NEW APPLICATIONS OF FAMOUS GEOMETRICAL INEQUALITIES (I)

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Abstract. In this paper are presented a few applications of famous geometrical inequalities.

Application 1.

In any ABC triangle having the semiperimeter s the following inequality holds:

$$\frac{1}{a(xb+yc)}+\frac{1}{b(xc+ya)}+\frac{1}{c(xa+yb)}\geq\frac{27}{4(x+y)s^2}$$

Proof. We have:

$$U(x,y)=\sum_{cyc}\frac{1}{a(xb+yc)}\overset{\text{Bergström}}{\geq}\frac{9}{\sum_{cyc}(abx+acy)}=\frac{9}{(x+y)\sum_{cyc}ab}$$

But, $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\geq 3(ab+bc+ca)\Rightarrow$
$\Rightarrow ab+bc+ca\leq\frac{(a+b+c)^2}{3}$ and we obtain:

$$U(x,y)\geq\frac{9}{(x+y)\cdot\frac{(a+b+c)^2}{3}}=\frac{27}{(x+y)(a+b+c)^2}=\frac{27}{(x+y)\cdot 4s^2}=\frac{27}{4(x+y)s^2}$$

□

Application 2.

If $m\in\mathbb{R}_+=[0,\infty), x,y\in(0,\frac{\pi}{4}), x+y=\frac{\pi}{4}$ then in any ABC triangle having the area F the following inequality holds:

$$\frac{a^{2(m+1)}}{\tan^m x}+\frac{b^{2(m+1)}}{\tan^m y}+\frac{c^{2(m+1)}}{\tan^m x\cdot\tan^m y}>2^{2m+2}\cdot(\sqrt{3})^{m+1}\cdot F^{m+1}$$

Proof. Let be $V(m,x,y)$ the left member of the relationship from enunciation. According to J. Radon's inequality we have:
(1)

$$V(m,x,y)\geq\frac{(a^2+b^2+c^2)^{m+1}}{(\tan x+\tan y+\tan x\tan y)^m}\overset{\text{Ionescu-Weitzenbök}}{\geq}\frac{(y\sqrt{3}F)^{m+1}}{(\tan x+\tan y+\tan x\tan y)^m}$$

But, $1=\tan\frac{\pi}{4}=\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}\Leftrightarrow 1=\tan x+\tan y+\tan x\tan y$ and then (1) leads us to the following inequality:

$$V(m,x,y)>2^{2(m+1)}\cdot(\sqrt{3})^{m+1}\cdot F^{m+1}$$

□

Key words and phrases. Ionescu-Weitzenböck, Radon, Klamkin, Ionescu-Nesbitt, Mitrinovič, Doucet, Tsintsifas, Bergström, Gordon, Bătineţu-Giurgiu, Goldner.

Application 3.
If $m \in \mathbb{R}_+ = [0, \infty)$, then in any ABC triangle the following inequality holds:

$$\frac{a^{2(m+1)}}{\tan^m 22^\circ} + \frac{b^{2(m+1)}}{\tan^m 22^\circ \cdot \tan^m 23^\circ} + \frac{c^{2(m+1)}}{\tan^m 23^\circ} > 2^{2m+2}(\sqrt{3})^{m+1} \cdot F^{m+1}$$

Proof. We have:
(1)

$$U = \frac{a^{2(m+1)}}{\tan^m 22^\circ} + \frac{b^{2(m+1)}}{\tan^m 22^\circ \cdot \tan^m 23^\circ} + \frac{c^{2(m+1)}}{\tan^m 23^\circ} \overset{\text{Radon}}{\geq} \frac{(a^2+b^2+c^2)^{m+1}}{(\tan 22^\circ + \tan 22^\circ \tan 23^\circ + \tan 23^\circ)^m}$$

According to Ionescu-Weitzenböck's inequalities we have:

