text{ al menos una solucion en } [0, 1]$

$$m > 1 \Rightarrow f(m > 1) > 0 \Rightarrow \exists \text{ solution for } m$$

Conclusion: como existe al menos una solucion para m en $[0, 1]$. Pero: $t \geq 2$

\Rightarrow para que $t \geq 2 \Rightarrow m \geq 2$ para $(x; y) \in \mathbb{R}$, como $m \in [0, 1] \Rightarrow (x, y) \notin \mathbb{R}$

$$\begin{aligned} (2) \rightarrow y^2(2x^2 - 2y) + 2 &= 2\sqrt{2x^2 - 2y + 1} \Rightarrow y^2\sqrt{2x^2 - 2y + 1}^2 + 2 - y^2 = \\ &= 2\sqrt{2x^2 - 2y + 1} \end{aligned}$$

$$\text{Sea: } t = \sqrt{2x^2 - 2y + 1} \Rightarrow t^2 y^2 + 2 - y^2 = 2t \Rightarrow (t - 1)(ty^2 + y^2 - 2) = 0 \Rightarrow$$

$$t = 1 \quad (*)$$

$$ty^2 = 2 - y^2 \quad (**)$$

$$(*) \rightarrow t = 1 \Rightarrow \sqrt{tx^2 - 2y + 1} = 1 \Rightarrow y = x^2 \quad (4)$$

$$\begin{aligned} (***) \rightarrow ty^2 = 2 - y^2 \Rightarrow t^2 y^4 &= y^4 - 4y^2 + 4 \Rightarrow y^4(2x^2 - 2y + 1) = y^4 - 4y^2 + 4 \\ \Rightarrow x^2 y^4 + 2y^2 &= y^5 + 2 \quad (5) \end{aligned}$$

$$\text{Ahora resolvemos: } \begin{cases} y = x \\ y = x^2 \end{cases} \Rightarrow x^2 - x = 0 \Rightarrow \begin{matrix} x=y=0 \\ x=y=1 \end{matrix} (x \wedge y \neq 0) \wedge \begin{cases} y = x \\ x^2 y^4 + 2y^2 = y^5 + 2 \end{cases}$$

$$\Rightarrow (x - 1)(x^5 + 2x + 2) = 0$$

$$\Rightarrow x = 1 \wedge x^5 + 2x + 2 = 0; \forall x \geq 0; \exists \text{ solucion: } \forall x \in \mathbb{R} \mid -1 \leq x < 0 \exists x < 0; y < 0.$$

$$\text{Pero: } (2) \rightarrow 2x^2 - 2y + 1 \geq$$

$$2x^2 + 1 \geq 2y \rightarrow y \geq 0$$

Como $y < 0 \Rightarrow \exists x \wedge y$ que satisfaga el Sistema.

$$\text{Unica solucion: } x = y = 1.$$

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SP.229. Let a, b, c be the lengths of a triangle such that $a + b + c = 3$. Prove that:

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} + 3 \cdot {}^{2018}\sqrt{abc} \geq 2(ab+bc+ca)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$\left(\sum_{cyc} \frac{1}{b+c-a}\right) + 3 {}^{2018}\sqrt{abc} \geq 2 \sum_{cyc} ab \quad (1)$$

WLOG, let $c \leq b \leq a$ (c) and let $(\sum_{cyc} a, \sum_{cyc} ab, abc) = (p, q, r)$. $p = 3$ and $r \leq 1$. As

$$a > b - c, a > c - b \text{ so } a > |b - c| \rightarrow a^2 > (b - c)^2 \quad (2)$$

Cyclic application of (2) and addition yields $2q \geq \sum_{cyc} a^2 = 9 - 2q$, so $q > \frac{9}{4}$

[Note: This is general without the condition ©]

Assume $c \leq \frac{1}{12}$. Then $|a - b| < c \leq 12$ and $\max(a + c - b, b + c - a) \leq \frac{1}{6}$ are one of

$\frac{1}{a+c-b}, \frac{1}{b+c-a}$ is > 6 . So, as $3 \cdot r^{\frac{1}{2018}} > 0$ and $2q \leq 6$ because $q \leq 3$ we are done!

Therefore, let $a \geq b \geq c \geq \frac{1}{12}$. In this case $r > \left(\frac{1}{12}\right)^3$. LHS (1) $\stackrel{BCS}{\geq} \frac{3^2}{\sum_{cyc}(b+c-a)} = 3$ so, it

suffices to show that (3) $3 + 3r^{\frac{1}{2018}} - 2q \geq 0$ for $\left(\frac{1}{12}\right)^3 < r \leq 1, \frac{9}{4} < q \leq 3$. From the

3rd degree Schur inequality we have: $q \leq \frac{9+3r}{4}$, so (3) reduces to:

$$f(r) = 2r^{\frac{1}{2018}} - r - 1 \geq 0 \text{ with } r > \left(\frac{1}{12}\right)^3 \text{ and } r \leq 1. f'(r) = \frac{1}{1009} \cdot \frac{1}{r^{\frac{2017}{2018}}} - 1 \text{ with only}$$

one real root $r_0 = \left(\frac{1}{1009}\right)^{\frac{2018}{2017}}$. It is easy to show that $f\left(\frac{1}{12^3}\right) > 0, f(r_0) > 0$ and

$f(1) = 0$ which means that $f(1) = 0$ is minimum for $f(r)$ on $\left(\frac{1}{12^3}, 1\right]$ and hence

$f(r) \geq 0$. Equality for $a = b = c = 1$. Done!

Solution 2 by Soumava Chakraborty-Kolkata-India

$${}^{2018}\sqrt{abc} = {}^{2018}\sqrt{abc \underbrace{1 \cdot 1 \cdot \dots \cdot 1}_{2015 \text{ times}}} \stackrel{G-H}{\geq} \frac{2018}{\sum \frac{1}{a} + 2015} = \frac{2018}{\sum \frac{ab}{4Rrs} + 2015} = \frac{2018}{\sum \frac{ab}{2Rr(3)} + 2015}$$

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$$\left(\because 2s = \sum a = 3\right) = \frac{12108Rr}{\sum ab + 12090Rr} \Rightarrow 3^{2018} \sqrt[2018]{abc} \stackrel{(1)}{\geq} \frac{36324Rr}{\sum ab + 12090Rr}$$

$$\begin{aligned} \text{Now, } \sum \frac{1}{b+c-a} &= \frac{1}{2} \sum \frac{1}{s-a} = \frac{\sum(s-b)(s-c)}{2r^2s} = \frac{\sum(s^2-s(b+c)+bc)}{2r^2s} \\ &= \frac{3s^2 - 4s^2 + s^2 + 4Rr + r^2}{2r^2s} = \frac{4R + r}{r(2s)} \stackrel{(2)}{=} \frac{4R + r}{3r} \left(\because 2s = \sum a = 3\right) \end{aligned}$$

$$\text{Again, } 2 \sum ab = \left(\frac{2}{3}\right) (3 \sum ab) \leq \frac{2}{3} (\sum a)^2 = \left(\frac{2}{3}\right) 9 = 6 \Rightarrow 2 \sum ab \stackrel{(3)}{\leq} 6$$

$$(1), (2) \Rightarrow LHS \stackrel{(4)}{\geq} \frac{4R+r}{3r} + \frac{36324Rr}{\sum ab + 12090Rr}$$

$$(4), (3) \Rightarrow \text{it suffices to prove: } \frac{4R+r}{3r} + \frac{36324Rr}{\sum ab + 12090Rr} \geq 6 \Leftrightarrow \frac{4R-17r}{3r} + \frac{36324Rr}{\sum ab + 12090Rr} \geq 0$$

$$\Leftrightarrow (4R - 17r) \left(\sum ab + 12090Rr\right) + 108972Rr^2 \geq 0$$

$$\Leftrightarrow (4R - 17r)(s^2 + 12094Rr + r^2) + 108972Rr^2 \geq 0$$

$$\Leftrightarrow (4R - 8r)s^2 + (4R - 17r)(12094Rr + r^2) + 108972Rr^2 \stackrel{(5)}{\geq} 9rs^2$$

$$\text{Now, LHS of (5)} \stackrel{\text{Gerretsen}}{\stackrel{(i)}{\geq}} (4R - 8r)(16Rr - 5r^2) + (4R - 17r)(12094Rr + r^2) +$$

$$+ 108972Rr^2 \text{ and, RHS of (5)} \stackrel{\text{Gerretsen}}{\stackrel{(ii)}{\leq}} 9r(4R^2 + 4Rr + 3r^2)$$

(i), (ii) \Rightarrow in order to prove (5), it suffices to prove:

$$(4R - 8r)(16Rr - 5r^2) + (4R - 17r)(12094Rr + r^2) + 108972Rr^2 \geq$$

$$\geq 9r(4R^2 + 4Rr + 3r^2) \Leftrightarrow 24202R^2 - 48403Rr - 2r^2 \geq 0$$

$$\Leftrightarrow (R - 2r)(24202R + r) \geq 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (5) \Rightarrow \text{proposed inequality is true (Proved)}$$

SP.230. Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of:

$$T = \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 3 \cdot \left(\frac{1}{abc + b^3 + c^3} + \frac{1}{abc + c^3 + a^3} + \frac{1}{abc + a^3 + b^3} \right)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Ruangkhaw Chaokha-Chiangrai-Thailand

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$$\text{Let } p = a + b + c = 3, q = abc + bc + ca, r = abc$$

$$\therefore (ab - bc)^2 + (bc - ca)^2 + (ca - ab)^2 \geq 0$$

$$(ab + bc + ca)^2 \geq 3abc(a + b + c) \Leftrightarrow q^2 \geq 9r \rightarrow (1) \text{ holds at } a = b = c = 1$$

$$\therefore (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0 \Leftrightarrow (a + b + c)^2 \geq 3(ab + bc + ca) \Leftrightarrow 3 \geq q \rightarrow$$

(2) hold at $a = b = c = 1$.

$$\text{Lemma } \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \stackrel{??}{\geq} \frac{2(a^3 + b^3 + c^3) + 3abc}{3} \text{ holds at } a = b = c = 1.$$

$$\text{Proof. } 3(a^2 + b^2 + c^2) \geq 2abc(a^3 + b^3 + c^3) + 3a^2b^2c^2$$

$$3(p^2 - 2q) \geq 2r(p^3 - 3pq + 3r) + 3r^2 \Leftrightarrow 3(9 - 2q) \geq 2r(27 - 9q + 3r) + 3r^2$$

$$0 \geq 9r^2 + 18(3 - q)r + 6q - 27 \Leftrightarrow 0 \geq \frac{q^4}{9} + 18(3 - q)\frac{q^2}{9} + 6q - 27 \stackrel{(1),(2)}{\geq}$$

$$\geq 9r^2 + 18(3 - q)r + 6q - 27$$

$$0 \geq q^4 + 18(3 - q)q^2 + 9(6q - 27) \Leftrightarrow 0 \geq (q - 3)^2(q^2 - 12q - 27) \text{ True!!}$$

$$\therefore (2); 0 < q \leq 3 \Leftrightarrow -6 < q - 6 \leq -3 \Leftrightarrow 9 \leq (q - 6)^2 < 36 \Leftrightarrow q^2 - 12q - 27 < -27 < 0$$

Holds at $q = 3 \Leftrightarrow a = b = c = 1$.

$$\begin{aligned} T &\geq \frac{2(a^3 + b^3 + c^3) + 3abc}{3} + 3 \left(\frac{1}{abc + b^3 + c^3} + \frac{1}{a^3 + abc + c^3} + \frac{1}{a^3 + b^3 + abc} \right) \\ &= \left(\frac{abc + b^3 + c^3}{3} + \frac{3}{abc + b^3 + c^3} \right) + \left(\frac{a^3 + abc + c^3}{3} + \frac{3}{a^3 + abc + c^3} \right) + \\ &\quad + \left(\frac{a^3 + b^3 + abc}{3} + \frac{3}{a^3 + b^3 + abc} \right) \stackrel{AM-GM}{\geq} 2 + 2 + 2 = 6 \end{aligned}$$

$$\text{Holds at } \frac{abc + b^3 + c^3}{3} = \frac{a^3 + abc + c^3}{3} = \frac{a^3 + b^3 + abc}{3} = 1 \Leftrightarrow a = b = c = 1$$

SP.231. In ΔABC the following relationship holds:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + 2 \log(a^{ab^2} \cdot b^{bc^2} \cdot c^{ca^2})$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\text{We must show: } a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + 2ab^2 \ln a + 2bc^2 \ln b + 2ca^2 \ln c \quad (1)$$

$$\text{Now we have } 2 \ln x \leq \frac{x^2 - 1}{x} \Leftrightarrow 2 \ln x \leq x - \frac{1}{x} \quad (2)$$

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(demonstration with derivative)

From (2) $\Rightarrow 2 \ln a \leq a - \frac{1}{a} \Rightarrow 2ab^2 \ln a \leq a^2b^2 - b^2$ and similarly (3)

From (1)+(3) we must show:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + a^2b^2 + b^2c^2 + c^2a^2 - (a^2 + b^2 + c^2) \Leftrightarrow$$

$$\Leftrightarrow a^2 + b^2 + c^2 \geq 4\sqrt{3}F \text{ which its true in any } \Delta ABC$$

Observation: equality for $a = b = c = 1$.

SP.232. If $a, b \geq 1$ then:

$$\frac{2}{2a+b} \cdot 3^{\frac{a+b}{2}} + 2\sqrt{ab} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2\frac{a+b}{2} \cdot 3^{\frac{2}{a+b}} + 2\sqrt{ab} \cdot 3^{\sqrt{ab}} \geq 24$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{Let } x \geq 1 \Rightarrow 2^x \cdot 3^{\frac{1}{x}} + 2^{\frac{1}{x}} \cdot 3^x \geq 2\sqrt{2^{x+\frac{1}{x}} \cdot 3^{x+\frac{1}{x}}} \geq 2\sqrt{2^2 \cdot 3^2}$$

$$\Rightarrow 2^x \cdot 3^{\frac{1}{x}} + 2^{\frac{1}{x}} \cdot 3^x \geq 12$$

$$\left. \begin{aligned} x = \frac{a+b}{2} &\Rightarrow 2^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} + 2^{\frac{2}{a+b}} \cdot 3^{\frac{a+b}{2}} \geq 12 \\ x = \sqrt{ab} &\Rightarrow 2^{\sqrt{ab}} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{1}{\sqrt{ab}}} \cdot 3^{\sqrt{ab}} \geq 12 \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow 2\frac{2}{2a+b} \cdot 3^{\frac{a+b}{2}} + 2\sqrt{ab} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2\frac{a+b}{2} \cdot 3^{\frac{2}{a+b}} + 2\sqrt{ab} \cdot 3^{\sqrt{ab}} \geq 24$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{For } a, b \geq 1, \text{ we have: } 2\frac{2}{2a+b}3^{\frac{a+b}{2}} + 2\sqrt{ab}3^{\frac{1}{\sqrt{ab}}} + 2\frac{a+b}{2}3^{\frac{2}{a+b}} + 2\sqrt{ab} - 3^{\sqrt{ab}}$$

$$\geq 4\sqrt[4]{\left(2\frac{2}{2a+b} \cdot 2^{\frac{a+b}{2}} \cdot 2\sqrt{ab} \cdot 2^{\frac{1}{\sqrt{ab}}}\right)\left(3^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} \cdot 3^{\sqrt{ab}}3^{\frac{1}{\sqrt{ab}}}\right)}$$

$$= 4\sqrt[4]{2^{\frac{2}{2a+b} + \frac{a+b}{2} + \sqrt{ab} + \frac{1}{\sqrt{ab}}} \cdot 3^{\frac{a+b}{2} + \frac{2}{a+b} + \sqrt{ab} + \frac{1}{\sqrt{ab}}}} \geq 4\sqrt[4]{2^4 3^4} = 2 \times 3 \times 4 = 24 \text{ ok}$$

$$\text{Because } \frac{a}{a+b} + \frac{a+b}{2} + \sqrt{ab} + \frac{1}{\sqrt{ab}} \geq 4\sqrt[4]{\left(\frac{2}{a+b}\right)\left(\frac{a+b}{2}\right)(\sqrt{ab})\left(\frac{1}{\sqrt{ab}}\right)} = 4 \times 1 = 4$$

