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PROBLEMS FOR JUNIORS

JP.226. Let a, b, c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.227. Prove that in any ABC triangle the following relationship holds:

$$3(r_a r_b^3+r_b r_c^3+r_c r_a^3)\geq r(4R+r)^3$$

Proposed by Nguyen Viet Hung- Hanoi - Vietnam

JP.228. Prove that if a, b, c are the lengths of the sides of a triangle, then:

$$\sqrt{\frac{a}{b+c-a}}+\sqrt{\frac{b}{c+a-b}}+\sqrt{\frac{c}{a+b-c}}\geq\frac{(a+b+c)^2}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung- Hanoi - Vietnam

JP.229. Let a, b, c be positive real numbers. Find the k_{max} such that the inequality is true:

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}-1\geq k\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.230. Prove that for any ABC triangle the following relationship holds:

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}+\frac{3\sqrt[4]{3}}{\sqrt{2}}\leq 2\left(\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\right)$$

Proposed by Vasile Mircea Popa - Romania

JP.231. Prove that for any positive real numbers a, b, c the following relationship holds:

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}\ge\frac{a+b+c}{2}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.232. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{\frac{r_a}{r_b r_c}}+\sqrt{\frac{r_b}{r_c r_a}}+\sqrt{\frac{r_c}{r_a r_b}}\ge\sqrt{\frac{3}{r}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.233. Find the maximum and minimum possible value of:

$$\frac{1}{\sin^4 x+\cos^2 x}+\frac{1}{\cos^4 x+\sin^2 x}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.234. Let x, y be positive real numbers such that $x+y\le 1$. Prove that:

$$\left(1-\frac{1}{x^4}\right)\left(1-\frac{1}{y^4}\right)\ge 225.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.235. In $\Delta ABC; I$ - incenter; A', B', C' - lies on circumcircle such that: $(A, I, A'); (B, I, B'); (C, I, C')$ are collinear. Prove that:

$$\frac{a}{IA'}+\frac{b}{IB'}+\frac{c}{IC'}\ge\frac{a+b+c}{R}$$

Proposed by Marian Ursărescu - Romania

JP.236. If $a, b, c, x, y, z > 0; a+b+c=3$ then:

$$a^a\cdot b^b\cdot c^c\cdot(x+y+z)^3\ge 27x^a y^b z^c$$

When does the equality holds?

Proposed by Daniel Sitaru - Romania

JP.237. If $a, b, c, x, y, z > 0; a+b+c\ge x+y+z$ then:

$$a^a b^b c^c\ge x^a y^b z^c$$

Proposed by Daniel Sitaru - Romania

JP.238. If $a, b, c, d, x, y, z, t > 0$ then:

$$\frac{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d}{(a+b+c+d)^{a+b+c+d}} \geq \left(\frac{xyzt}{xyz + xyt + xzt + yzt}\right)^{a+b+c+d}$$

Proposed by Daniel Sitaru - Romania

JP.239. In ΔABC the following relationship holds:

$$4(m_a + m_b + m_c) \leq 3\left(\frac{r_a}{\cos^2 \frac{A}{2}} + \frac{r_b}{\cos^2 \frac{B}{2}} + \frac{r_c}{\cos^2 \frac{C}{2}}\right)$$

Proposed by Marin Chirciu - Romania

JP.240. In ΔABC the following relationship holds:

$$3(m_a + m_b + m_c) \leq r_a \cot^2 \frac{A}{2} + r_b \cot^2 \frac{B}{2} + r_c \cot^2 \frac{C}{2}$$

Proposed by Marin Chirciu - Romania

PROBLEMS FOR SENIORS

SP.226. If $a, b > 0$ then:

$$\left(\sqrt{ab} - \frac{a+b}{2}\right)\arctan\left(\frac{2ab}{a+b}\right) + \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan(\sqrt{ab}) +$$

$$+\left(\frac{2ab}{a+b} - \sqrt{ab}\right)\arctan\left(\frac{a+b}{2}\right) \geq 0$$

Proposed by Daniel Sitaru - Romania

SP.227. Prove that for any positive real numbers a, b, c:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{8(a+b+c)^3}{3(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.228. Find the positive real numbers (x, y) such that:

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{x} = 2\sqrt[4]{\frac{x^4+y^4}{2}} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.229. Let a, b, c be the lengths of a triangle such that $a + b + c = 3$. Prove that:

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} + 3 \cdot \sqrt[2018]{abc} \geq 2(ab+bc+ca)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.230. Let a, b, c be positive real numbers such that $a+b+c = 3$. Find the minimum value of:

$$T = \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+3\cdot\left(\frac{1}{abc+b^3+c^3}+\frac{1}{abc+c^3+a^3}+\frac{1}{abc+a^3+b^3}\right)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.231. In ΔABC the following relationship holds:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 4\sqrt{3}F + 2\log(a^{ab^2} \cdot b^{bc^2} \cdot c^{ca^2})$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

SP.232. If $a, b \geq 1$ then:

$$2^{\frac{2}{a+b}} \cdot 3^{\frac{a+b}{2}} + 2^{\sqrt{ab}} \cdot 3^{\frac{1}{\sqrt{ab}}} + 2^{\frac{a+b}{2}} \cdot 3^{\frac{2}{a+b}} + 2^{\frac{1}{\sqrt{ab}}} \cdot 3^{\sqrt{ab}} \geq 24$$

Proposed by Daniel Sitaru - Romania

SP.233. If $A \in M_3(\mathbb{R}); \mathrm{Tr}(A^2) = 0; \det A = 1$ then:

$$\det(A^2 + A + I_3) \geq (\mathrm{Tr}\,A)^3$$

Proposed by Marian Ursărescu - Romania

SP.234. In $\Delta ABC; I$ - incentre; A' - is the intersection between AI and circumcircle of $\Delta BIC; B'$ - is the intersection between BI and circumcircle of $\Delta AIC; C'$ - is the intersection between CI and circumcircle of ΔAIB. Prove that:

$$\frac{IA}{IA'} + \frac{IB}{IB'} + \frac{IC}{IC'} \geq 2\left(1 - \frac{r}{R}\right)$$

Proposed by Marian Ursărescu - Romania

SP.235. Let be $A(z_1); B(z_1); C(z_3); z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$; $|z_1| = |z_2| = |z_3|; AB = c; BC = a; CA = b$. If $(b+c)z_Bz_C+(c+a)z_Cz_A+(a+b)z_Az_B = 0$ then $AB = BC = CA$.

Proposed by Marian Ursărescu - Romania

SP.236. In ΔABC the following relationship holds:

$$m_a + m_b + m_c \leq 3R^2\left(\frac{r_a}{a^2} + \frac{r_b}{b^2} + \frac{r_c}{c^2}\right)$$

Proposed by Marin Chirciu - Romania

SP.237. In ΔABC the following relationship holds:

$$3r(m_a + m_b + m_c) \leq Rs\left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c}\right)$$

Proposed by Marin Chirciu - Romania

SP.238. Let a, b, c be positive real numbers such that $\min\{b+c, c+a, a+b\} > \max\{a, b, c\}$. Prove that:

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.239. If a, b, c, d are sides in a cyclic quadrilateral, r_a, r_b, r_c, r_d - exradii, s - semiperimeter, then:

$$\frac{a}{r_a^2} + \frac{b}{r_b^2} + \frac{c}{r_c^2} + \frac{d}{r_d^2} \geq \frac{32}{s}$$

Proposed by Daniel Sitaru - Romania

SP.240. If a, b, c, d are sides in a bicentric quadrilateral; r_a, r_b, r_c, r_d - exradii; s - semiperimeter, then:

$$\frac{r_a^2}{a^3} + \frac{r_b^2}{b^3} + \frac{r_c^2}{c^3} + \frac{r_d^2}{d^3} \geq \frac{2}{s}$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.226. Find all positive real numbers (x, y, z) such that:

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x^3y + y^3z + z^3x = \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.227. If $x_n = \sum_{k=1}^{n} \sqrt[k+1]{1+\frac{1}{k}}; n \in \mathbb{N}; n \geq 1$, then find:

$$\Omega = \lim_{n\to\infty}\left(\log\left(\sum_{k=1}^{n} x_k^2\right) - 3H_n\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.228. If $f : \mathbb{R} \to (0,\infty)$, f continuous; $a, b \in \mathbb{R}; a \leq b$ then:

$$\int_a^b \int_a^b \left(\frac{(f^2(x)+f(y))(f^2(y)+f(x))}{(1+f(x))(1+f(y))}\right) dxdy \geq \left(\int_a^b f(x)dx\right)^2$$