(I-W) $$a + b + c \geq F$$

Also, we have:
(2)

$$1 = \tan 45^\circ = \tan(22^\circ + 23^\circ) = \frac{\tan 22^\circ + \tan 23^\circ}{1 - \tan 22^\circ \tan 23^\circ} \Leftrightarrow 1 = \tan 22^\circ + \tan 22^\circ \tan 23^\circ + \tan 23^\circ$$

So, from (1), (I-W), (2), we deduce that:

$$U > \frac{(4\sqrt{3}F)^{m+1}}{1} = 2^{2m+2}(\sqrt{3})^{m+1} F^{m+1}$$

□

Application 4.
In any ABC triangle having the area F the following inequality holds:

$$(a^2 r_b + b^2 r_c + c^2 r_a)^2 \geq 48\sqrt{3} \cdot F^3$$

Proof. We have M.S. Klamkin's inequality:

(K) $$(a^2 x + b^2 y + c^2 z)^2 \geq 16(xy + yz + zx)F^2, \forall x, y, z \in \mathbb{R}_+^*$$

If in (K) we take $x = r_b, y = r_c, z = r_a$ and we take into account that $r_a r_b + r_b r_c + r_c r_a = s^2$, where s is the semiperimeter, we obtain:

$$(a^2 r_b + b^2 r_c + c^2 r_a)^2 \geq 16(r_a r_b + r_b r_c + r_c r_a)F^2 = 16 s^2 F^2 =$$

$$= 16 s \cdot s \cdot F^2 \overset{\text{Mitrinovič}}{\geq} 16 s (3\sqrt{3} r) \cdot F^2 = 48\sqrt{3}(sr)F^2 = 48\sqrt{3}F^3$$

□

Application 5.
If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle with the area F and the semiperimeter s, then:

$$\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)(r_a^4 + r_b^4 + r_c^4) \geq \frac{s^4}{3} \geq 9F^2$$

Proof. We have, according to Nesbitt-Ionescu's inequality:

(N-I) $$\sum \frac{x}{y+z} \geq \frac{3}{2}$$

and $$r_a^4 + r_b^4 + r_c^4 \overset{\text{Radon}}{\geq} \frac{(r_a + r_b + r_c)^4}{27} = \frac{(4R + r)^4}{27} \overset{\text{Doucet}}{\geq} \frac{(s\sqrt{3})^4}{27} =$$

$$= \frac{s^4 \cdot 9}{27} = \frac{s^4}{3} = \frac{s^2 \cdot s^2}{3} \overset{\text{Mitrinovič}}{\geq} \frac{s^2 (3\sqrt{3} r)^2}{3} = 9(sr)^2 = 9F^2$$

□

Application 6.
If $m, n \in \mathbb{R}_+ = [0, \infty), m \geq n$ and ABC is a triangle having the area F, then:

$$m(ab + bc + ca) - nr(r_a + r_b + r_c) \geq (4m - n)\sqrt{3}F$$

Proof. We know that:

$$ab + bc + ca = s^2 + r^2 + 4Rr \tag{1}$$

$$\text{and } r_a + r_b + r_c = 4R + r \tag{2}$$

So, $m(ab + bc + ca) - nr(r_a + r_b + r_c) = m(s^2 + r^2 + 4Rr) - nr(4R + r) =$

$$= m \cdot s^2 + (m - n)r(4R + r) \overset{\text{Mitrinovič}}{\geq} m \cdot s(3\sqrt{3}r) + (m - n)r(4R + r) =$$

$$= 3m\sqrt{3}sr + (m - n)r(4R + r) \overset{\text{Doucet}}{\geq} 3m\sqrt{3}F + (m - n) \cdot r \cdot s\sqrt{3} =$$

$$= 3m\sqrt{3}F + (m - n)\sqrt{3} \cdot sr = 3m\sqrt{3}F + (m - n)\sqrt{3}F = (3m + m - n)\sqrt{3}F =$$

$= (4m - n)\sqrt{3}F$. So, s is the triangle's semiperimeter.