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SP.233. If $A \in M_3(\mathbb{R})$; $\text{Tr}(A^2) = 0$; $\det = 1$ then:

$$\det(A^2 + A + I_3) \geq (\text{Tr } A)^3$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vişescu – Romania

If $\text{Tr}(A^2) = 0$ and $\det A = 1$

$$P_A(x) = x^3 - \text{Tr } A x^2 + \frac{(\text{Tr } A)^2}{2} x - 1$$

$$\det(A^2 + A + I_3) = \det(A - \varepsilon I_3)(A - \bar{\varepsilon} I_3) =$$

$$= P_A(\varepsilon) \cdot P_A(\bar{\varepsilon}) = \left(\varepsilon^3 - \text{Tr } A \varepsilon^2 + \frac{(\text{Tr } A)^2}{2} \varepsilon - 1 \right) \cdot \left(\bar{\varepsilon}^3 - \text{Tr } A \bar{\varepsilon}^2 + \frac{(\text{Tr } A)^2}{2} \bar{\varepsilon} - 1 \right) =$$

$$= (\text{Tr } A)^2 \varepsilon \cdot \bar{\varepsilon} \left(\varepsilon - \frac{\text{Tr } A}{2} \right) \left(\bar{\varepsilon} - \frac{\text{Tr } A}{2} \right) = (\text{Tr } A)^2 \cdot \left(\varepsilon \bar{\varepsilon} - \varepsilon \frac{\text{Tr } A}{2} - \bar{\varepsilon} \frac{\text{Tr } A}{2} + \frac{(\text{Tr } A)^2}{4} \right)$$

$$= (\text{Tr } A)^2 \cdot \left(1 - \frac{\text{Tr } A}{2} (\varepsilon + \bar{\varepsilon}) + \frac{(\text{Tr } A)^2}{4} \right) = (\text{Tr } A)^2 \cdot \left(1 + \frac{\text{Tr } A}{2} + \frac{(\text{Tr } A)^2}{4} \right) \geq (\text{Tr } A)^3$$

$$1 + \frac{\text{Tr } A}{2} + \frac{(\text{Tr } A)^2}{4} \geq \text{Tr } A$$

$$4 + 2 \text{Tr } A + (\text{Tr } A)^2 \geq 4 \text{Tr } A$$

$$(\text{Tr } A)^2 - 2 \text{Tr } A + 4 \geq 0$$

$$(\text{Tr } A - 1)^2 + 3 \geq 0$$

SP.234. In ΔABC , I – incentre; A' - is the intersection between AI and circumcircle of ΔBIC ; B' - is the intersection between BI and circumcircle of ΔAIC ; C' - is the intersection between CI and circumcircle of ΔAIB . Prove that:

$$\frac{IA}{IA'} + \frac{IB}{IB'} + \frac{IC}{IC'} \geq 2 \left(1 - \frac{r}{R} \right)$$

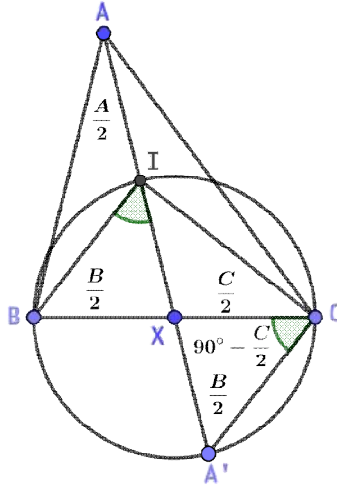
Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

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Let AA' intersect BC at X . In $\triangle ABX$, $\angle AXB = 180^\circ - \left(B + \frac{A}{2}\right)$ using $\triangle BIX$, $\angle BIX =$

$$= 180^\circ - \left(\frac{B}{2} + 180^\circ - \left(B + \frac{A}{2}\right)\right) = \frac{A+B}{2} = 90^\circ - \frac{C}{2} \Rightarrow$$

$$\Rightarrow \angle BCA' = \angle BIX = 90^\circ - \frac{C}{2}$$

$$\therefore \angle ICA' = 90^\circ. \text{ Also, } \angle IA'C = \angle IBC = \frac{B}{2}$$

Using $\triangle IA'C$, $\sin \frac{B}{2} = \frac{IC}{IA'} = \frac{r}{IA' \sin \frac{C}{2}} \Rightarrow \frac{r}{4R \sin \frac{A}{2}} = \frac{r}{IA'} \Rightarrow IA' = 4R \sin \frac{A}{2} \Rightarrow$

$$\Rightarrow \frac{IA}{IA'} \stackrel{(a)}{=} \frac{r}{4R \sin^2 \frac{A}{2}} \text{ Similarly, } \frac{IB}{IB'} \stackrel{(b)}{=} \frac{r}{4R \sin^2 \frac{B}{2}} \text{ and } \frac{IC}{IC'} \stackrel{(c)}{=} \frac{r}{4R \sin^2 \frac{C}{2}}$$

$$(a)+(b)+(c) \Rightarrow LHS = \frac{r}{4R} \sum \csc^2 \frac{A}{2} = \frac{r}{4R} \sum \frac{bc(s-a)}{r^2 s} =$$

$$= \frac{s(s^2 + 4Rr + r^2) - 12Rrs}{4Rrs} = \frac{s^2 - 8Rr + r^2}{4Rr} \stackrel{\text{Gerretsen}}{\geq} \frac{8Rr - 4r^2}{4Rr} =$$

$$= 2 - \frac{r}{R} = 2 \left(1 - \frac{r}{2R}\right)$$

SP.235. Let be $A(z_1); B(z_1); C(z_3); z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$;

$$|z_1| = |z_2| = |z_3|; AB = c; BC = a; CA = b.$$

If $(b+c)z_B z_C + (c+a)z_C z_A + (a+b)z_A z_B = 0$ then $AB = BC = CA$.

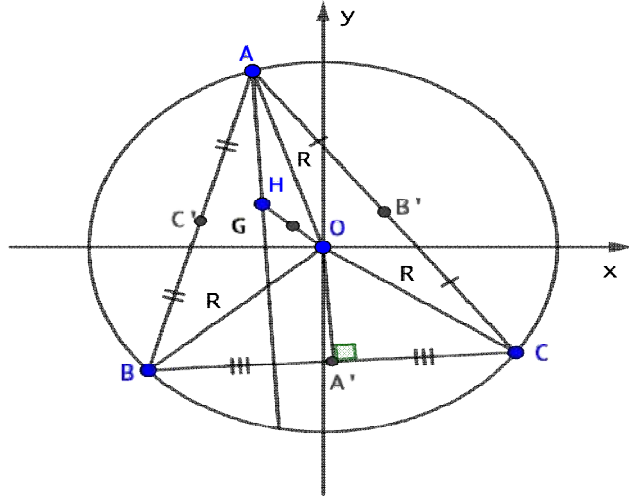
Proposed by Marian Ursărescu – Romania

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Solution by Khaled Abd Imouti-Damascus-Syria



Let be $A(z_1), B(z_2), C(z_3), z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$

$|z_1| = |z_2| = |z_3|, AB = c, BC = a, CA = b$

If $(b + c) \cdot z_B \cdot z_C + (c + a) \cdot z_C \cdot z_A + (a + b) \cdot z_A \cdot z_B = 0$

Then $AB = BC = CA$.

$|z_A| = |z_B| = |z_C| = R$ (R radius of circle)

$z_A \cdot \bar{z}_A = R^2, z_B \cdot \bar{z}_B = R^2, z_C \cdot \bar{z}_C = R^2$

$z_A = \frac{R^2}{\bar{z}_A}, z_B = \frac{R^2}{\bar{z}_B}, z_C = \frac{R^2}{\bar{z}_C}$

$$(b + c) \cdot \left(\frac{R^2}{\bar{z}_B} \cdot \frac{R^2}{\bar{z}_C} \right) + (c + a) \left(\frac{R^2}{\bar{z}_C} \cdot \frac{R^2}{\bar{z}_A} \right) + (a + b) \left(\frac{R^2}{\bar{z}_A} \cdot \frac{R^2}{\bar{z}_B} \right) = 0$$

$$(b + c) \cdot \left(\frac{1}{\bar{z}_B \cdot \bar{z}_C} \right) + (c + a) \left(\frac{1}{\bar{z}_C \cdot \bar{z}_A} \right) + (a + b) \left(\frac{1}{\bar{z}_A \cdot \bar{z}_B} \right) = 0$$

$$\frac{(b + c)}{z_B \cdot z_C} + \frac{(c + a)}{z_C \cdot z_A} + \frac{a + b}{z_A \cdot z_B} = 0 \quad (z_A \cdot z_B \cdot z_C \neq 0)$$

$$(b + c)z_A + (c + a)z_B + (a + b) \cdot z_C = 0$$

$$(b + c + a - a)z_A + (c + a + b - b)z_B + (a + b + c - c)z_C = 0$$

$$2p(z_A + z_B + z_C) = a \cdot z_A + b \cdot z_B + c \cdot z_C$$

$$6p \cdot z_G = a \cdot z_A + b \cdot z_B + c \cdot z_C$$

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$$z_G = \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{6p}$$

Suppose H is orthocenter:

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \quad (*)$$

$$\overrightarrow{OH} = \overrightarrow{OA} + (\overrightarrow{OB} + \overrightarrow{OC}) \Rightarrow \overrightarrow{ON} - \overrightarrow{OA} = 2\overrightarrow{OA}'$$

$$\text{So: } \overrightarrow{AH} = 2 \cdot \overrightarrow{OA}', (AH) \parallel (OA')$$

$$\text{but } OA' \perp BC \Rightarrow AH \perp BC$$

in a similar way, $BH \perp AC$ and $CH \perp AB$

so, H is orthocenter in triangle ΔABC

from (*): $\overrightarrow{OH} = 3\overrightarrow{OG}$, so:

O, H, G is collinear.

$$\text{not: } z_H = 3 \cdot z_G$$

$$z_G - z_H = \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{6p} - \frac{z_A + z_B + z_C}{1}$$

$$z_G - z_H = \frac{(a - 6p)z_A + (6p - b)z_B + (c - 6p)z_C}{6p}$$

$$-z_G + z_H = \frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{6p}$$

$$6p(z_H - z_G) = (6p - a)z_A + (6p - b)z_B + (6p - c)z_C$$

$$\frac{6p(z_H - z_G)}{16p} = \frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{16p}$$

$$\frac{3}{9}(z_H - z_G) = \frac{1}{16p} [(6p - a)z_A + (6p - b)z_B + (6p - c)z_C]$$

$$\text{but: } (6p - a)z_A + (6p - b)z_B + (6p - c)z_C = 6p(z_A + z_B + z_C) - (az_A + bz_B + cz_C)$$

$$(6p - a)z_A + (6p - b)z_B + (6p - c)z_C$$

$$= 18p \cdot \left(\frac{z_A + z_B + z_C}{3} \right) - (a \cdot z_A + b \cdot z_B + c \cdot z_C)$$

$$(6p - a) \cdot z_A + (6p - b)z_B + (6p - c)z_C = 18p \cdot z_G - (a \cdot z_A + b \cdot z_B + c \cdot z_C)$$

$$\frac{(6p - a)z_A + (6p - b)z_B + (6p - c)z_C}{16p} = \frac{18p \cdot z_G}{16p} - \frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{16p}$$

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$$= \frac{9}{4}z_G - \frac{1}{8} \left(\frac{a \cdot z_A + b \cdot z_B + c \cdot z_C}{2p} \right)$$

$$= \frac{9}{4}z_G - \frac{1}{8} \cdot z_H$$

$$\text{So: } \frac{3}{8}(z_H - z_G) = \frac{9}{4}z_G - \frac{1}{8}z_H$$

$$\frac{3}{8}z_H + \frac{1}{8}z_H = \frac{9}{4}z_G + \frac{3}{8}z_G$$

$$\frac{1}{2}z_H = \frac{21}{8}z_G \Rightarrow z_H = \frac{21}{4}z_G$$

$$\text{and: } z_H = 3 \cdot z_G$$

$$\frac{21}{4}z_G - 3z_G = 0$$

$$\frac{9}{4}z_G = 0 \Rightarrow z_G = 0$$

$$G \equiv O$$

So ΔABC is equilateral triangle.

SP.236. In ΔABC the following relationship holds:

$$m_a + m_b + m_c \leq 3R^2 \left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow x^2 + y^2 + z^2 \geq xy + yz + zx \Rightarrow x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{2} - \frac{(x^2+y^2+z^2)}{2}$$

$$\Rightarrow 3(x^2 + y^2 + z^2) \geq (x+y+z)^2 \Leftrightarrow \text{put } x = m_a, y = m_b, z = m_c$$

$$\Rightarrow 3(m_a^2 + m_b^2 + m_c^2) \geq (m_a + m_b + m_c)^2 \Rightarrow \frac{9}{4}(a^2 + b^2 + c^2) \geq (m_a + m_b + m_c)^2$$

$$\Rightarrow m_a + m_b + m_c \leq \frac{3}{2}\sqrt{a^2 + b^2 + c^2} \stackrel{\text{Leibnitz}}{\leq} \frac{3}{2}\sqrt{9R^2} = \frac{9R}{2}$$

$$\Leftrightarrow \text{Need to show} \Leftrightarrow \frac{9R}{2} \leq 3R^2 \left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \right) \Rightarrow \Omega = \frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \geq \frac{3}{2R}$$

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$$\Leftrightarrow \Omega \stackrel{AM-GM}{\geq} 3 \left(\frac{r_a r_b r_c}{a^2 b^2 c^2} \right)^{\frac{1}{3}} = 3 \left(\frac{s^2 r}{16R^2 \Delta^2} \right)^{\frac{1}{3}} = 3 \left(\frac{s^2 r}{16R^2 r^2 s^2} \right)^{\frac{1}{3}} = 3 \left(\frac{1}{16R^2 r} \right)^{\frac{1}{3}}$$

$$\Leftrightarrow 3 \left(\frac{1}{16R^2 r} \right)^{\frac{1}{3}} \geq \frac{3}{2R} \Rightarrow \frac{1}{16R^2 r} \geq \frac{1}{8R^3} \Rightarrow \frac{R}{r} \geq 2 \Rightarrow R \geq 2r \Rightarrow (\text{Euler * true})$$

$$\Leftrightarrow m_a + m_b + m_c \leq 3R^2 \left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \right) \quad (\text{Proved})$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{r_a}{a^2} = \sum \frac{\left(\frac{1}{a}\right)^2}{\frac{1}{r_a}} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\sum \frac{1}{a}\right)^2}{\sum \frac{1}{r_a}} = \frac{\left(\frac{\sum ab}{4Rrs}\right)^2}{\frac{1}{r}} = \frac{r(\sum ab)^2}{16R^2 r^2 s^2} = \frac{(s^2 + 4Rr + r^2)^2}{16R^2 r s^2}$$

$$\Rightarrow 3R^2 \left(\sum \frac{r_a}{a^2} \right) \geq \frac{3[s^4 + 2s^2(4Rr + r^2) + (4Rr + r^2)^2]}{16rs^2} \stackrel{?}{\geq} 4R + r$$

$$\Leftrightarrow 3s^4 + 6s^2(4Rr + r^2) + 3(4Rr + r^2)^2 \stackrel{?}{\underset{(1)}}{\geq} 16s^2(4Rr + r^2)$$

$$\text{Now, LHS of (1)} \stackrel{\text{Gerretsen}}{\geq} 3s^2(16Rr - 5r^2) + 6s^2(4Rr + r^2) + 3(4Rr + r^2)^2 \stackrel{?}{\geq}$$

$$\geq 16s^2(4Rr + r^2) \Leftrightarrow s^2(8Rr - 16r^2) + 3(4Rr + r^2)^2 \stackrel{?}{\underset{(2)}}{\geq} 9r^2 s^2$$

$$\text{Now, LHS of (2)} \stackrel{\text{Gerretsen}}{\underset{(i)}}{\geq} (16Rr - 5r^2)(8Rr - 16r^2) + 3(4Rr + r^2)^2 \text{ and}$$

$$\text{RHS of (2)} \stackrel{\text{Gerretsen}}{\underset{(ii)}}{\leq} 9r^2(4R^2 + 4Rr + 3r^2)$$