Proposed by Daniel Sitaru - Romania

UP.229. Calculate the limit:

$$\Omega = \lim_{x\to\infty}\left(x^2 \int_x^{x+\frac{2}{x}} \arctan\left(\frac{1}{t}\right) dt\right)$$

Proposed by Vasile Mircea Popa - Romania

UP.230. If $a, b > 1$ then:

$$\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}} \cdot (\sqrt{ab})^{\frac{a+b}{2}-\frac{2ab}{a+b}} \cdot \left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}} \geq 1$$

Proposed by Daniel Sitaru - Romania

UP.231. Prove that for any acute triangle ABC the following inequality holds:

$$\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}+\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C} \geq 3\sqrt[4]{3}+\frac{3}{\sqrt[4]{3}}$$

Proposed by Vasile Mircea Popa - Romania

UP.232. If $a, b, c > 0; a+b+c = 3$ then:

$$\left(1+\frac{1}{a}\right)^{a^2+2ac} \cdot \left(1+\frac{1}{b}\right)^{b^2+2ba} \cdot \left(1+\frac{1}{c}\right)^{c^2+2cb} \geq 512$$

Proposed by Daniel Sitaru - Romania

UP.233. If $m, p \in \mathbb{N} - \{0\}; m \geq p$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^m} - \sqrt[n]{((2n-1)!!)^m}}{n^{m-p}(\sqrt[n+1]{((n+1)!)^p} - \sqrt[n]{(n!)^p})}\right)$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

UP.234. If $m, p \geq 0$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(\sqrt[n+1]{(n+1)!})^{m+p+1} - (\sqrt[n]{n!})^{m+p+1}}{n^m \cdot (\sqrt[n]{(2n-1)!!})^p}\right)$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

UP.235. If $a > 0; r \geq 0; (b_n)_{n\geq 0} \subset (0,\infty); \lim_{n\to\infty} \frac{b_{n+1}}{n^r \cdot b_n} = b > 0$ then find:

$$\Omega = \lim_{n\to\infty}\left(\left(\left(\sqrt[n+1]{b_{n+1}}\right)^a - \left(\sqrt[n]{b_n}\right)^a\right) \cdot n^{1-ar}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.236. If $f, g : (0,\infty) \to (0,\infty)$ are such that exists:

$$\lim_{x\to\infty} \frac{f(x+1)}{x \cdot f(x)} = a > 0; \lim_{x\to\infty} \frac{g(x+1)}{x \cdot g(x)} = b > 0; \lim_{x\to\infty} \frac{f(x)^{\frac{1}{x}}}{x};$$

$$\lim_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x} \text{ then find:}$$

$$\Omega = \lim_{x\to\infty}\left((f(x))^{\frac{2}{x}} \cdot \left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(x+1)^2} - \frac{(g(x))^{\frac{1}{x}}}{x^2}\right)\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.237. Let be $f : (0,\infty) \to (0,\infty)$ a continuous function; $(a_n)_{n\geq 1} \subset (0,\infty); (b_n)_{n\geq 1} \subset (0,\infty)$; $\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{b_n \cdot n^2} = b > 0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\int_{\sqrt[n]{n!\cdot a_n}}^{\sqrt[n+1]{(n+1)!\cdot a_{n+1}}} f\left(\frac{x}{\sqrt[n]{b_n}}\right)dx\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.238. If $m \geq 0$ then find:

$$\Omega = \lim_{x\to\infty}\left(\left((x+1)^m \cdot \Gamma(x+2)\right)^{\frac{1}{x+1}} - \left(x^m \cdot \Gamma(x+1)\right)^{\frac{1}{x}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

UP.239. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{1}{2}\int_a^b\int_a^b (1+\tan x)(1+\tan y)(1+\tan x \tan y)dxdy \geq (\tan b - \tan a)^2$$

Proposed by Daniel Sitaru - Romania

UP.240. Let a, b, c be positive real numbers such that $a+b+c = 3$. Find the minimum value of:

$$T = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

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