□

Application 7.
If $m \in \mathbb{R}_+ = [0, \infty)$, then in any ABC triangle having the area F the following inequality holds:

$$\frac{a^{m+4}}{b^m + c^m} + \frac{b^{m+4}}{c^m + a^m} + \frac{c^{m+4}}{a^m + b^m} \geq 8F^2$$

Proof. According to Tsintsifas triangle, we have:

$$\sum_{cyc} \frac{x}{y + z} a^4 \geq 8F^2 \tag{T}$$

□

So, in (T) we take: $x = a^m, y = b^m, z = c^n$, we obtain:

$$\sum_{cyc} \frac{a^{m+4}}{b^m + c^m} \geq 8F^2$$

Application 8.
In any ABC triangle having the area F the following inequality holds:

$$(a + b)^4 \cdot m_a \cdot w_b + (b + c)^4 m_b w_c + (c + a)^4 m_c \cdot w_c \geq 256F^3\sqrt{3}$$

where m_a is the mediator, and w_a the length of the bisector from A and the analogs.

Proof. Because $m_a \geq h_{a_c}, w_a \geq h_a$ and the analogs, it follows that:

$$V = \sum_{cyc} (a + b)^4 m_a \cdot w_b \geq \sum_{cyc} (a + b)^4 h_a h_b = \sum_{cyc} \frac{(a + b)^4 (ah_a \cdot bh_b)}{ab} =$$

$$= 4F^2 \cdot \sum_{cyc} \frac{(a + b)^4}{ab} \overset{\text{AM-GM}}{\geq} 4F^2 \cdot \sum_{cyc} \frac{(a + b)^4}{\frac{(a+b)^2}{4}} =$$

$$= 16F^2 \cdot \sum_{cyc} (a + b)^2 \overset{\text{Bergström}}{\geq} 16F^2 \cdot \frac{(\sum_{cyc}(a + b))^2}{3} =$$

$$= \frac{16}{3} F^2 \cdot 4 \cdot (a + b + c)^2 = \frac{64}{3} \cdot F^2 \cdot 4 \cdot s^2 \overset{\text{Mitrinovič}}{\geq}$$

$\geq \frac{256}{3} \cdot F^2 \cdot s \cdot (3\sqrt{3}r) = 256\sqrt{3}F^2(s \cdot r) = 256\sqrt{3} \cdot F^3$, where s is the semiperimeter.

□

Application 9.
If $m, n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle with the area F, then:

$$\frac{x \cdot a^4}{my + nz} + \frac{y \cdot b^4}{my + nz} + \frac{z \cdot z^4}{mz + nx} \geq \frac{16F^2}{m + n}$$

Proof. We have:

$$U(m,n) = \sum_{cyc} \frac{xa^4}{my + nz} = \sum_{cyc} \frac{x^2a^4}{mxy + nxz} = \sum_{cyc} \frac{(xa^2)^2}{mxy + nxz} \overset{\text{Bergström}}{\geq}$$

$$\geq \frac{(xa^2 + yb^2 + zc^2)^2}{\sum_{cyc}(mxy + nxz)} = \frac{(xa^2 + yb^2 + zc^2)}{(m+n)(xy + yz + zx)} \overset{\text{Klamkin}}{\geq} \frac{16(xy + yz + zx)F^2}{(m+n)(ny + yz + zx)} = \frac{16F^2}{m+n}$$

Observation. If $m = n = 1$ we obtain one of Tsintsifas's inequalities. □

Application 10.
If ABC is a triangle having the area F, then:

$$(a+b)\sqrt{a^2 + b^2 - ab} + (b+c)\sqrt{b^2 + c^2 - bc} + (c+a)\sqrt{c^2 + a^2 - ca} \geq 8\sqrt{3}F$$