(i), (ii) \Rightarrow in order to prove (2), it suffices to show:

$$(16Rr - 5r^2)(8Rr - 16r^2) + 3(4Rr + r^2)^2 \geq 9r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 5R^2 - 11Rr + 2r^2 \geq 0 \Leftrightarrow (R - 2r)(5R - r) \geq 0$$

$$\rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (2) \Rightarrow (1) \text{ is true} \Rightarrow 3R^2 \left(\sum \frac{r_a}{a^2} \right) \geq 4R + r \stackrel{\text{Bager}}{\geq} \sum m_a$$

$$\therefore \sum m_a \leq 3R^2 \left(\sum \frac{r_a}{a^2} \right) \quad (\text{Proved})$$

Solution 3 by Bogdan Fustei-Romania

We know that: $m_a + m_b + m_c \leq r_a + r_b + r_c = 4R + r \leq \frac{9}{2}R$

$(4R + r \leq \frac{9}{2}R \Rightarrow 8R + 2r \leq 9R \Rightarrow 2r \leq R - \text{Euler's inequality})$

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$$a^2 = 2R \frac{h_b h_c}{h_a} \text{ (and the analogs)} \Rightarrow \frac{r_a}{a^2} = \frac{h_a r_a}{2R h_b h_c} \text{ (and the analogs)}$$

$$\begin{aligned} 3R^2 \left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \right) &= 3R^2 \cdot \frac{1}{2R} \left(\frac{h_a r_a}{h_b h_c} + \frac{h_b r_b}{h_a h_c} + \frac{h_c r_c}{h_a h_b} \right) = \\ &= \frac{3}{2} R \cdot \left(\frac{h_a r_a}{h_b h_c} + \frac{h_b r_b}{h_a h_c} + \frac{h_c r_c}{h_a h_b} \right). \text{ We will prove that:} \end{aligned}$$

Using the inequality between the arithmetic mean and the geometric mean:

$$\frac{h_a r_a}{h_b h_c} + \frac{h_b r_b}{h_a h_c} + \frac{h_c r_c}{h_a h_b} \geq 3 \cdot \sqrt[3]{\frac{h_a r_a h_b r_b h_c r_c}{h_a h_b h_b h_c h_a h_c}} = 3 \sqrt[3]{\frac{r_a r_b r_c}{h_a h_b h_c}}$$

$$h_a = \frac{2S}{a}; h_b = \frac{2b}{b}; h_c = \frac{2b}{c}; abc = 4RS; S^2 = r r_a r_b r_c$$

$$h_a h_b h_c = \frac{4S \cdot 2S^2}{4RS} = \frac{2S^2}{R} = \frac{2r}{R} \cdot r_a r_b r_c \Leftrightarrow h_a h_b h_c = \frac{2r}{R} r_a r_b r_c$$

$$\Rightarrow \frac{R}{2r} = \frac{r_a r_b r_c}{h_a h_b h_c}; R \geq 2r \text{ (Euler's inequality)} \Rightarrow \frac{r_a r_b r_c}{h_a h_b h_c} \geq 1$$

$$\text{So, we have: } \frac{h_a r_a}{h_b h_c} + \frac{h_b r_b}{h_a h_c} + \frac{h_c r_c}{h_a h_b} \geq 3 \sqrt[3]{\frac{r_a r_b r_c}{h_a h_b h_c}} = 3 \sqrt[3]{\frac{R}{2r}} \geq 3$$

$$\text{So, } 3R^2 \left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2} \right) \geq \frac{3}{2} R \cdot 3 = \frac{9}{2} R \geq 4R + r \geq \sum m_a$$

SP.237. In $\triangle ABC$ the following relationship holds:

$$3r(m_a + m_b + m_c) \leq Rs \left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{We have: } \frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{s^2 + (4R+r)^2}{4Rs} \quad (1)$$

$$\text{From Doucet's inequality: } (4R + r)^2 \geq 3s^2 \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow \frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \frac{s}{R} \Rightarrow \text{we must show: } 3r(m_a + m_b + m_c) \leq s^2 \quad (3)$$

$$\text{But } \frac{m_a}{h_a} \leq \frac{R}{2r} \Rightarrow m_a \leq \frac{R}{2r} h_a \Rightarrow m_a + m_b + m_c \leq \frac{R}{2r} (h_a + h_b + h_c) \quad (4)$$

$$\text{From (3)+(4) we must show: } \frac{3}{2} R (h_a + h_b + h_c) \leq s^2 \quad (5)$$

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$$\text{But } h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \quad (6)$$

From (5)+(6) we must show: $\frac{3}{4}(s^2 + r^2 + 4Rr) \leq s^2 \Leftrightarrow 3s^2 + 3r^2 + 12Rr \leq 4s^2 \Leftrightarrow$

$$s^2 \geq 12Rr + 3r^2 \quad (7)$$

From Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2 \quad (8)$

From (7)+(8) we must show: $16Rr - 5r^2 \geq 12Rr + 3r^2$

$$\Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r, \text{ true (Euler)}$$

Solution 2 by Bogdan Fustei-Romania

In ΔABC we have the following relationship:

$$3r(m_a + m_b + m_c) \leq Rs \left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \right)$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{r_a h_a}{2S} + \frac{r_b h_b}{2S} + \frac{r_c h_c}{2S} = \frac{r_a h_a + r_b h_b + r_c h_c}{2S}$$

$$2S = h_a \cdot a = h_b \cdot b = h_c \cdot c, h_a = \left(1 + \frac{b+c}{a} \right) \cdot r \text{ (and the analogs)}$$

$$\frac{b+c}{a} = \frac{r_a + h_a}{r_a} \text{ (and the analogs); } r_a h_a = \left(1 + \frac{b+c}{a} \right) r_a r$$

$$r_a h_a = \left(1 + \frac{r_a + h_a}{r_a} \right) r_a r = (2r_a + h_a) r \text{ (and the analogs)}$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{[2(r_a + r_b + r_c) + h_a + h_b + h_c] \cdot r}{2S}; S = s \cdot r$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{[2(r_a + r_b + r_c) + h_a + h_b + h_c]}{2s} \cdot Rs$$

$$Rs \left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \right) = \frac{R}{2} [(2r_a + 2r_b + 2r_c) + h_a + h_b + h_c]$$

The inequality from enunciation becomes:

$$3r(m_a + m_b + m_c) \leq \frac{R}{2} [2(r_a + r_b + r_c) + h_a + h_b + h_c]$$

$$6r(m_a + m_b + m_c) \leq 2R(r_a + r_b + r_c) + R(h_a + h_b + h_c)$$

$$\frac{R}{2r} \geq \frac{m_a}{h_a} \text{ (and the analogs)} \Rightarrow R \cdot h_a \geq 2r m_a \Rightarrow R(h_a + h_b + h_c) \geq 2r \sum m_a$$

We will prove that $4R(m_a + m_b + m_c) \leq 2R(r_a + r_b + r_c)$

$$m_a + m_b + m_c \leq r_a + r_b + r_c = 4R + r - \text{true.}$$

$$4r \leq 2R \Rightarrow 2r \leq R \text{ (Euler's inequality)}$$

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So, $4r(m_a + m_b + m_c) \leq 2R(r_a + r_b + r_c)$ is true.

$2r(m_a + m_b + m_c) \leq R(h_a + h_b + h_c)$ is true.

Finally, the inequality from enunciation is proved.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Firstly, } \sum ar_a &= \sum 4R \sin \frac{A}{2} \cos \frac{A}{2} s \tan \frac{A}{2} = 2Rs \sum 2 \sin^2 \frac{A}{2} = 2Rs \sum (1 - \cos A) \\ &= 2Rs \left(3 - 1 - \frac{r}{R} \right) \stackrel{(1)}{=} 2s(2R - r) \end{aligned}$$

$$\text{Now, } Rs \sum \frac{r_a}{a} = Rs \sum \frac{ra^2}{ar_a} \stackrel{\text{Bergstrom}}{\geq} \frac{Rs(4R+r)^2}{\sum ar_a}$$

$$\stackrel{\text{by (1)}}{=} \frac{Rs(4R+r)^2}{2s(2R-r)} \stackrel{\text{Bager}}{\geq} \frac{R(4R+r)(\sum m_a)}{2(2R-r)} \stackrel{?}{\geq} 3r \sum m_a \Leftrightarrow 4R^2 + Rr \stackrel{?}{\geq} 12Rr - 6r^2$$

$$\Leftrightarrow 4R^2 - 11Rr + 6r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R-2r)(4R-3r) \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \therefore 3r \sum m_a \leq Rs \sum \frac{r_a}{a} \text{ (proved)}$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$\text{We have: } r_a + r_b + r_c = 4R + r$$

$$m_a + m_b + m_c \leq r_a + r_b + r_c; ar_a + br_b + cr_c =$$

$$= s \left(a \tan \left(\frac{A}{2} \right) + b \tan \left(\frac{B}{2} \right) + c \tan \left(\frac{C}{2} \right) \right) =$$

$$= 2s(2R - r). \text{ Using Schwarz's inequality, we have:}$$

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{r_a^2}{ar_a} + \frac{r_b^2}{br_b} + \frac{r_c^2}{cr_c} \geq \frac{(r_a + r_b + r_c)^2}{ar_a + br_b + cr_c} = \frac{(4R + r)^2}{2s(2R - r)}$$

$$\rightarrow \text{RHS} \geq \frac{R(4R + r)^2}{2(2R - r)}$$

$$\text{We must show that: } \frac{R(4R+r)^2}{2(2R-r)} \geq 3r(4R + r) \geq \text{LHS}$$

$$R(4R + r) \geq 6r(2R - r)$$

$$4R^2 - 11Rr + 6r^2 \geq 0$$

$$(4R - 3r)(R - 2r) \geq 0$$

It is true because: $R \geq 2r \rightarrow 4R - 3r \geq 8r - 3r = 5r > 0$ (proved)

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SP.238. Let a, b, c be positive real numbers such that

$\min\{b + c, c + a, a + b\} > \max\{a, b, c\}$. Prove that:

$$\frac{a}{b + c - a} + \frac{b}{c + a - b} + \frac{c}{a + b - c} \geq 3 \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $\min\{a + b, b + c, c + a\} > \max\{a, b, c\}$, we have:

$a + b > c, b + c > a, c + a > b$ consider because $2(a^2 + b^2 + c^2)(ab + bc + ca) \geq$

$$\geq 3((ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2) \Rightarrow$$

$$\Rightarrow 4((a^2 + b^2 + c^2)(ab + bc + ca))^2 \geq$$

$$\geq 9((ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2)^2$$

$$\Rightarrow 4(a^2 + b^2 + c^2)^3(ab + bc + ca) \geq$$

$$\geq 9((ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2)^2$$

$$\Rightarrow \frac{4(a^2 + b^2 + c^2)^4}{((ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2)^2} \geq \frac{9(a^2 + b^2 + c^2)}{(ab + bc + ca)}$$

$$\Rightarrow \frac{2(a^2 + b^2 + c^2)^2}{(ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2} \geq 3 \sqrt{\frac{(a^2 + b^2 + c^2)}{(ab + bc + ca)}}$$

$$\Rightarrow \frac{4(a^2 + b^2 + c^2)^2}{2(ab)^2 + (bc)^2 + (ca)^2 + a^2bc + ab^2c + abc^2} \geq 3 \sqrt{\frac{(a^2 + b^2 + c^2)}{(ab + bc + ca)}}$$

$$\Rightarrow 4 \left(\frac{a^2}{(b + c)^2} + \frac{b^2}{(c + a)^2} + \frac{c^2}{(a + b)^2} \right) \geq 3 \sqrt{\frac{(a^2 + b^2 + c^2)}{(ab + bc + ca)}}$$

$$\Rightarrow \left(\frac{2a}{b + c} \right)^2 + \left(\frac{2b}{c + a} \right)^2 + \left(\frac{2c}{a + b} \right)^2 \geq 3 \sqrt{\frac{(a^2 + b^2 + c^2)}{(ab + bc + ca)}}$$

$$\Rightarrow \left(\sqrt{\frac{a}{b + c - a}} \right)^2 + \left(\sqrt{\frac{b}{c + a - b}} \right)^2 + \left(\sqrt{\frac{c}{a + b - c}} \right)^2 \geq 3 \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}}$$

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$$\Rightarrow \frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3 \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \text{ ok. Therefore, it is to be true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\min\{b+c, c+a, a+b\} \geq \max\{a, b, c\} \Rightarrow b+c > a, c+a > b, a+b > c \\ \Rightarrow a, b, c \text{ are sides of a triangle.}$$

$$\sum \frac{a}{b+c-a} = \frac{1}{2} \sum \frac{a}{s-a} = \frac{1}{2} \sum \frac{a-s+s}{s-a} = \frac{1}{2} \left(-3 + \frac{s \sum (s-b)(s-c)}{sr^2} \right) \\ = \frac{1}{2} \left(-3 + \frac{\sum s(s^2 - s(b+c) + bc)}{r^2} \right) = \frac{1}{2} \left(-3 + \frac{3s^2 - 4s^2 + s^2 + 4Rr + r^2}{r^2} \right)$$

$$= \frac{1}{2} \left(-3 + \frac{4R+r}{r} \right) = \frac{2R-r}{r} \geq 3 \sqrt{\frac{\sum a^2}{\sum ab}} \Leftrightarrow \left(\frac{2R-r}{r} \right)^2 \geq \frac{18(s^2 - 4Rr - r^2)}{s^2 + 4Rr + r^2}$$

$$\Leftrightarrow (2R-r)^2(s^2 + 4Rr + r^2) \geq 18r^2(s^2 - 4Rr - r^2)$$

$$\Leftrightarrow (4R^2 - 4Rr - 8r^2)s^2 + (2R-r)^2(4Rr + r^2) + 18r^2(4Rr + r^2) \stackrel{(1)}{\geq} 9r^2s^2$$

$$\because 4R^2 - 4Rr - 8r^2 = 4(R+r)(R-2r) \stackrel{\text{Euler}}{\geq} 0 \therefore \text{LHS of (1)} \stackrel{\text{Gerretsen}}{\underset{(a)}{\geq}}$$

$$\geq (4R^2 - 4Rr - 8r^2)(16Rr - 5r^2) + (2R-r)^2(4R+r^2) + 18r^2(4Rr+r^2) \text{ and,}$$

$$\text{RHS of (1)} \stackrel{\text{Gerretsen}}{\underset{(a)}{\leq}} 9r^2(4R^2 + 4Rr + 3r^2)$$

