R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

UP.224. If $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty}\left(\frac{a_n}{n}\right) = a > 0; \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\right) = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{a_n}{n} = a > 0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{a_n b_n} = b > 0$$

Now, $$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{CAUCHY-D'ALEMBERT}{=} \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\cdot\frac{a_n}{n}\right) = \frac{ab}{e}$$

Let $u_n = \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}$ *for all* $n\in\mathbb{N}$ *then* $$\lim_{n\to\infty}u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\left(1+\frac{1}{n}\right)\right) = 1$$

Hence $\frac{u_n-1}{\ln u_n}\to 1$ *for* $n\to\infty$. $$\mathrm{Lim}_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n b_n}\cdot\frac{a_n}{n}\cdot\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\right) = e$$

$$\therefore \lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = \frac{ab}{e}\cdot 1\cdot\ln e = \frac{ab}{e} \text{ (Answer)}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

$$\Omega = \lim_{n\to+\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to+\infty}\left((n+1)\sqrt[n]{\frac{b_{n+1}}{(n+1)^{n+1}}} - n\sqrt[n]{\frac{b_n}{n^n}}\right) =$$

$$= \lim_{n\to+\infty}\frac{\frac{b_{n+1}}{(n+1)^{n+1}}}{\frac{b_n}{n^n}} = \lim_{n\to+\infty}\frac{b_{n+1}}{b_n a_n}\cdot\frac{a_n}{n}\left(1-\frac{1}{n+1}\right)^{n+1} = \frac{ab}{e}$$