Proof. We have:

$$U = \sum_{cyc}(a+b)\sqrt{a^2 + b^2 - ab} \overset{\text{AM-GM}}{\geq} \sum_{cyc}(a+b)\sqrt{2ab - ab} =$$

$$= \sum_{cyc}(a+b)\sqrt{ab} \overset{\text{AM-GM}}{\geq} \sum_{cyc} 2\sqrt{ab} \cdot \sqrt{ab} = 2 \cdot \sum_{cyc} ab \overset{\text{Gordon}}{\geq} 2 \cdot 4\sqrt{3}F = 8\sqrt{3}F$$

□

Application 11.
In any ABC triangle with the area F the following inequality holds:

$$\frac{r_a^2}{\cot \frac{B}{2} \cot \frac{C}{2}} + \frac{r_b^2}{\cot \frac{C}{2} \cot \frac{A}{2}} + \frac{r_c^2}{\cot \frac{A}{2} \cot \frac{B}{2}} \geq \sqrt{3}F$$

Proof. We have:

$$U = \sum_{cyc} \frac{r_a^2}{\cot \frac{B}{2} \cot \frac{C}{2}} \overset{\text{Bergström}}{\geq} \frac{(r_a + r_b + r_c)^2}{\sum_{cyc} \cot \frac{A}{2} \cot \frac{B}{2}} = \frac{(4R + r)^2}{\frac{4R+r}{r}} =$$

$$= (4R + r) \cdot r \overset{\text{Doucet}}{\geq} \sqrt{3} \cdot s \cdot r = \sqrt{3} \cdot F, \text{ where } s \text{ is the semiperimeter.}$$

□

Application 12.
If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle with the area F then:

$$\left(\frac{(x+y)^3a^2}{z} + \frac{(y+z)^3b^2}{x} + \frac{(z+x)^3c^2}{y}\right) \cdot \left(\frac{z^3a^2}{x+y} + \frac{x^3b^2}{y+z} + \frac{y^3c^2}{z+x}\right) \geq 192 \cdot F^2$$

Proof. We have:

$$U(x,y,z)=\left(\sum_{cyc}\frac{(x+y)^3a^2}{z}\right)\cdot\sum_{cyc}\frac{z^3a^2}{x+y}\overset{\text{C-B-S}}{\geq}$$

$$\geq\left(\sum_{cyc}\sqrt{\frac{(x+y)^3a^2}{z}\cdot\frac{za^2}{x+y}}\right)^2=\left(\sum_{cyc}\frac{x+y}{z}a^2\right)^2 \tag{1}$$

$$\text{But, } V=\sum_{cyc}\frac{x+y}{z}a^2\Leftrightarrow V+\sum_{cyc}a^2=(x+y+z)\cdot\sum_{cyc}\frac{a^2}{z}\geq$$

$$\overset{\text{Bergström}}{\geq}(x+y+z)\cdot\frac{(a+b+c)^2}{z+x+y}=(a+b+c)^2=4s^2\Leftrightarrow V\geq 4s^2-(a^2+b^2+c^2)=$$

$$=4s^2-2(s^2-r^2-4Rr)=2(s^2+v^2+4Rr)=2(ab+bc+ca)\geq$$

$$\overset{\text{Gordon}}{\geq}2\cdot 4\sqrt{3}F=8\sqrt{3}F,\text{ namely Bătineţu-Giurgiu inequality.} \tag{B-G}$$

From (1) and (B-G) we deduce that:

$$U(x,y,z)\geq(8\sqrt{3}F)^2=192F^2$$

□

Application 13.

If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and ABC is a triangle with the area F, then:

$$\left(\frac{x+y}{z}a+\frac{y+z}{x}\cdot b+\frac{z+x}{y}c\right)\left(\frac{x+y}{z}a^3+\frac{y+z}{x}b^3+\frac{z+x}{y}c^3\right)\geq 192F^2$$