(a), (b) \Rightarrow in order to prove (1), it suffices to prove:

$$(4R^2 - 4Rr - 8r^2)(16Rr - 5r^2) + (2R-r)^2(4Rr + r^2) + 18r^2(4Rr + r^2) \geq$$

$$\geq 9r^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow 20t^3 - 33t^2 - 18t + 8 \geq 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t-2)(20t^2 + 7(t-2) + 10) \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (1) \Rightarrow \text{proposed inequality} \\ \text{is true (Proved)}$$

SP.239 If a, b, c, d are sides in a cyclic quadrilateral, r_a, r_b, r_c, r_d - exradii, s - semiperimeter then:

$$\frac{a}{r_a^2} + \frac{b}{r_b^2} + \frac{c}{r_c^2} + \frac{d}{r_d^2} \geq \frac{32}{s}$$

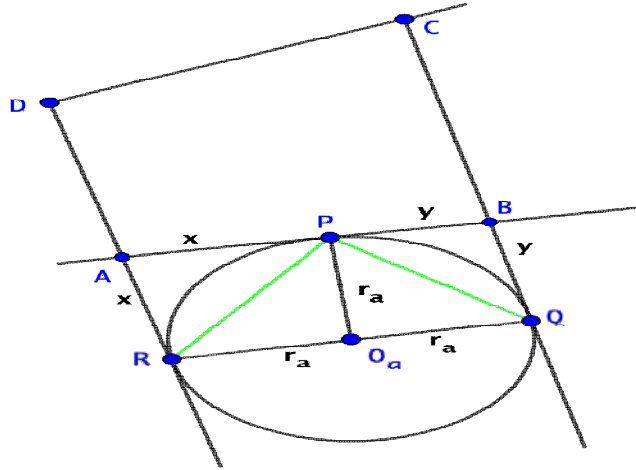
Proposed by Daniel Sitaru - Romania

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Solution by Soumava Chakraborty-Kolkata-India



Let $AB = a, BC = b, CD = c, DA = d$

$$\therefore \angle O_a P A + \angle O_a R A = 90^\circ + 90^\circ = 180^\circ$$

$$\therefore \angle P O_a R = 180^\circ - \angle P A R$$

($\because A P O_a R$ is a cyclic quadrilateral)

$$= 180^\circ - (180^\circ - A) = A$$

using cosine law on $\Delta P A R$ and $\Delta P O_a R$,

$$x^2 + x^2 - 2x^2 \cos(180^\circ - A) = r_a^2 + r_a^2 - 2r_a^2 \cos A \Rightarrow 2x^2 \cdot 2 \cos^2 \frac{A}{2} = 2r_a^2 \cdot 2 \sin^2 \frac{A}{2}$$

$$\Rightarrow x \stackrel{(1)}{=} r_a \tan \frac{A}{2}. \text{ Similarly, } \angle P O_a Q = B \text{ and } \therefore y \stackrel{(2)}{=} r_a \tan \frac{B}{2}$$

$$(1)+(2) \Rightarrow r_a \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) = x + y = a \Rightarrow \frac{1}{r_a} \stackrel{(a)}{=} \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a}$$

$$\text{Similarly, } \frac{1}{r_b} \stackrel{(b)}{=} \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{b}, \frac{1}{r_c} \stackrel{(c)}{=} \frac{\tan \frac{C}{2} + \tan \frac{D}{2}}{c} \text{ and } \frac{1}{r_d} \stackrel{(d)}{=} \frac{\tan \frac{D}{2} + \tan \frac{A}{2}}{d}$$

(a), (b), (c), (d) \Rightarrow LHS

$$= \frac{\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right)^2}{a} + \frac{\left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)^2}{b} + \frac{\left(\tan \frac{C}{2} + \tan \frac{D}{2} \right)^2}{c} + \frac{\left(\tan \frac{D}{2} + \tan \frac{A}{2} \right)^2}{d}$$

$$\stackrel{A-G}{\geq} 4 \left(\frac{\tan \frac{A}{2} \tan \frac{B}{2}}{a} + \frac{\tan \frac{B}{2} \tan \frac{C}{2}}{b} + \frac{\tan \frac{C}{2} \tan \frac{D}{2}}{c} + \frac{\tan \frac{D}{2} \tan \frac{A}{2}}{d} \right)$$

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$$\begin{aligned} & \stackrel{A-G}{\geq} 16 \sqrt[4]{\frac{\tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} \tan^2 \frac{D}{2}}{abcd}} \\ &= 16 \sqrt[4]{\frac{(s-a)(s-d)}{(s-b)(s-c)} \cdot \frac{(s-b)(s-a)}{(s-c)(s-d)} \cdot \frac{(s-c)(s-b)}{(s-a)(s-d)} \cdot \frac{(s-d)(s-c)}{(s-a)(s-b)} \cdot \frac{1}{abcd}} \\ &= 16 \sqrt[4]{\frac{1}{abcd}} \stackrel{?}{\geq} \frac{32}{s} \Leftrightarrow s^4 \stackrel{?}{\geq} 16abcd \Leftrightarrow (a+b+c+d)^4 \stackrel{?}{\geq} 256abcd \\ &\Leftrightarrow a+b+c+d \stackrel{?}{\geq} 4\sqrt[4]{abcd} \rightarrow \text{true by A-G (Hence proved)} \end{aligned}$$

SP.240 If a, b, c, d are sides in a cyclic quadrilateral, r_a, r_b, r_c, r_d – exradii, s – semiperimeter then:

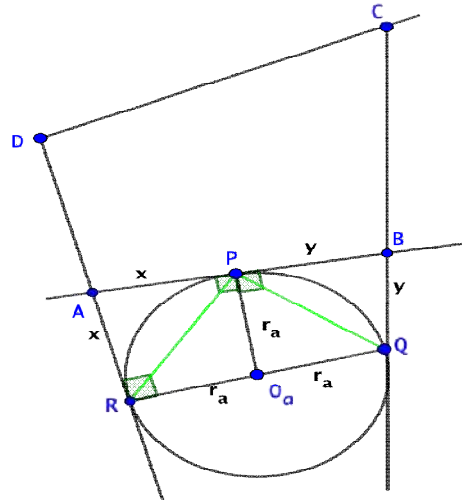
$$\frac{r_a^2}{a^3} + \frac{r_b^2}{b^3} + \frac{r_c^2}{c^3} + \frac{r_d^2}{d^3} \geq \frac{2}{s}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{r_a^2}{a^3} \geq \frac{2}{s} = \frac{4}{\sum a} \Leftrightarrow (\sum a) \left(\sum \frac{r_a^2}{a^3} \right) \stackrel{(i)}{\geq} 4$$

$$\text{Now, } (\sum a) \left(\sum \frac{r_a^2}{a^3} \right) \stackrel{CBS}{\geq} \left(\sum \frac{r_a}{a} \right)^2 \stackrel{?}{\geq} 4 \Leftrightarrow \sum \frac{r_a}{a} \stackrel{?}{\geq} 2 \quad (ii)$$



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Let $AB = a, BC = b, CD = c, DA = d$

$$\because \angle O_a PA + \angle O_a RA = 90^\circ + 90^\circ = 180^\circ$$

($\therefore APO_aR$ is a cyclic quadrilateral)

$$\therefore \angle PO_a R = 180^\circ - (180^\circ - A) = A$$

using cosine law on ΔPAR and $\Delta PO_a R$,

$$x^2 + x^2 - 2x^2 \cos(180^\circ - A) = r_a^2 + r_a^2 - 2r_a^2 \cos A \Rightarrow 2x^2 \cdot 2 \cos^2 \frac{A}{2} = 2r_a^2 \cdot 2 \sin^2 \frac{A}{2}$$

$$\Rightarrow x \stackrel{(1)}{=} r_a \tan \frac{A}{2}. \text{ Similarly, } \angle PO_a Q = B \text{ and } \therefore y \stackrel{(2)}{=} r_a \tan \frac{B}{2}$$

$$\begin{aligned} (1)+(2) \Rightarrow r_a \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) &= x + y = a \Rightarrow \frac{r_a}{a} = \frac{1}{\tan \frac{A}{2} + \tan \frac{B}{2}} = \frac{1}{\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} + \sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}} \\ &= \frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(s-a)(s-d) + (s-a)(s-b)} = \frac{\Delta}{(s-a)(a+c)} \Rightarrow \frac{r_a}{a} = \frac{\Delta}{(s-a)(a+c)} \end{aligned}$$

$$\text{Now, } \frac{r_b}{b} = \frac{1}{\tan \frac{B}{2} + \tan \frac{C}{2}} \text{ (similarly)}$$

$$= \frac{1}{\sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}} + \sqrt{\frac{(s-c)(s-b)}{(s-a)(s-d)}}} \stackrel{(b)}{=} \frac{\Delta}{(s-b)(b+d)}$$

$$\text{Again, } \frac{r_c}{c} = \frac{1}{\tan \frac{C}{2} + \tan \frac{D}{2}} = \frac{1}{\sqrt{\frac{(s-c)(s-b)}{(s-a)(s-d)}} + \sqrt{\frac{(s-d)(s-c)}{(s-a)(s-b)}}} \stackrel{(c)}{=} \frac{\Delta}{(s-c)(a+c)}$$

$$\text{Also, } \frac{r_d}{d} = \frac{1}{\tan \frac{D}{2} + \tan \frac{A}{2}} = \frac{1}{\sqrt{\frac{(s-d)(s-c)}{(s-a)(s-b)}} + \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}} \stackrel{(d)}{=} \frac{\Delta}{(s-d)(b+d)}$$

$$(a)+(b)+(c)+(d) \Rightarrow$$

$$\begin{aligned} \sum \frac{r_a}{a} &= \frac{\Delta}{a+c} \left(\frac{1}{s-a} + \frac{1}{s-c} \right) + \frac{\Delta}{b+d} \left(\frac{1}{s-b} + \frac{1}{s-d} \right) = \\ &= \frac{\Delta(b+d)}{(s-a)(s-c)(a+c)} + \frac{\Delta(a+c)}{(s-b)(s-d)(b+d)} \end{aligned}$$

$$\stackrel{A-G}{\geq} 2 \sqrt{\frac{\Delta^2}{(s-a)(s-b)(s-c)(s-d)}} = 2 \sqrt{\frac{\Delta^2}{\Delta^2}} = 2$$

$\Rightarrow (ii) \Rightarrow (i) \Rightarrow$ given inequality is true (Proved)

UP.226. Find all positive real numbers (x, y, z) such that:

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$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x^3y + y^3z + z^3x = \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Ruangkhaw Chaoka-Chiangrai-Thailand

$$x^2 + y^2 + z^2 = 3 \quad (a)$$

$$x^3y + y^3z + z^3x = \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \quad (b)$$

Lemma. $\frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq 3 \quad (1) \text{ (hold at } x = y = z = 1)$

Proof. $\frac{2x}{y^2+z^2} = \frac{2x}{3-x^2} \stackrel{??}{\geq} x^2 \Leftrightarrow x^4 - 3x^2 + 2x = x(x-1)^2(x+2) \geq 0 \text{ true}$

(hold at $x = 1$)

Similarly: $\frac{2y}{z^2+x^2} \geq y^2 \text{ (holds at } y = 1)$ and $\frac{2z}{x^2+y^2} \geq z^2 \text{ (holds at } z = 1)$

$\therefore \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq x^2 + y^2 + z^2 = 3 \text{ (holds at } x = y = z = 1)$

(Vasile Cîrtoaje, 1992) $(x^2 + y^2 + z^2)^2 \geq 3(x^3y + y^3z + z^3x)$

(holds at $x^2 - xy + yz = y^2 - yz + zx = z^2 - zx + xy$)

$$\frac{1}{2} \left\{ \begin{aligned} & [((x^2 - xy + yz) - (y^2 - yz + zx))]^2 + [(y^2 - yz + zx) - (z^2 - zx + xy)]^2 + \\ & + [(z^2 - zx + xy) - (x^2 - xy + yz)]^2 \end{aligned} \right\} \geq 0 \quad (2)$$

(1), (2): $\frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \geq 3 = \frac{(x^2+y^2+z^2)^2}{3} \geq (x^3y + y^3z + z^3x)$

(holds at $x = y = z = 1$)

\therefore (b) is the hold point of inequality at $x = y = z = 1$

UP.227. If $x_n = \sum_{k=1}^n \sqrt[k+1]{1 + \frac{1}{k}}$; $n \in \mathbb{N}; n \geq 1$, then find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\log \left(\sum_{k=1}^n x_k^2 \right) - 3H_n \right)$$

Proposed by D.M. Băţineţu-Giurgiu, Daniel Sitaru – Romania

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Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n \rightarrow \infty} \left(\ln \left(\sum_{k=1}^n x_k^2 \right) - 3H_n \right) =$$

$$= \lim_{n \rightarrow \infty} \ln \left(\sum_{k=1}^n x_k^2 \right) - 3 \ln n = 3 \lim_{n \rightarrow \infty} (H_n - \ln n) \quad (1)$$

$$\lim_{n \rightarrow \infty} (H_n - \ln n) = \gamma \quad (2)$$

$$\lim_{n \rightarrow \infty} \ln \left(\sum_{k=1}^n x_k^2 \right) - \ln n^3 = \lim_{n \rightarrow \infty} \ln \left(\frac{\sum_{k=1}^n x_k^2}{n^3} \right) = \ln \left(\lim_{n \rightarrow \infty} \left(\frac{\sum_{k=1}^n x_k^2}{n^3} \right) \right) \quad (3)$$

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n x_k^2}{n^3} \stackrel{C.S.}{=} \lim_{n \rightarrow \infty} \frac{x_{n+1}^2}{(n+1)^3 - n^3} = \lim_{n \rightarrow \infty} \frac{x_{n+1}^2}{3n^2 + 3n + 1} = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{n+1} \right)^2 \cdot \frac{(n+1)^2}{3n^2 + 3n + 1}$$

$$= \frac{1}{3} \left(\lim_{n \rightarrow \infty} \frac{x_{n+1}}{n+1} \right)^2 = \frac{1}{3} \left(\lim_{n \rightarrow \infty} \frac{x_n}{n} \right)^2 \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{x_n}{n} \stackrel{C.S.}{=} \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{n+1 - n} = \lim_{n \rightarrow \infty} \sqrt[n+2]{1 + \frac{1}{n+1}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1} \right)^{\frac{1}{n+2}} = 1 \quad (5)$$

$$\text{From (1)+(2)+(3)+(4)+(5)} \Rightarrow \Omega = \ln \frac{1}{3} - 3\gamma = -(\ln 3 + 3\gamma)$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n \rightarrow \infty} \left(\ln \left(\sum_{k=1}^n x_k^2 \right) - 3(H_n - \ln(n) + \ln(n)) \right) = -3\gamma + \lim_{n \rightarrow \infty} \ln \left(\frac{\sum_{k=1}^n x_k^2}{n^3} \right) \stackrel{\text{Stolz Cesaro}}{=} =$$

$$= -3\gamma + \lim_{n \rightarrow \infty} \ln \left(\frac{x_{n+1}^2}{3n^2 + 3n + 1} \right) = -3\gamma + 2 \lim_{n \rightarrow \infty} \ln \left(\frac{x_{n+1}}{\sqrt{3n^2 + 3n + 1}} \right) \stackrel{\text{Stolz Cesaro}}{=} =$$

$$= -3\gamma + 2 \lim_{n \rightarrow \infty} \ln \left(\frac{\sqrt[n+3]{1 + \frac{1}{n+1}}}{\frac{6n+6}{\sqrt{3n^2 + 9n + 7} + \sqrt{3n^2 + 3n + 1}}} \right) =$$

$$= -3\gamma + 2 \lim_{n \rightarrow \infty} \ln \left(\frac{1}{\frac{n \left(6 + \frac{6}{n} \right)}{n \left(\sqrt{3 + \frac{9}{n} + \frac{7}{n^2}} + \sqrt{3 + \frac{3}{n} + \frac{1}{n^2}} \right)}} \right) =$$

$$= -3\gamma + 2 \ln \left(\frac{1}{\sqrt{3}} \right) = -3\gamma - \ln 3 \Rightarrow \Omega = -3\gamma - \ln(3)$$