Proof. We have:

$$U(x,y,z)=\left(\sum_{cyc}\frac{x+y}{z}\cdot a\right)\cdot\sum_{cyc}\frac{x+y}{z}a^3\overset{\text{C-B-S}}{\geq}\left(\sum_{cyc}\sqrt{\frac{x+y}{z}\cdot a\cdot\frac{x+y}{z}a^3}\right)^2=$$

$$=\sum_{cyc}\frac{x+y}{z}a^2\overset{\text{Bătineţu-Giurgiu}}{\geq}(8\sqrt{3}F)^2=192F^2$$

and we prove Bătineţu-Giurgiu's inequality:

$$\textbf{I.}V=\sum_{cyc}\frac{x+y}{z}\cdot a^2\Leftrightarrow V+\sum_{cyc}a^2=(x+y+z)\sum_{cyc}\frac{a^2}{z}\overset{\text{Bergström}}{\geq}$$

$$\geq(x+y+z)\cdot\frac{(a+b+c)^2}{z+x+y}=4s^2\Leftrightarrow V\geq 4s^2-\sum_{cyc}a^2=$$

$$=4s^2-2(s^2-r^2-4Rr)=2(s^2+r^2+4Rr)=$$

$$=2(ab+bc+ca)\overset{\text{Gordon}}{\geq}2(4\sqrt{3}F)=8\sqrt{3}F \tag{B-G}$$

$$\textbf{II. } V=\sum_{cyc}\frac{x+y}{z}a^2\overset{\text{AM-GM}}{\geq}\sqrt[3]{\prod_{cyc}\frac{x+y}{z}a^2}\overset{\text{AM-GM}}{\geq}3\cdot\sqrt[3]{\prod_{cyc}\cdot\frac{2\sqrt{xy}}{t}a^2}=$$

$$=6\sqrt[3]{(abc)^2}=6\sqrt[3]{(4R)^2}=6\sqrt[3]{16R^2F^2}\overset{\text{Euler}}{\geq}$$

$$\geq 6\cdot\sqrt[3]{32RrF^2}\overset{\text{Mitrinovič}}{\geq}6\sqrt[3]{64\frac{s}{3\sqrt{3}}\cdot r\cdot F^2}=\frac{29}{\sqrt{3}}\sqrt[3]{srF^2}=$$
$$=8\sqrt{3}F \text{ where } s \text{ is the semiperimeter.}$$

□

Application 14.
If $m\in\mathbb{R}_+=[0,\infty); n\in\mathbb{N}^*; x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in a triangle ABC having the area F the following inequality holds:

$$(3n)^{m+1}+\left(\frac{xa^2}{y+z}\right)^{(m+1)(n+1)}+\left(\frac{yb^2}{z+x}\right)^{(m+1)(n+1)}+\left(\frac{zc^2}{x+y}\right)^{(m+1)(n+1)}\geq\frac{(n+1)\sqrt{3}}{2^{2m-1}}F$$

Proof. We have:

$$U(m,n)=(3n)^{m+1}+\sum_{cyc}\left(\left(\frac{xa^2}{y+z}\right)^{n+1}\right)^{m+1}\overset{\text{Radon}}{\geq}$$
$$\geq\frac{(3n+\sum_{cyc}(\frac{xa^2}{y+z})^{n+1})^{m+1}}{4^m}=\frac{1}{4^m}\cdot\sum_{cyc}\left(n+\left(\frac{xa}{y+z}\right)^{n+1}\right)\geq$$
$$\overset{\text{AM-GM}}{\geq}\frac{1}{4^m}\cdot\sum_{cyc}(n+1)\sqrt[n+1]{\underbrace{1\cdot1\cdot\ldots\cdot1\cdot1}_{"n"\text{times}}\left(\frac{xa^2}{y+z}\right)^{n+1}}=$$
$$=\frac{1}{4^m}\cdot(n+1)\cdot\sum_{cyc}\frac{xa^2}{y+z}\overset{\text{Tsintsifas}}{\geq}\frac{1}{4^m}(n+1)\cdot2\sqrt{3}F=\frac{(n+1)^{\sqrt{3}}}{2^{2m-1}}F$$

□

Application 15.
If $x\in\mathbb{R}$ then in ABC triangle with the area F, the following inequality holds:

$$\frac{a^{m+2}}{(a\cos^2x+b\sin^2x)^m}+\frac{b^{m+2}}{(b\cos^2x+c\sin^2x)^m}+\frac{c^{m+2}}{(c\cdot\cos^2x+a\sin^2x)^m}\geq$$
$$\geq4\sqrt{3}F,\forall m\in\mathbb{R}_+=[0,\infty)$$

Proof. We have:

$$W(m)=\sum_{cyc}\frac{a^{m+2}}{(a\cos^2x+b\sin^2x)^m}=\sum_{cyc}\frac{(a^2)^{m+1}}{(a^2\cos^2x+b\sin^2x)^m}\geq$$
$$\overset{\text{Radon}}{\geq}\frac{(a^2+b^2+c^2)^{m+1}}{(\sum_{cyc}(a^2\cos^2x+ab\cdot\sin^2x))^m}=\frac{(a^2+b^2+c^2)^{m+1}}{((a^2+b^2+c^2)\cdot\cos^2x+(ab+bc+ca)\sin^2x)^m}\geq$$
$$\geq\frac{(a^2+b^2+c^2)^{m+1}}{((a^2+b^2+c^2)(\sin^2x+\cos^2x))^m}=\frac{(a^2+b^2+c^2)^{m+1}}{(a^2+b^2+c^2)^m}=$$
$$=a^2+b^2+c^2\overset{\text{Ionescu-Weitzenböck}}{\geq}4\sqrt{3}F,\forall m\in\mathbb{R}_+$$

□

Application 16.
If $m,n\in\mathbb{N}$ and ABC is a triangle having the area F, then:

$$(m+a^{2m+2})(n+b^{2m+2})+(m+b^{2m+2})(n+c^{2n+2})+(m+c^{2m+2})(n+a^{2n+2})\geq$$
$$\geq16(m+1)(n+1)F^2$$

Proof. We have:

$$V(m,n)=\sum_{cyc}(m+(a^2)^{m+1})(n+(b^2)^{n+1})\overset{\text{AM-GM}}{\geq}$$

$$\geq\sum_{cyc}\left((m+1)\sqrt[m+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1\cdot 1}_{\text{"}m\text{" times}}(a^2)^{m+1}}\cdot(n+1)\cdot\sqrt[n+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1\cdot 1}_{\text{"}n\text{" times}}(b^2)^{n+1}}\right)=$$

$$=(m+1)(n+1)\cdot\sum_{cyc}a^2b^2\overset{\text{Bergström}}{\geq}\frac{(m+1)(n+1)}{3}\left(\sum_{cyc}ab\right)^2\geq$$

$$\overset{\text{V.O. Gordon}}{\geq}\frac{(m+1)(n+1)}{3}(4\sqrt{3}F)^2=\frac{48(m+1)(n+1)F^2}{3}=$$

$$=16(m+1)(n+1)F^2$$

□

Observation. If $m=n=0$ then we obtain:

$$\sum_{cyc}a^2b^2\geq 26F^2$$

and if we take into account that: $a^4+b^4+c^4\geq a^2b^2+b^2c^2+c^2a^2$ it follows: $a^4+b^4+c^4\geq 16F^2$ namely we've obtained Goldner's inequality.

Application 17.
If $m\in\mathbb{N}^*, x,y\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle with the area F the following inequality holds:

$$3m+(ax+by)^{2m+2}+(bx+cy)^{2m+2}+(cx+ay)^{2m+2}\geq 4(m+1)(x+y)^2\sqrt{3}F$$

Proof. We have:

$$V(m)=3m+\sum_{cyc}(ax+by)^{2m+2}=\sum_{cyc}\left(m+(ax+by)^{2(m+1)}\right)\geq$$

$$\overset{\text{AM-GM}}{\geq}\sum_{cyc}(m+1)\sqrt[m+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1\cdot 1}_{\text{"m" times}}((ax+by)^2)^{m+1}}=(m+1)\sum_{cyc}(ax+by)^2\geq$$