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Solution 3 by Ali Jaffal-Lebanon

By Bernoulli's inequality we have:

$$(1+n)^{\alpha} \leq 1 + \alpha n \text{ for } x \geq -1 \text{ and } 0 \leq \alpha \leq 1, \text{ so,}$$

$$\left(1 + \frac{1}{k}\right)^{\frac{1}{k+1}} \leq 1 + \frac{1}{k(k+1)} = 1 + \frac{1}{k} - \frac{1}{k+1}$$

$$\text{then } 1 \leq \left(1 + \frac{1}{k}\right)^{\frac{1}{k+1}} \leq 1 + \frac{1}{k} - \frac{1}{k+1} \text{ then } n \leq x_n \leq n + 1 - \frac{1}{n+1}$$

$$\text{so, } n \leq x_n \leq n + 1 \text{ and } n^2 \leq x_n^2 \leq (n+1)^2$$

$$\text{then } \sum_{k=1}^{k=n} k^2 = \sum_{k=1}^{k=n} x_k^2 \leq \sum_{k=1}^{k=n} (k+1)^2$$

$$\frac{n(n+1)(2n+1)}{6} \leq \sum_{k=1}^{k=n} x_k^2 \leq \frac{n(n+1)(2n+1)}{6} - 14(n+1)^2$$

$$\text{We have: } \lim_{n \rightarrow +\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}. \text{ Then } \lim_{n \rightarrow \infty} n^{\frac{1}{3}} \sum_{k=1}^{k=n} x_k^2 = \frac{1}{3}$$

$$\text{by } \Omega_n = \log\left(\sum_{k=1}^{k=n} x_k^2\right) - 3H_n = \log\left(\sum_{k=1}^{k=n} x_k^2\right) - 3 \ln n - 3\gamma + \varphi'(n)$$

$$\text{Since } H_n = \ln n + \gamma + \varphi(n) = \log\left(n^{\frac{1}{3}} \sum_{k=1}^{k=n} x_k^2\right) - 3\gamma + \varphi'(n)$$

$$\lim_{n \rightarrow +\infty} \Omega_n = \log\left(\frac{1}{3}\right) - 3\gamma$$

UP.228. If $f: \mathbb{R} \rightarrow (0, \infty)$, f continuous; $a, b \in \mathbb{R}$; $a \leq b$ then:

$$\int_a^b \int_a^b \left(\frac{(f^2(x) + f(y))(f^2(y) + f(x))}{(1+f(x))(1+f(y))} \right) dx dy \geq \left(\int_a^b f(x) dx \right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$f: \mathbb{R} \rightarrow (0, \infty)$ and f is continuous. Hence $f(x) > 0$ for all $x \in \mathbb{R}$

$$(f^2(x) + f(y)) \left(1 + \frac{1}{f(y)}\right) \stackrel{\text{CAUCHY}}{\geq} (1+f(x))^2 \text{ where } x, y \in \mathbb{R}$$

$$\Rightarrow \frac{f^2(x)+f(y)}{1+f(y)} \geq f(y) \left(\frac{1+f(x)}{1+f(y)}\right)^2 \text{ similarly, } \frac{f^2(y)+f(x)}{1+f(x)} \geq f(x) \left(\frac{1+f(y)}{1+f(x)}\right)^2$$

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$$\therefore \int_a^b \int_a^b \frac{(f^2(x) + f(y))(f^2(y) + f(x))}{(1 + f(x))(1 + f(y))} dx dy \geq \int_a^b \int_a^b f(y) \left(\frac{1 + f(x)}{1 + f(y)}\right)^2 \cdot f(x) \left(\frac{1 + f(y)}{1 + f(x)}\right)^2 dx dy$$

$$= \left(\int_a^b f(x) dx\right) \left(\int_a^b f(y) dy\right) = \left(\int_a^b f(x) dx\right)^2$$

(proved)

Solution 2 by Nassim Nicholas Taleb-USA

$$g = \frac{(f(x)^2 + f(y))(f(x) + f(y)^2)}{(f(x) + 1)(f(y) + 1)}. \text{ Rewrite, with } f(x) = v, f(y) = w:$$

$$g = \frac{(v^2 + w)(v + w^2)}{(1 + v)(1 + w)}, v, w > 0$$

We have $g \geq vw$, with identity for $v = w$

$$\text{So, } \int_a^b \int_a^b \left(\frac{(f(x)^2 + f(y))(f(x) + f(y)^2)}{(f(x) + 1)(f(y) + 1)}\right) dy dx \geq \int_a^b \int_a^b f(x) f(y) dx dy =$$

$$\left(\int_a^b f(x) dx\right) \left(\int_a^b f(y) dy\right) = \left(\int_a^b f(x) dx\right)^2 \text{ as required.}$$

Note:

$$\frac{(v^2 + w)(v + w^2)}{(1 + v)(1 + w)} - vw = \frac{(v - w)^2(v + w)}{(1 + v)(1 + w)} \geq 0$$

UP.229. Calculate the limit:

$$\Omega = \lim_{x \rightarrow \infty} \left(x^2 \int_x^{x + \frac{2}{x}} \arctan\left(\frac{1}{t}\right) dt \right)$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Florentin Vişescu-Romania

$$\exists \alpha \in \left(x; x + \frac{2}{x}\right) \alpha x$$

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$$\int_x^{x+\frac{2}{x}} \arctan \frac{1}{t} dt = \left(x + \frac{2}{x} - x\right) \arctan \alpha = \frac{2}{x} \arctan \alpha$$

$$0 < x < \alpha < x + \frac{2}{x} \quad \frac{1}{x} > \frac{1}{\alpha} > \frac{x}{x^2 + 2}$$

$$\arctan \frac{1}{x} > \arctan \frac{1}{\alpha} > \arctan \frac{x}{x^2 + 2}$$

$$2x \arctan \frac{1}{x} > \underbrace{2x \arctan \frac{1}{\alpha}}_{x^2 \int_x^{x+\frac{2}{x}} \arctan \frac{1}{t} dt} > 2x \arctan \frac{x}{x^2 + 2}$$

$$\lim_{n \rightarrow \infty} x^2 \int_x^{x+\frac{2}{x}} \arctan \frac{1}{t} dt = 2$$

$$\Omega = 2$$

Solution 2 by Ali Jaffal-Lebanon

$$\text{Let } I(x) = x^2 \int_x^{x+\frac{2}{x}} \arctan \left(\frac{1}{t}\right) dt, x > 0$$

We have: $x < t < x + \frac{2}{x}$ **then:**

$$x < t < \frac{x^2 + 2}{x}$$

$$\frac{x}{x^2 + 2} < \frac{1}{t} < \frac{1}{x}$$

But $\varphi(t) = \arctan t$ **is increasing on** $]-\infty, \infty[$

$$\text{So, } \arctan \left(\frac{x}{x^2+2}\right) < \arctan \left(\frac{1}{t}\right) < \arctan \left(\frac{1}{x}\right)$$

$$\text{then } 2x \arctan \left(\frac{x}{x^2+2}\right) < I(x) < 2x \arctan \left(\frac{1}{x}\right)$$

$$\text{we have } \lim_{u \rightarrow 0} \frac{\arctan(u)}{u} = 1$$

$$\text{so, } \lim_{x \rightarrow +\infty} 2x \arctan \left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} 2 \frac{\arctan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow 0} 2 \frac{\arctan(u)}{u} = 2$$

$$\text{and } \lim_{x \rightarrow +\infty} 2x \arctan \left(\frac{x}{x^2+2}\right) = \lim_{u \rightarrow 0^+} \frac{2}{u} \arctan \left(\frac{u}{2u^2+1}\right)$$

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$$= \lim_{u \rightarrow 0} \frac{2}{u} \times \frac{u}{2u^2 + 1} \times \frac{\arctan\left(\frac{u}{2u + 1}\right)}{\frac{2a}{2u^2 + 1}} = 2 \times 1 = 2$$

Then by sandwich theorem: $\lim_{n \rightarrow \infty} I(x) = \Omega = 2$

UP.230. If $a, b > 1$ then:

$$\left(\frac{2ab}{a+b}\right)^{\sqrt{ab} - \frac{a+b}{2}} \cdot (\sqrt{ab})^{\frac{a+b}{2} - \frac{2ab}{a+b}} \cdot \left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b} - \sqrt{ab}} \geq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$. We are to prove: $H^{G-A} \cdot G^{A-H} \cdot A^{H-G} \geq 1$

$$\Leftrightarrow \frac{H^G}{H^A} \cdot \frac{G^A}{G^H} \cdot \frac{A^H}{A^G} \geq 1 \Leftrightarrow \left(\frac{H}{A}\right)^G \left(\frac{G}{H}\right)^A \left(\frac{A}{G}\right)^H \geq 1$$

$$\Leftrightarrow \left(\frac{H}{A}\right)^G \left(\frac{A}{G}\right)^A \left(\frac{A}{G}\right)^H \geq 1 \left(\because \frac{G}{H} = \frac{A}{G} \text{ as } G^2 = AH\right)$$

$$\Leftrightarrow \left(\frac{H}{A}\right)^G \left(\frac{A}{\sqrt{AH}}\right)^{A+H} \geq 1 \left(\because G = \sqrt{AH}\right) \Leftrightarrow \left(\frac{H}{A}\right)^G \left(\frac{\sqrt{A}}{\sqrt{H}}\right)^{A+H} \stackrel{(1)}{\geq} 1$$

$$\text{Now, } \left(\frac{\sqrt{A}}{\sqrt{H}}\right)^{A+H} \stackrel{(2)}{\geq} \left(\frac{\sqrt{A}}{\sqrt{H}}\right)^{2G} \Leftrightarrow (A+H) \log \sqrt{\frac{A}{H}} \geq 2G \log \sqrt{\frac{A}{H}}$$

$$\Leftrightarrow \left(\log \sqrt{\frac{A}{H}}\right)(A+H-2G) \geq 0 \rightarrow \text{true} \because \log \sqrt{\frac{A}{H}} \geq 0 \text{ (as } \sqrt{\frac{A}{H}} \geq 1)$$

$$\text{and } A+H-2G \stackrel{A-G}{\geq} 2\sqrt{AH} - 2G = 2G - 2G = 0$$

$$\therefore \left(\frac{H}{A}\right)^G \left(\frac{\sqrt{A}}{\sqrt{H}}\right)^{A+H} \stackrel{\text{by (2)}}{\geq} \left(\frac{H}{A}\right)^G \left(\frac{\sqrt{A}}{\sqrt{H}}\right)^{2G} = \left(\frac{H}{A}\right)^G \left(\frac{A}{H}\right)^G = 1 \Rightarrow (1) \Rightarrow \text{given inequality is true}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = \ln x$ for all $x > 1$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2} < 0$ for all $x > 1$ hence f is a

concave function and $\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$

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$$\begin{aligned} & \frac{f(\sqrt{ab}) - f\left(\frac{2ab}{a+b}\right)}{\sqrt{ab} - \frac{2ab}{a+b}} \geq \frac{f\left(\frac{a+b}{2}\right) - f\left(\frac{2ab}{a+b}\right)}{\frac{a+b}{2} - \frac{2ab}{a+b}} \\ \Rightarrow & \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) f(\sqrt{ab}) + f\left(\frac{2ab}{a+b}\right) \left(\frac{2ab}{a+b} - \frac{a+b}{2}\right) \geq \\ & \geq \left(\sqrt{ab} - \frac{2ab}{a+b}\right) f\left(\frac{a+b}{2}\right) - \left(\sqrt{ab} - \frac{2ab}{a+b}\right) f\left(\frac{2ab}{a+b}\right) \\ \Rightarrow & \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) f(\sqrt{ab}) + \left(\sqrt{ab} - \frac{a+b}{2}\right) f\left(\frac{2ab}{a+b}\right) + \left(\frac{2ab}{a+b} - \sqrt{ab}\right) f\left(\frac{a+b}{2}\right) \geq 0 \\ & \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right) \ln(\sqrt{ab}) + \left(\sqrt{ab} - \frac{a+b}{2}\right) \ln\left(\frac{2ab}{a+b}\right) + \left(\frac{2ab}{a+b} - \sqrt{ab}\right) \ln\left(\frac{a+b}{2}\right) \geq 0 \\ \Rightarrow & \left(\frac{2ab}{a+b}\right)^{\sqrt{ab} - \frac{a+b}{2}} \cdot (\sqrt{ab})^{\frac{a+b}{2} - \frac{2ab}{a+b}} \cdot \left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b} - \sqrt{ab}} \geq 1 \quad (\text{Proved}) \end{aligned}$$

Solution 3 by Marian Ursărescu Romania

Let $\frac{2ab}{a+b} = x, \sqrt{ab} = y, \frac{a+b}{2} = z \Rightarrow x < y < z$. We must show: $x^{y-z} \cdot y^{z-x} \cdot z^{x-y} \geq 1 \Leftrightarrow$

$$\ln(x^{y-z} \cdot y^{z-x} \cdot z^{x-y}) \geq 0 \Leftrightarrow (y-z) \ln x + (z-x) \ln y + (x-y) \ln z \geq 0 \quad (1)$$

Theorem: Let $f: I \rightarrow \mathbb{R}, I \subset \mathbb{R}, I$ - internal f its concave $\Leftrightarrow \forall x, y, z \in I$ with: $x < y < z$

$$\text{we have: } \frac{f(x)-f(y)}{x-y} \geq \frac{f(y)-f(z)}{y-z}. \text{ In our case } f(x) = \ln x$$

$$\text{From theorem } \Rightarrow (y-z)f(x) + (z-x)f(y) + (x-y)f(z) \geq 0$$

Let $f(x) = \ln x \Rightarrow (1)$ it's true.