$$\overset{\text{Bergström}}{\geq}\frac{m+1}{3}\cdot\left(\sum_{cyc}(ax+by)\right)^2=\frac{(m+1)(x+y)^2}{3}(a+b+c)^2=$$

$$=\frac{4(m+1)(x+y)^2\cdot s^2}{3}\overset{\text{Mitrinovič}}{\geq}\frac{4(m+1)(x+y)^2 s\cdot 3\sqrt{3}r}{3}=$$

$$=4(m+1)(x+y)^2\sqrt{3}\cdot s\cdot r=4(m+1)(x+y)^2\sqrt{3}F$$

□

Application 18.
In any ABC triangle with the semiperimeter s we have:

$$\frac{1}{a(xb+yc)}+\frac{1}{b(xc+ya)}+\frac{1}{c(xa+yb)}\geq\frac{27}{4(x+y)s^2},\forall x,y\in\mathbb{R}_+^*=(0,\infty)$$

Proof. We have:

$$V=\sum_{cyc}\frac{1}{a(xb+yc)}=\sum_{cyc}\frac{1}{abx+acy}\overset{\text{Bergström}}{\geq}$$

$$\geq\frac{(1+1+1)^2}{\sum_{cyc}(abx+acy)}=\frac{9}{(x+y)(ab+bc+ca)}\geq\frac{9}{(x+y)\frac{(a+b+c)^2}{3}}=$$

$$=\frac{27}{(x+y)\cdot 4s^2}=\frac{27}{4(x+y)s^2}$$

□

Application 19.
If $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{m^2x^2}{(y+z)^2}+\frac{m^2y^2}{(z+x)^2}+\frac{m^2z^2}{(x+y)^2}+n^2(a^4+b^4+c^4)\geq 4mn\sqrt{3}F$$

Proof. According to means inequality we have:

$$\sum_{cyc}\frac{m^2x^2}{(y+z)^2}+\sum_{cyc}n^2a^4\geq 2\cdot\sqrt{\left(\sum_{cyc}\frac{m^2x^2}{(y+z)^2}\right)\cdot\sum_{cyc}n^2a^4}\geq$$

$$\overset{\text{C-B-S}}{\geq}2\sqrt{m^2n^2\cdot\left(\sum_{cyc}\frac{x}{y+z}a^2\right)^2}=2mn\cdot\sum_{cyc}\frac{x}{y+z}a^2\geq$$

$$\overset{\text{Tsintsifas}}{\geq}2\cdot m\cdot n\cdot 2\cdot\sqrt{3}F=4mn\cdot\sqrt{3}\cdot F$$

□

Application 20.
If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$x^2a^3+y^2b^3+z^2c^3+\frac{a^5}{(y+z)^2}+\frac{b^5}{(z+x)^2}+\frac{c^5}{(x+y)^2}\geq 16F^2$$

Proof. We have:

$$\sum_{cyc}x^2a^3+\sum_{cyc}\frac{a^5}{(y+z)^2}\overset{\text{AM-GM}}{\geq}2\sqrt{\left(\sum_{cyc}x^2a^3\right)\left(\sum_{cyc}\frac{a^5}{(y+z)^2}\right)}\geq$$

$$\overset{\text{C-B-S}}{\geq}2\cdot\sqrt{\left(\sum_{cyc}\frac{xa^4}{y+z}\right)^2}=2\cdot\sum_{cyc}\frac{xa^4}{y+z}\overset{\text{Tsintsifas}}{\geq}2\cdot 8\cdot F^2=16F^2$$

□

References

[1] Daniel Sitaru, Mihály Bencze, *699 Olympic Mathematical Challenges.* Studis Publishing House, Iaşi, 2017.
[2] Mihály Bencze, Daniel Sitaru, *Quantum Mathematical Power.* Studis Publishing House, Iaşi, 2018.
[3] Mihály Bencze, Daniel Sitaru, Marian Ursărescu, *Olympic Mathematical Energy.* Studis Publishing House, Iaşi, 2018.
[4] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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