UP.231. Prove that for any acute triangle ABC the following inequality holds:

$$\sqrt{\tan A} + \sqrt{\tan B} + \sqrt{\tan C} + \sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \geq 3\sqrt[4]{3} + \frac{3}{\sqrt[4]{3}}$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Let } \sin A = x, \sin B = y, \sin C = z$$

$$\cos A = a, \cos B = b, \cos C = c$$

We have:

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$$1. x + y + z \leq \frac{3\sqrt{3}}{2}$$

$$2. a + b + c \leq \frac{3}{2}$$

$$3. \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geq 3\sqrt{3}$$

$$4. \frac{a}{x} + \frac{b}{y} + \frac{c}{z} \geq \sqrt{3}$$

$$\text{Consider } \frac{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2}{3} \leq x + y + z \leq \frac{3\sqrt{3}}{2} \Rightarrow \sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3\sqrt[4]{3}}{\sqrt{2}}$$

$$\text{and } \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{3} \leq a + b + c \leq \frac{3}{2} \Rightarrow \sqrt{a} + \sqrt{b} + \sqrt{c} \leq \frac{3}{\sqrt{2}}$$

$$\text{by using a method above, we have: } \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} \geq 3\sqrt[4]{3} \text{ and } \sqrt{\frac{a}{x}} + \sqrt{\frac{b}{y}} + \sqrt{\frac{c}{z}} \geq \frac{3}{\sqrt[4]{3}}$$

$$\text{Hence } \sqrt{\tan A} + \sqrt{\tan B} + \sqrt{\tan C} + \sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \geq 3\sqrt[4]{3} + \frac{3}{\sqrt[4]{3}}$$

Therefore, it is true.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\text{Let } f(x) = \sqrt{\tan x} + \sqrt{\cot x}, \left(0 < x < \frac{\pi}{2}\right)$$

$$\rightarrow f'(x) = \frac{1}{2} \csc x \sec x (\sqrt{\tan x} - \sqrt{\cot x})$$

$$\rightarrow f''(x) = \frac{1}{4} \left(\sqrt{\cot x} \left[(\csc x \sec x)^2 + \frac{2 \cos 2x}{(\cos x \sin x)^2} \right] + \sqrt{\tan x} \left[(\csc x \sec x)^2 - \frac{2 \cos 2x}{(\cos x \sin x)^2} \right] \right)$$

$$= \frac{1}{4} \left((\csc x \sec x)^2 [\sqrt{\cot x} + \sqrt{\tan x}] + \frac{2 \cos 2x}{(\cos x \sin x)^2} [\sqrt{\cot x} - \sqrt{\tan x}] \right)$$

$$= \frac{1}{4} \left((\csc x \sec x)^2 [\sqrt{\cot x} + \sqrt{\tan x}] + \frac{2 \cos 2x}{(\cos x \sin x)^2} \cdot \frac{\cot x - \tan x}{\sqrt{\cot x} + \sqrt{\tan x}} \right)$$

$$= \frac{1}{4} \left((\csc x \sec x)^2 [\sqrt{\cot x} + \sqrt{\tan x}] + \frac{2 \cos^2 2x}{(\cos x \sin x)^3} \cdot \frac{1}{\sqrt{\cot x} + \sqrt{\tan x}} \right) > 0$$

(with $0 < x < \frac{\pi}{2}$)

Using Jensen's inequality with $0 < A, B, C < \frac{\pi}{2}$ we have:

$$f(A) + f(B) + f(C) \geq 3f\left(\frac{A+B+C}{3}\right) = 3f\left(\frac{\pi}{3}\right)$$

$$= 3 \left(\sqrt{\tan \frac{\pi}{3}} + \sqrt{\cot \frac{\pi}{3}} \right) = 3 \left(\frac{1}{\sqrt[4]{3}} + \sqrt[4]{3} \right) = \text{RHS}$$

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Proved. Equality if and only if $A = B = C = \frac{\pi}{3}$.

UP.232. If $a, b, c > 0; a + b + c = 3$ then:

$$\left(1 + \frac{1}{a}\right)^{a^2+2ac} \cdot \left(1 + \frac{1}{b}\right)^{b^2+2ba} \cdot \left(1 + \frac{1}{c}\right)^{c^2+2cb} \geq 512$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\left. \begin{aligned} \left(1 + \frac{1}{a}\right)^{a^2+2ac} &= \left[\left(1 + \frac{1}{a}\right)^a\right]^{1+2c} \\ \left(1 + \frac{1}{a}\right)^a &\geq 2 \text{ (Bernoulli)} \end{aligned} \right\} \Rightarrow \left(1 + \frac{1}{a}\right)^{a^2+2ac} \geq 2^{1+2c} \Rightarrow$$

$$\left(1 + \frac{1}{a}\right)^{a^2+2ac} \cdot \left(1 + \frac{1}{b}\right)^{b^2+2ab} \cdot \left(1 + \frac{1}{c}\right)^{c^2+2bc} \geq 2^{1+2c} \cdot 2^{1+2a} \cdot 2^{1+2b} \Rightarrow$$

$$\Rightarrow \left(1 + \frac{1}{a}\right)^{a^2+2ac} \cdot \left(1 + \frac{1}{b}\right)^{b^2+2ab} \cdot \left(1 + \frac{1}{c}\right)^{c^2+2cb} \geq 2^{3+2(a+b+c)} = 2^9 = 512$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c > 0$ and $a + b + c = 3$, we get as follows:

$$1. a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2 = 3^2 = 9$$

$$2. abc \leq 1 \Rightarrow (abc)^2 \leq abc \leq 1$$

$$\text{Hence } a^{(a^2+2ac)} b^{(b^2+2ba)} c^{(c^2+2cb)} \geq a^2(a^2+2ac) b^2(b^2+2ba) c^2(c^2+2cb)$$

$$\Rightarrow a^{\frac{a^2+2ac}{2}} b^{\frac{b^2+2bc}{2}} c^{\frac{c^2+2cb}{2}} \geq a^{(a^2+2ac)} b^{(b^2+2ba)} c^{(c^2+2cb)}$$

$$\Rightarrow 2^9 \left[a^{\frac{(a^2+2ac)}{2}} b^{\frac{(b^2+2ba)}{2}} c^{\frac{(c^2+2cb)}{2}} \right] \geq 512 a^{(a^2+2ac)} b^{(b^2+2ba)} c^{(c^2+2cb)}$$

$$\Rightarrow 2^{[(a^2+2ac)+(b^2+2ba)+(c^2+2cb)]} a^{\frac{(a^2+2ac)}{2}} b^{\frac{(b^2+2ba)}{2}} c^{\frac{(c^2+2cb)}{2}} \geq$$

$$\geq 512 a^{(a^2+2ac)} b^{(b^2+2ba)} c^{(c^2+2cb)}$$

$$\Rightarrow (a + 1)^{(a^2+2ac)} \cdot (b + 1)^{(b^2+2ba)} \cdot (c + 1)^{(c^2+2cb)} \geq 512 a^{(a^2+2ac)} b^{(b^2+2ba)} c^{(c^2+2cb)}$$

$$\Rightarrow \left(\frac{a+1}{a}\right)^{(a^2+2ac)} \cdot \left(\frac{b+1}{b}\right)^{(b^2+2ba)} \cdot \left(\frac{c+1}{c}\right)^{(c^2+2cb)} \geq 512$$

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$$\Rightarrow \left(1 + \frac{1}{a}\right)^{(a^2+2ac)} \cdot \left(1 + \frac{1}{b}\right)^{(b^2+2ba)} \cdot \left(1 + \frac{1}{c}\right)^{(c^2+2cb)} \geq 512 \text{ ok}$$

Therefore, it is true.

UP.233. If $m, p \in \mathbb{N} - \{0\}$; $m \geq p$ then find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{((2n+1)!!)^m} - \sqrt[n]{((2n-1)!!)^m}}{n^{m-p} \left(\sqrt[n+1]{((n+1)!)^p} - \sqrt[n]{(n!)^p} \right)} \right)$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

Solution 1 by Marian Ursărescu-Romania

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{((2n-1)!)^m} \left(\frac{\sqrt[n+1]{(2n+1)!!^m}}{\sqrt[n]{(2n-1)!!^m}} - 1 \right)}{n^{m-p} \sqrt[n]{n!^p} \left(\frac{\sqrt[n+1]{(n+1)!^p}}{\sqrt[n]{n!^p}} - 1 \right)} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{(2n-1)!!}}{n} \right)^m \cdot \left(\frac{n}{\sqrt[n]{n!}} \right)^p \cdot \frac{n \left(\left(\frac{\sqrt[n]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right)^m - 1 \right)}{n \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^p - 1 \right)} \quad (1) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \left(\frac{n}{n+1} \right)^n = \frac{2}{e} \quad (2) \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e \quad (3)$$

$$\lim_{n \rightarrow \infty} n \left(\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right)^m - 1 \right) = \lim_{n \rightarrow \infty} \frac{n \left(e^{\ln \left(\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right)^m \right) - 1} \right)}{\ln \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right)^m} \cdot \ln \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right)^m$$

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$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} m \ln^{n+1} \frac{\sqrt{(2n+1)!!}^n}{(2n-1)!!} = m \ln \left(\lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{n+1\sqrt{(2n+1)!!}} \right) = \\
 &= m \ln \left(\lim_{n \rightarrow \infty} \left(\frac{2n+1}{n+1} \cdot \frac{n+1}{n+1\sqrt{(2n+1)!!}} \right) \right)^{(2)} = m \ln \left(2 \cdot \frac{e}{2} \right) = m \quad (4)
 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n \left(\left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p - 1 \right) &= \lim_{n \rightarrow \infty} n \frac{\left(e^{\ln \left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p} - 1 \right)}{\ln \left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p} \cdot \ln \left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p \\
 &= \lim_{n \rightarrow \infty} p \ln \frac{n+1\sqrt{(n+1)!}}{n!} = p \ln \left(\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{n+1\sqrt{(n+1)!}} \right) \\
 &= p \ln \left(\lim_{n \rightarrow \infty} \frac{n+1}{n+1\sqrt{(n+1)!}} \right)^{(3)} = p \ln e = p \quad (5)
 \end{aligned}$$

$$\text{From (1)+(2)+(3)+(4)+(5)} \Rightarrow \Omega = \left(\frac{2}{e} \right)^m \cdot e^p \cdot m \cdot p = 2^m \cdot m \cdot p \cdot e^{p-m}$$

Solution 2 by Remus Florin Stanca-Romania

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \frac{\left(\frac{n+1\sqrt{(2n+1)!!}}{n\sqrt{(2n-1)!!}} \right)^m \cdot \left(\frac{n+1\sqrt{(2n+1)!!}}{n\sqrt{(2n-1)!!}} \right)^m - 1}{\left(\frac{n\sqrt{(2n-1)!!}}{n} \right)^p \cdot \left(\left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p - 1 \right)} = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{n\sqrt{(2n-1)!!}}{n} \right)^m \cdot \left(\frac{n}{n\sqrt{n}} \right)^p \cdot \frac{\left(\frac{n+1\sqrt{(2n+1)!!}}{n\sqrt{(2n-1)!!}} \right)^m - 1}{\left(\frac{n+1\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^p - 1} \quad (1)
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{n\sqrt{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \frac{n\sqrt{(2n-1)!!}}{n^n} = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = \frac{2}{e}$$

(2)

$$\lim_{n \rightarrow \infty} \frac{n\sqrt{n!}}{n} = \lim_{n \rightarrow \infty} \frac{n\sqrt{n!}}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{n+1}{n+1} = \frac{1}{e} \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{n\sqrt{n!}} =$$

e (3)

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$$\begin{aligned}
 \stackrel{(1);(2);(3)}{\Rightarrow} \Omega &= \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^m - 1}{\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} - 1} \cdot \frac{\frac{\sqrt[n+1]{(n+1)!} - 1}{\sqrt[n]{n!}}}{\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{p-1}}. \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{e^{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} - 1}{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} \cdot \frac{\frac{\sqrt[n+1]{(n+1)!} - 1}{\sqrt[n]{n!}}}{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}. \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{e^{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} - 1}{\ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)} \cdot \frac{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)}{e^{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)} - 1} \cdot \frac{1}{\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)} = \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot 1 \cdot 1 \cdot \lim_{n \rightarrow \infty} \frac{n \ln\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)}{n \ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)} = \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{\ln\left(\frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)}{\ln\left(\frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}}\right)} = \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{\ln\left(\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}\right)}{\ln\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)} = mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \frac{\ln\left(\sqrt[n+1]{\frac{(2n+1)^{n+1}}{(2n+1)!!}}\right)}{\ln\left(\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}\right)} \\
 &= mp \left(\frac{2}{e}\right)^m \cdot e^p \cdot \lim_{n \rightarrow \infty} \ln\left(\frac{(2n+3)^{n+2}}{(2n+3)!!} \cdot \frac{(2n+1)!!}{(2n+1)^{n+1}}\right) \cdot \frac{1}{\ln\left(\frac{(n+2)^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}\right)} = \\
 &= mp \cdot 2^m \cdot e^{p-m} \ln(e) \cdot \frac{1}{\ln(e)} = mp 2^m e^{p-m} \Rightarrow \Omega = mp 2^m e^{p-m}
 \end{aligned}$$

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UP.234. If $m, p \geq 0$ then find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{\left(\frac{(n+1)!}{\sqrt[n+1]{(n+1)!}} \right)^{m+p+1} - \left(\frac{n!}{\sqrt[n]{n!}} \right)^{m+p+1}}{n^m \cdot \left(\frac{n!}{\sqrt[n]{(2n-1)!!}} \right)^p} \right)$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution 1 by Marian Ursărescu – Romania

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}^{m+p+1} \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{m+p+1} - 1 \right)}{n^m \sqrt[n]{(2n-1)!!}^p} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{n!}}{n} \right)^{m+p+1} \cdot \left(\frac{n}{\sqrt[n]{(2n-1)!!}} \right)^p \cdot n \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{m+p+1} - 1 \right) \quad (1) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e} \quad (2) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{(2n-1)!!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} = \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} \cdot \left(\frac{n+1}{n} \right)^n = \frac{e}{2} \quad (3) \end{aligned}$$

$$\begin{aligned} &\lim_{n \rightarrow \infty} n \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{m+p+1} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\left(e^{\ln \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{m+p+1}} - 1 \right)}{\ln \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)} \cdot \ln \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{m+p+1} = \\ &= \lim_{n \rightarrow \infty} (m+p+1) \cdot \ln \frac{\sqrt[n+1]{(n+1)!}}{n!} = (m+p+1) \ln \left(\lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{(n+1)!}}{n!} \right)^n \end{aligned}$$

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$$\begin{aligned}
 &= (m + p + 1) \ln \left(\lim_{n \rightarrow \infty} \frac{(n + 1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n + 1)!}} \right) = \\
 &= (m + p + 1) \ln \left(\lim_{n \rightarrow \infty} \frac{n + 1}{\sqrt[n+1]{(n + 1)!}} \right) = (m + p + 1) \ln \left(\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} \right) \\
 &\stackrel{(2)}{=} (m + p + 1) \ln e = m + p + 1 \quad (4)
 \end{aligned}$$

$$\text{From (1)+(2)+(3)+(4): } \Omega = \left(\frac{1}{e}\right)^{m+p+1} \cdot \left(\frac{e}{2}\right)^p \cdot (m + p + 1) = \frac{(m+p+1)}{2^p \cdot e^{m+1}}$$

Solution 2 by Remus Florin Stanca – Romania

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \frac{\left(\sqrt[n]{n!}\right)^{m+p+1} \cdot \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1} - 1\right)}{n^m \left(\sqrt[n]{(2n-1)!!}\right)^p} = \\
 &= \lim_{n \rightarrow \infty} \left(\sqrt[n]{n!}\right)^{m+p+1} \cdot \frac{1}{n^m} \cdot \left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^p \cdot \frac{1}{n^p} \cdot \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1} - 1\right) = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{n!}}{n}\right)^{m+p+1} \cdot \left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^p \cdot n \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{m+p+1} - 1\right) \quad (1)
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} = \frac{e}{2}$$

$$\stackrel{(1)}{\Rightarrow} \Omega = \frac{m + p + 1}{e^{p+m+1}} \cdot \left(\frac{e}{2}\right)^p \cdot \lim_{n \rightarrow \infty} \ln \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n =$$

$$= \frac{m + p + 1}{e^{m+1}} \cdot \frac{1}{2^p} \cdot \lim_{n \rightarrow \infty} \left((n + 1) \cdot \frac{1}{\sqrt[n+1]{(n + 1)!}} \right) = \frac{m + p + 1}{2^p \cdot e^{m+1}} \Rightarrow \Omega = \frac{m + p + 1}{2^p \cdot e^{m+1}}$$

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UP.235. If $a > 0; r \geq 0; (b_n)_{n \geq 0} \subset (0, \infty); \lim_{n \rightarrow \infty} \frac{b_{n+1}}{n^r \cdot b_n} = b > 0$ then find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \sqrt[n]{b_{n+1}} \right)^a - \left(\sqrt[n]{b_n} \right)^a \right) \cdot n^{1-ar}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu – Romania

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \frac{n}{n^{ar}} \cdot \left(\sqrt[n]{b_n} \right)^a \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{b_n}}{n^r} \right)^a \cdot n \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a - 1 \right) \quad (1) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n^r} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{(n^n)^r}} \stackrel{c.D.}{=} \lim_{n \rightarrow \infty} \frac{b_{n+1}}{(n+1)^{(n+1)r}} \cdot \frac{n^{nr}}{b_n} = \\ &= \lim_{n \rightarrow \infty} \frac{b_{n+1}}{n^r \cdot b_n} \cdot \frac{n^r}{(n+1)^r} \cdot \left(\frac{n}{n+1} \right)^{nr} = b \cdot 1 \cdot \left(\frac{1}{e} \right)^r = \frac{b}{e^r} \quad (2) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a - 1 \right) &= \lim_{n \rightarrow \infty} n \frac{\left[e^{\ln \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a} - 1 \right]}{\ln \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a} \cdot \ln \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a \\ &= \lim_{n \rightarrow \infty} na \cdot \ln \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = a \lim_{n \rightarrow \infty} \ln \frac{\sqrt[n+1]{b_{n+1}}}{b_n} = \\ &= a \cdot \ln \left(\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} \cdot \frac{1}{\sqrt[n+1]{b_{n+1}}} \right) = a \ln \left(\lim_{n \rightarrow \infty} \frac{b_{n+1}}{n^r \cdot b_n} \cdot \frac{n^r}{\sqrt[n+1]{b_{n+1}}} \right) \\ &= a \ln \left(b \lim_{n \rightarrow \infty} \frac{(n+1)^r}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{n^r}{(n+1)^r} \right) \stackrel{(2)}{=} a \ln \left(b \frac{e^r}{b} \right) = a \ln e^r = ar \quad (3) \end{aligned}$$

$$\text{From (1) + (2) + (3)} \Rightarrow \Omega = \frac{b^a}{e^{ar}} \cdot ar = \frac{arb^a}{e^{ar}}$$

Solution 2 by Remus Florin Stanca – Romania

$$\Omega = \lim_{n \rightarrow \infty} \left(\sqrt[n]{b_n} \right)^a \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^a - 1 \right) n^{1-ar} = a \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{b_n}}{n^r} \right)^a n \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1 \right) \quad (1)$$

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$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n^r} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^{rn}}} = \lim_{n \rightarrow \infty} \frac{b_{n+1}}{(n+1)^{r(n+1)}} \cdot \frac{n^{nr}}{b_n} = \left(\frac{1}{e}\right)^r \cdot b \Rightarrow \\ &\Rightarrow \Omega = a \cdot \left(\frac{1}{e}\right)^{ra} b^a \lim_{n \rightarrow \infty} \ln \left(\frac{b_{n+1}}{n^r b_n} \cdot \frac{n^r}{\sqrt[n+1]{b_{n+1}}} \right) = \\ &= ar \left(\frac{1}{e}\right)^{ra} b^a = \frac{arb^a}{e^{ar}} \Rightarrow \Omega = \frac{arb^a}{e^{ar}} \end{aligned}$$

UP.236. If $f, g: (0, \infty) \rightarrow (0, \infty)$ are such that exists:

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{x \cdot f(x)} = a > 0; \lim_{x \rightarrow \infty} \frac{g(x+1)}{x \cdot g(x)} = b > 0; \lim_{x \rightarrow \infty} \frac{f(x)^{\frac{1}{x}}}{x}, \lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x}$$

then find:

$$\Omega = \lim_{x \rightarrow \infty} \left((f(x))^{\frac{2}{x}} \cdot \left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(x+1)^2} - \frac{(g(x))^{\frac{1}{x}}}{x^2} \right) \right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu-Romania

$$\text{We have (from Heine): } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^{a_n}} = a_1, \lim_{n \rightarrow \infty} \frac{b_{n+1}}{n b_n} = b$$

$$\text{and we must find: } \Omega = \lim_{n \rightarrow \infty} \left[\sqrt[n]{a_n^2} \left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2} - \frac{\sqrt[n]{b_n}}{n^2} \right) \right] \quad (1)$$

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \sqrt[n]{a_n^2} \cdot \frac{\sqrt[n]{b_n}}{n^2} \left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\sqrt[n]{b_n}} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a_n}}{n} \right)^2 \cdot \frac{\sqrt[n]{b_n}}{n} \cdot n \left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\sqrt[n]{b_n}} - 1 \right) \quad (2) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} \cdot \frac{n}{n+1} \cdot \left(\frac{n}{n+1} \right)^n = \frac{a}{e} \quad (3) \end{aligned}$$

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$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} \right) = \\
 &= \lim_{n \rightarrow \infty} \frac{b_{n+1}}{nb_n} \cdot \frac{n}{n+1} \cdot \left(\frac{n}{n+1} \right)^n = \frac{b}{e} \quad (4) \\
 \lim_{n \rightarrow \infty} n \left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\sqrt[n]{b_n}} - 1 \right) &= \\
 &= \lim_{n \rightarrow \infty} n \frac{\left(e^{\ln \left(\frac{\sqrt[n]{b_{n+1}}}{\sqrt[n]{b_n}} \right)} - 1 \right)}{\ln \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)} \cdot \ln \left(\frac{\sqrt[n]{b_{n+1}}}{\sqrt[n]{b_n}} \right) = \\
 &= \lim_{n \rightarrow \infty} n \ln \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right) = \lim_{n \rightarrow \infty} \ln \frac{\sqrt[n+1]{b_{n+1}}^n}{b_n} = \\
 &= \ln \left(\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n \sqrt[n+1]{b_{n+1}}} \right) = \ln \left(\lim_{n \rightarrow \infty} \frac{b_{n+1}}{nb_n} \cdot \frac{n}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{b_{n+1}}} \right) = \\
 &\stackrel{(4)}{=} \ln \left(b \cdot \frac{e}{b} \right) = \ln e = 1 \quad (5)
 \end{aligned}$$

From (1)+(2)+(3)+(4)+(5) ⇒

$$\Omega = \left(\frac{a}{e} \right)^2 \frac{b}{e} \cdot 1 = \frac{a^2 b}{e^3}$$

Solution 2 by Remus Florin Stanca-Romania

We know that there are two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a > 0 \text{ and } \lim_{n \rightarrow \infty} \frac{b_{n+1}}{nb_n} = b > 0$$

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left((a_n)^{\frac{2}{n}} \cdot \left(\frac{\frac{1}{b_{n+1}^{\frac{n+1}}}{(n+1)^2} - \frac{1}{b_n^{\frac{n}}{n^2}}}{\frac{1}{b_n^{\frac{n}}{n^2}}} \right) \right) = \lim_{n \rightarrow \infty} (a_n)^{\frac{2}{n}} \cdot \frac{1}{n^2} \left(\frac{\frac{1}{b_{n+1}^{\frac{n+1}}}}{\frac{1}{b_n^{\frac{n}}}} \cdot \left(\frac{n}{n+1} \right)^2 - 1 \right) = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^2 \cdot \frac{1}{n} \cdot n \cdot \lim_{n \rightarrow \infty} \ln \left(\frac{\frac{1}{b_{n+1}^{\frac{n+1}}}}{\frac{1}{b_n^{\frac{n}}}} \left(\frac{n}{n+1} \right)^2 \right) \quad (1)
 \end{aligned}$$

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$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{n} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \frac{a}{e} \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{b_n}{n} = \lim_{n \rightarrow \infty} \left(\frac{b_n}{n^n} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} = \frac{b}{e} \Rightarrow$$

$$\begin{aligned} \Rightarrow \Omega &= \frac{a^2 b}{e^2} \ln \left(\frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{n b_n} \cdot \frac{n+1}{b_{n+1}^{n+1}} \right) \right) = \frac{a^2 b}{e^2} \ln \left(\frac{1}{e^2} \cdot b \cdot \frac{e}{b} \right) = -\frac{a^2 b}{e^2} \Rightarrow \\ &\Rightarrow \Omega = -\frac{a^2 b}{e^2} \end{aligned}$$

We also notice that if $h(x) = \ln(f(x)) - x \ln(x) - x$ and $h_1(x) = \ln(g(x)) - x \ln(x) - x$ were continuous and differentiable functions on $(0; +\infty)$ then we would apply Lagrange's theorem on $(x; x+1)$:

$$\frac{h(x+1)-h(x)}{x+1-x} = h'(c_x) \text{ and } \frac{h_1(x+1)-h_1(x)}{x+1-x} = h'_1(c'_x) \Leftrightarrow$$

$$\begin{aligned} \Leftrightarrow \ln(f(x+1)) - (x+1) \ln(x+1) - x - 1 - \ln(f(x)) + x \ln(x) + x &= \\ &= \frac{f'(c_x)}{f(c_x)} - \ln(c_x) - 2 \end{aligned}$$

Where $c_x \in (x, x+1) \Rightarrow \lim_{x \rightarrow \infty} \left(\ln \left(\frac{f(x+1)}{f(x)} \right) - 2 - \ln(x) \right) = \lim_{x \rightarrow \infty} \left(\frac{f'(x)}{f(x)} - \ln(x) - 2 \right)$
(1)

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{x f(x)} = e^{\lim_{x \rightarrow \infty} \left(\ln \left(\frac{f(x+1)}{f(x)} \right) - \ln(x) \right)} = a \Rightarrow \lim_{x \rightarrow \infty} \ln \left(\frac{f(x+1)}{f(x)} \right) - \ln(x) = \ln(a)$$

$$\stackrel{(1)}{\Rightarrow} \lim_{x \rightarrow \infty} \frac{f'(x)}{f(x)} - \ln(x) = \ln(a) \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{(f(x))^{\frac{1}{x}}}{x} = \lim_{x \rightarrow \infty} e^{\frac{\ln(f(x))}{x}} \stackrel{L'H}{\underset{\infty}{\infty}} \lim_{x \rightarrow \infty} e^{\frac{f'(x)}{f(x)} - \ln(x) - 1} \stackrel{(2)}{\Rightarrow} \lim_{x \rightarrow \infty} \frac{(f(x))^{\frac{1}{x}}}{x} = e^{\ln(a) - 1} = \frac{a}{e}$$

and

in the same way $\lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} = \frac{b}{e}$ so,

$$\Omega = \lim_{x \rightarrow \infty} \frac{(g(x))^{\frac{1}{x}}}{x} \cdot \left(\frac{(f(x))^{\frac{1}{x}}}{x} \right)^2 x \ln \left(\frac{g(x+1)^{\frac{1}{x+1}}}{g(x)^{\frac{1}{x}}} \cdot \left(\frac{x}{x+1} \right)^2 \right) = \frac{ba^2}{e^2}.$$

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$$\begin{aligned} \cdot \ln \left(\frac{1}{e^2} \lim_{x \rightarrow \infty} \left(\frac{g(x+1)}{g(x)} \cdot \frac{1}{g(x+1)^{\frac{1}{x+1}}} \right) \right) &= \frac{ba^2}{e^2} \ln \left(\frac{1}{e^2} \cdot b \cdot \frac{e}{b} \right) = -\frac{ba^2}{e^2} \\ \Rightarrow \Omega &= -\frac{ba^2}{e^2} \end{aligned}$$

UP.237. Let be $f: (0, \infty) \rightarrow (0, \infty)$ a continuous function; $(a_n)_{n \geq 1} \subset (0, \infty)$;

$(b_n)_{n \geq 1} \subset (0, \infty)$; $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a > 0$; $\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n \cdot n^2} = b > 0$. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \int_{\frac{n}{\sqrt[n]{a_n}}}^{\frac{n+1}{\sqrt[n+1]{(n+1)! a_{n+1}}}} f \left(\frac{x}{\sqrt[n]{b_n}} \right) dx \right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania

Solution by Remus Florin Stanca – Romania

$f = \text{continuous function} \Rightarrow \exists \zeta \in [\sqrt[n]{n! a_n} \text{ and } \sqrt[n+1]{(n+1)! a_{n+1}}]$ such that:

$$\int_{\frac{n}{\sqrt[n]{a_n}}}^{\frac{n+1}{\sqrt[n+1]{(n+1)! a_{n+1}}}} f \left(\frac{x}{\sqrt[n]{b_n}} \right) dx = \left(\sqrt[n+1]{(n+1)! a_{n+1}} - \sqrt[n]{n! a_n} \right) f \left(\frac{\zeta}{\sqrt[n]{b_n}} \right) \Rightarrow$$

$$\Rightarrow \Omega = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt[n+1]{(n+1)! a_{n+1}} - \sqrt[n]{n! a_n} \right) f \left(\frac{\zeta}{\sqrt[n]{b_n}} \right) \quad (a)$$

$$\zeta \in \left[\sqrt[n]{n! a_n}; \sqrt[n+1]{(n+1)! a_{n+1}} \right] \Rightarrow \sqrt[n]{n! a_n} \leq \zeta \leq \sqrt[n+1]{(n+1)! a_{n+1}} \Rightarrow$$

$$\Rightarrow \frac{\sqrt[n]{n! a_n}}{\sqrt[n]{b_n}} \leq \frac{\zeta}{\sqrt[n]{b_n}} \leq \frac{\sqrt[n+1]{(n+1)! a_{n+1}}}{\sqrt[n]{b_n}} \quad (1)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n! a_n}}{\sqrt[n]{b_n}} &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{n^2}{\sqrt[n]{b_n}} \cdot \frac{\sqrt[n]{a_n}}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{n^n}} \cdot \lim_{n \rightarrow \infty} \frac{n^{2n}}{\sqrt[n]{b_n}} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{\sqrt[n]{n^n}} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{2n+2}}{b_{n+1}} \cdot \frac{b_n}{n^{2n}} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \\ &= \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \left(e^2 \cdot \frac{(n+1)^2}{b_{n+1}} b_n \right) \lim_{n \rightarrow \infty} \left(\frac{1}{e} \cdot \frac{a_{n+1}}{na_n} \right) = \frac{1}{e} e^2 \frac{1}{b} \cdot \frac{1}{e} a = \frac{a}{b} \quad (2) \end{aligned}$$

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$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{{}^{n+1}\sqrt{(n+1)! a_{n+1}}}{\sqrt[n]{b_n}} &= \lim_{n \rightarrow \infty} \frac{{}^{n+1}\sqrt{(n+1)!}}{n+1} \cdot \frac{{}^{n+1}\sqrt{a_{n+1}}}{n+1} \cdot \frac{(n+1)^2}{\sqrt[n]{b_n}} = \\
 &= \lim_{n \rightarrow \infty} \frac{{}^{n+1}\sqrt{(n+1)!}}{\sqrt{(n+1)^{n+1}}} \lim_{n \rightarrow \infty} \frac{{}^{n+1}\sqrt{a_{n+1}}}{\sqrt{(n+1)^{n+1}}} \lim_{n \rightarrow \infty} \frac{n^{2n}}{\sqrt[n]{b_n}} = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{(n+2)!}{(n+2)^{n+2}} \cdot \frac{(n+1)^{n+1}}{(n+1)!} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{a_{n+2}}{(n+2)^{n+2}} \cdot \frac{(n+1)^{n+1}}{a_{n+1}} \right) \cdot \\
 &\cdot \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{2n+2}}{b_{n+1}} \cdot \frac{b_n}{n^{2n}} \right) = \frac{1}{e} \cdot \left(\frac{1}{e} \lim_{n \rightarrow \infty} \frac{a_{n+2}}{(n+1)a_{n+1}} \right) \left(e^2 \lim_{n \rightarrow \infty} \frac{nb_n}{b_{n+1}} \right) = \frac{1}{e^2} a e^2 \frac{1}{b} = \frac{a}{b} \quad (3) \\
 &\stackrel{(1):(2):(3)}{\Rightarrow} \lim_{n \rightarrow \infty} \frac{\zeta}{\sqrt[n]{b_n}} = \frac{a}{b}, f = \text{continuous} \Rightarrow \lim_{n \rightarrow \infty} f\left(\frac{\zeta}{\sqrt[n]{b_n}}\right) = f\left(\frac{a}{b}\right) \stackrel{(a)}{\Rightarrow} \\
 &\Rightarrow \Omega = f\left(\frac{a}{b}\right) \lim_{n \rightarrow \infty} \frac{1}{n} \sqrt[n]{n! a_n} \left(\frac{{}^{n+1}\sqrt{(n+1)! a_{n+1}}}{\sqrt[n]{n! a_n}} - 1 \right) = \\
 &= f\left(\frac{a}{b}\right) \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{\sqrt[n]{a_n}}{n} \ln \left(\frac{e^{\frac{\ln({}^{n+1}\sqrt{(n+1)! a_{n+1}})}{\sqrt[n]{n! a_n}}} - 1}{\ln\left(\frac{{}^{n+1}\sqrt{(n+1)! a_{n+1}}}{\sqrt[n]{n! a_n}}\right)} \right) \cdot \ln \left(\frac{{}^{n+1}\sqrt{(n+1)! a_{n+1}}}{\sqrt[n]{n! a_n}} \right) = \\
 &= \frac{1}{e} \cdot f\left(\frac{a}{b}\right) a \cdot 1 \cdot \lim_{n \rightarrow \infty} \ln \left(\left(\frac{{}^{n+1}\sqrt{(n+1)! a_{n+1}}}{\sqrt[n]{n! a_n}} \right)^n \right) \\
 &= \frac{a}{e} f\left(\frac{a}{b}\right) \lim_{n \rightarrow \infty} \ln \left(\frac{(n+1)! a_{n+1}}{n! \cdot a_n} \cdot \frac{1}{\sqrt[n]{(n+1)! a_{n+1}}} \right) = \\
 &= \frac{a}{e} f\left(\frac{a}{b}\right) \lim_{n \rightarrow \infty} \ln \left(\frac{a_{n+1}}{n a_n} \cdot \frac{n^2}{\sqrt[n]{b_n}} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{(n+1)! a_{n+1}}} \right) = \frac{a}{e} f\left(\frac{a}{b}\right) \ln \left(a \frac{e^2}{b} \cdot \frac{b}{a} \right) = \\
 &= \frac{2a}{e} f\left(\frac{a}{b}\right) \Rightarrow \Omega = \frac{2a}{e} f\left(\frac{a}{b}\right)
 \end{aligned}$$

UP.238. If $m \geq 0$ then find:

$$\Omega = \lim_{x \rightarrow \infty} \left(((x+1)^m \cdot \Gamma(x+2))^{\frac{1}{x+1}} - (x^m \cdot \Gamma(x+1))^{\frac{1}{x}} \right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

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Solution 1 by Marian Ursărescu-Romania

(another approach by sequence)

Because $\Gamma(n+1) = n!$ we must calculate (from Heine):

$$\begin{aligned}\Omega &= \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(n+1)^m (n+1)!} - \sqrt[n]{n^m n!} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^m \cdot n!}}{n} \cdot n \left(\frac{\sqrt[n+1]{(n+1)^m (n+1)!}}{\sqrt[n]{n^m n!}} - 1 \right) \quad (1)\end{aligned}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^m \cdot n!}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^m n!}{n^n}} \stackrel{c.s.}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^m (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n^m - n!} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^m \cdot \left(\frac{n}{n+1} \right)^n = \frac{1}{e} \quad (2)\end{aligned}$$

$$\lim_{n \rightarrow \infty} n \left(\frac{\sqrt[n+1]{(n+1)^m (n+1)!}}{\sqrt[n]{n^m n!}} - 1 \right) =$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(e^{\frac{\ln \sqrt[n+1]{(n+1)^m (n+1)!}}{\sqrt[n]{n^m \cdot n!}}} - 1 \right)}{\ln \frac{\sqrt[n+1]{(n+1)^m (n+1)!}}{\sqrt[n]{n^m n!}}} \cdot \ln \frac{\sqrt[n+1]{(n+1)^m (n+1)!}}{\sqrt[n]{n^m n!}}$$

$$= \lim_{n \rightarrow \infty} \ln \frac{\sqrt[n+1]{(n+1)^m (n+1)!}^n}{n^m n!} = \ln \left(\lim_{n \rightarrow \infty} \frac{(n+1)^m (n+1)!}{n^m n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)^m (n+1)!}} \right) =$$

$$= \ln \left(\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^m \cdot \frac{n+1}{\sqrt[n+1]{(n+1)^m (n+1)!}} \right) \stackrel{(2)}{=} \ln e = 1 \quad (3) \text{ From (1)+(2)+(3)} \Rightarrow \Omega = \frac{1}{e}$$

Solution 2 by Remus Florin Stanca-Romania

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+1)}{x\Gamma(x)} \stackrel{n \in \mathbb{N}}{=} \lim_{n \rightarrow \infty} \frac{n!}{n \cdot (n-1)!} = 1 \quad (1)$$

$\Gamma(x)$ is continuous and differentiable so the function $f(x) = \ln(\Gamma(x)) - x \ln x$ is

differentiable and continuous \Rightarrow

$$\Rightarrow \frac{\ln(\Gamma(x+1)) - (x+1) \ln(x+1) - \ln(\Gamma(x)) + x \ln(x)}{x+1-x} = \frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln c_x,$$

$$c_x \in (x; x+1) \text{ (Lagrange)} \Rightarrow \ln(\Gamma(x+1)) - \ln(\Gamma(x)) - x \ln \left(\frac{x+1}{x} \right) - \ln(x+1) =$$

$$= \frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln(c_x) \Rightarrow \lim_{x \rightarrow \infty} \left(\ln \left(\frac{\Gamma(x+1)}{\Gamma(x)} \right) - \ln(x+1) \right) - 1 =$$

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$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{\Gamma'(c_x)}{\Gamma(c_x)} - 1 - \ln(c_x) \Rightarrow \\
 &\Rightarrow \lim_{x \rightarrow \infty} \left(\ln \left(\frac{\Gamma(x+1)}{\Gamma(x)} \right) - \ln(x) \right) - 1 = \lim_{x \rightarrow \infty} \left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1 \right) \Rightarrow \\
 &\Rightarrow \lim_{x \rightarrow \infty} \ln \left(\frac{\Gamma(x+1)}{x\Gamma(x)} \right) - 1 = \lim_{x \rightarrow \infty} \left(\frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1 \right) \\
 &\stackrel{(1)}{\Rightarrow} \lim_{x \rightarrow \infty} \frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) - 1 = -1 \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} &= \lim_{x \rightarrow \infty} \left(\frac{\Gamma(x+1)}{x^x} \right)^{\frac{1}{x}} = \lim_{x \rightarrow \infty} e^{\frac{\ln(\Gamma(x+1)) - x \ln(x)}{x}} = \\
 &\stackrel{L'H}{\lim_{x \rightarrow \infty}} e^{\frac{\Gamma'(x)}{\Gamma(x)} - 1 - \ln(x)} \stackrel{(2)}{=} e^{-1} = \frac{1}{e} \Rightarrow
 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1} = \frac{1}{e} \quad (3)$$

$$\begin{aligned}
 \Gamma &= \lim_{x \rightarrow \infty} (x^m \Gamma(x+1))^{\frac{1}{x}} \left(\frac{((x+1)^m \Gamma(x+2))^{\frac{1}{x+1}}}{(x^m \Gamma(x+1))^{\frac{1}{x}}} - 1 \right) = \\
 &= \lim_{x \rightarrow \infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} x \ln \left(\frac{((x+1)^m \Gamma(x+2))^{\frac{1}{x+1}}}{(x^m \Gamma(x+1))^{\frac{1}{x}}} \right) = \\
 &= \frac{1}{e} \lim_{x \rightarrow \infty} \ln \left(\frac{(x+1)^m}{x^m} \cdot \frac{\Gamma(x+2)}{\Gamma(x+1)} \cdot \frac{1}{((x+1)^m \Gamma(x+2))^{\frac{1}{x+1}}} \right) = \\
 &= \frac{1}{e} \lim_{x \rightarrow \infty} \ln \left(\frac{\Gamma(x+2)}{(x+1)\Gamma(x+1)} \cdot \frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}} \right) = \frac{1}{e} \ln(e) = \frac{1}{e} \Rightarrow \Omega = \frac{1}{e}
 \end{aligned}$$

Solution 3 by Rohan Shinde-India

Using Lemma 1 of Stolz Cesaro theorem that for two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of real numbers, if $0 < b_1 < b_2 < \dots < b_n < \dots$ and $\lim_{n \rightarrow \infty} b_n = \infty$

then if $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \Omega$ then also $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \Omega$

Here if $a_n = (n^m \Gamma(n+1))^{\frac{1}{n}}$ and $b_n = n$ then

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$$\lim_{n \rightarrow \infty} \left\{ ((n+1)^m \Gamma(n+2))^{\frac{1}{n+1}} - (n^m \Gamma(n+1))^{\frac{1}{n}} \right\} = \Omega$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n^m \Gamma(n+1))^{\frac{1}{n}}}{n} = \Omega \Rightarrow \lim_{n \rightarrow \infty} \left(n^m \cdot \frac{\Gamma(n+1)}{n^n} \right)^{\frac{1}{n}}$$

Using Stirling's approximation for Gamma function,

$$\Omega = \lim_{n \rightarrow \infty} \left(n^m \times \frac{\sqrt{2\pi n}}{n^n} \times \left(\frac{n}{e}\right)^n \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(n^{\frac{m}{n}} \times (\sqrt{2\pi n})^{\frac{1}{n}} \times \frac{1}{e} \right)$$

But

$$\lim_{n \rightarrow \infty} n^{\frac{m}{n}} = \lim_{n \rightarrow \infty} e^{\frac{m \ln n}{n}} = e^{\lim_{n \rightarrow \infty} \frac{m \ln n}{n}} = e^0 = 1$$

$$\text{and } \lim_{n \rightarrow \infty} (\sqrt{2\pi n})^{\frac{1}{n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln(2\pi n)}{2n}} = e^{\lim_{n \rightarrow \infty} \left(\frac{\ln(2\pi n)}{2n}\right)} = e^0 = 1$$

$$\Rightarrow \Omega = \frac{1}{e}$$

UP.239. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{1}{2} \int_a^b \int_a^b (1 + \tan x)(1 + \tan y)(1 + \tan x \tan y) dx dy \geq (\tan b - \tan a)^2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\begin{aligned} \frac{2}{\cos^2 x \cos^2 y} &= 2 \cdot \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 y} = 2 \cdot \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \cdot \frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \\ &= 2(\tan^2 x + 1)(\tan^2 y + 1) = \\ &= (\tan^2 x + 1)(\tan^2 y + 1) + (\tan^2 x + 1)(\tan^2 y + 1) \geq \\ &\stackrel{QM-AM}{\geq} 2 \left(\frac{\tan x + 1}{2} \right)^2 (\tan^2 y + 1) + (\tan^2 x + 1) \cdot 2 \left(\frac{\tan y + 1}{2} \right)^2 = \\ &= \frac{1}{2} [(\tan x + 1)^2 (\tan^2 y + 1) + (\tan^2 x + 1) (\tan y + 1)^2] = \\ &= \frac{1}{2} [(\tan^2 x + 2 \tan x + 1)(\tan^2 y + 1) + (\tan^2 x + 1)(\tan^2 y + 2 \tan y + 1)] = \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{2} \left(\tan^2 x \tan^2 y + \tan^2 x + 2 \tan x \tan^2 y + 2 \tan x + \right. \\
 &\quad \left. + \tan^2 y + 1 + \tan^2 x \tan^2 y + 2 \tan^2 x \tan y + \tan^2 x + \tan^2 y + 2 \tan y + 1 \right) = \\
 &= \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x + \tan y + 1 + \tan^2 x + \tan^2 y \geq \\
 &\stackrel{AM-GM}{\geq} \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x + \tan y + 1 + 2 \tan x \tan y \\
 &= \\
 &= \tan^2 x \tan^2 y + \tan x \tan^2 y + \tan^2 x \tan y + \tan x \tan y + \tan x \tan y + \tan x + \\
 &\quad + \tan y + 1 = \tan x \tan y (\tan x \tan y + \tan x + \tan y + 1) + \\
 &\quad + (\tan x \tan y + \tan x + \tan y + 1) = \\
 &= (\tan x \tan y + \tan x + \tan y + 1)(\tan x \tan y + 1) = \\
 &= [\tan x (\tan y + 1) + (\tan y + 1)](\tan x \tan y + 1) = \\
 &= (\tan x + 1)(\tan y + 1)(\tan x \tan y + 1) \\
 &\int_a^b \int_a^b (1 + \tan x)(1 + \tan y)(1 + \tan x \tan y) dx dy \geq \\
 &\geq \int_a^b \int_a^b \frac{2}{\cos^2 x \cos^2 y} dx dy = 2 \left(\int_a^b \frac{1}{\cos^2 x} dx \right) \left(\int_a^b \frac{1}{\cos^2 y} dy \right) \\
 &\frac{1}{2} \int_a^b \int_a^b (1 + \tan x)(1 + \tan y)(1 + \tan x \tan y) dx dy \geq \\
 &\geq \left(\tan x \Big|_a^b \right) \cdot \left(\tan y \Big|_a^b \right) = (\tan b - \tan a)^2
 \end{aligned}$$

Equality holds for $a = b$.

UP.240. Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of:

$$T = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Michael Sterghiou-Greece

$$T = \left(\sum_{cyc} \frac{a}{b} \right) + \sum_{cyc} \frac{1}{a^3 + b^3 + abc} \quad (1)$$

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We will prove that $T \geq 4$. We use the following lemma: With the problem's

conditions it holds that: $G\left(\sum_{cyc} \frac{a}{b}\right) + 3 \geq 7 \cdot \sum_{cyc} a^2$ (L). To prove this, we

homogenize the inequality by multiplying by $3abc$ (and taking into account that

$$\sum_{cyc} a = 3) \text{ as follows:}$$

(L) $\rightarrow 2(\sum_{cyc} a)^2 \cdot (\sum ab^2) + abc \cdot (\sum_{cyc} a)^2 - 21abc \sum_{cyc} a^2 \geq 0$ or the equivalent

$\sum_{cyc} a^4c + \sum_{cyc} a^3b^2 + 2 \sum_{cyc} a^2b^3 + 4abc \cdot \sum_{cyc} ab - 8abc \sum_{cyc} a^2 \geq 0$. The last reduces to: $\sum_{cyc} a(a-b)^2(b-2c)^2 \geq 0$ which is true. Now, as $\sum_{cyc} a^2 \geq 3$ be adding

this to (L) we get $\left(\sum_{cyc} \frac{a}{b}\right) \geq \sum_{cyc} a^2$ (2). Let $(p, q, r) = (\sum_{cyc} a, \sum_{cyc} ab, abc)$ with $p = 3$ (1) becomes the stronger inequality $(\sum_{cyc} a^2) + \frac{9}{2(\sum_{cyc} a^3) + 3abc} \geq 4$ [by (2) and

BCS] or $9 - 2q + \frac{1}{6-2q+r} - 4 \geq 0$ [Note that $\sum_{cyc} a^3 = p^3 - 3pq + 3r$] which reduces to $4q^2 - 22q + (5 - 2q)r + 31 \geq 0$ (3). This is either an increasing or decreasing function of r (depending on the sign of $5 - 2q$). In either case it suffices to hold when $r = \max$ or $r = \min$ which according to V. Cîrtoaje theorem happens (for any fixed

q) when any two of a, b, c are equal. Let WLOG $b = c \left(< \frac{3}{2} \right) \Rightarrow a = 3 - 2b$. Now (3)

becomes $-(b-1)^2(12b^3 - 54b^2 + 70b - 31) \geq 0$ for $0 < b < \frac{3}{2}$. This can be easily

show to be true because $12b^3 - 54b^2 + 70b - 31 < 0$ for $b \in \left(0, \frac{3}{2}\right)$. Done!

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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

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To be continued!

Daniel Sitaru