NEW APPLICATIONS OF FAMOUS ALGEBRAIC INEQUALITIES (I)

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Abstract. In the paper bellow, are presented new applications of famous inequalities.

Application 1.

If $m \in \mathbb{R}_+ = [0, \infty), n \in \mathbb{N}^*, x_h \in \mathbb{R}_+^* = (0, \infty), k = \overline{1, n}$, then:

$$\sum_{k=1}^{n} \left(\frac{x_k}{x_{h+1}}\right)^{m+1} \geq \sum_{k=1}^{n} \frac{x_h}{x_{k+1}}, \text{ where } x_{n+1} = x_1$$

Proof. We have:

$$U_n(m) = \sum_{k=1}^{n} \left(\frac{x_h}{x_{k+1}}\right)^{m+1} \overset{\text{RADON}}{\geq} \frac{1}{n^m} \left(\sum_{k=1}^{n} \frac{x_k}{x_{h+1}}\right)^{m+1} =$$

$$= \frac{1}{n^m} \left(\sum_{h=1}^{n} \frac{x_h}{x_{h+1}}\right)^m \cdot \sum_{h=1}^{n} \frac{x_h}{x_{h+1}} \overset{\text{AM-GM}}{\geq} \frac{1}{n^m} \left(n \cdot \sqrt[n]{\prod_{h=1}^{n} \frac{x_a}{x_{h+1}}}\right)^m \sum_{h=1}^{n} \frac{x_h}{x_{h+1}}$$

$$= \frac{1}{n^m} \cdot n^m \cdot \sum_{h=1}^{n} \frac{x_h}{x_{h+1}} = \sum_{h=1}^{n} \frac{x_h}{x_{h+1}}$$

□

Application 2.

If $u, v, w, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\frac{x^2 + y^2 + z^2 + 2yt}{xy + xz} u^2 + \frac{x^2 + y^2 + z^2 + 2zx}{yz + yz} v^2 + \frac{x^2 + y^2 + z^2 + 2xy}{zx + zy} w^2 \geq$$

$$\geq 3(uv + vw + wu) - \frac{1}{2}(u^2 + v^2 + w^2)$$

Proof. We have:

$$U = \sum_{cyc} \frac{x^2 + y^2 + z^2 + 2yz}{xy + xz} u^2 \Leftrightarrow U = 2\sum_{cyc} u^2 = \sum_{cyc} \left(\frac{x^2 + y^2 + z^2 + 2yz}{xy + xz} + 2\right) u^2 =$$

$$= \sum_{cyc} \frac{x^2 + y^2 + z^2 + 2(xy + yz + zx)}{xy + xz} \cdot u^2 = (x + y + z)^2 \cdot \sum_{cyc} \frac{u^2}{xy + xz} \overset{\text{Bergström}}{\geq}$$

$$\geq (x + y + z)^2 \frac{(u + v + w)^2}{\sum_{cyc}(xy + xz)} = \frac{(x + y + z)^2}{2(xy + yz + zx)} (u + v + w)^2 \geq$$

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$$\geq \frac{3(xy+yz+zx)}{2(xy+yz+zx)}(u+v+w)^2 = \frac{3}{2}(u+v+w)^2 \Leftrightarrow$$

$$\Leftrightarrow U \geq \frac{3}{2}(u^2+v^2+w^2+2(uv+vw+wu)) - 2(u^2+v^2+w^2) =$$

$$= 3(uv+vw+wu) - \frac{1}{2}(u^2+v^2+w^2)$$

□

Application 3.
If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\frac{2x^2+y^2+z^2+2yz}{xy+xz} + \frac{2y^2+z^2+x^2+2zx}{yz+xy} + \frac{2z^2+x^2+y^2+2xy}{xz+yz} \geq 9$$

Proof. According to Nesbitt-Ionescu's inequality, we have:

$$\text{(N-I)} \qquad U = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

$$U = \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \Leftrightarrow U + 3 = (x+y+z)\sum_{cyc}\frac{1}{x} \overset{\text{Bergström}}{\geq}$$

$$\geq (x+y+z)\frac{9}{x+y+z} = 9 \Leftrightarrow V \geq 9 - 3 = 6. \text{ So,}$$

$$(*) \qquad \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq 6$$

We have:

$$2U + V = \sum_{cyc}\left(\frac{2x}{y+z} + \frac{y+z}{x}\right) = \sum_{cyc}\frac{2x^2+(y+z)^2}{x(y+z)} =$$

$$= \sum_{cyc}\frac{2x^2+y^2+z^2+2yz}{x(y+z)} \geq 2\cdot\frac{3}{2} + 6 = 3 + 6 = 9$$

which proves the relationship from enunciation. □

Application 4.
If $m, n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$8(x+y+z)^9\left(\left(\frac{xy}{myz+uzx}\right)^3 + \left(\frac{yz}{mzx+nxy}\right)^3 + \left(\frac{zx}{mx+xyz}\right)^3\right) \geq \frac{8\cdot 3^{10}}{(m+n)^3}\cdot x^3\cdot y^3\cdot z^3$$

Proof. We have:

$$\sum_{cyc}\frac{u}{mv+xw} = \sum_{cyc}\frac{u^2}{muv+nuw} \overset{\text{Bergström}}{\geq} \frac{(u+v+w)^2}{(m+u)(uv+vw+wu)} \geq$$

$$\geq \frac{3(uv+vw+wa)}{(m+n)(uv+vw+wu)} = \frac{3}{n}, \forall u, v, w \in \mathbb{R}_+^*$$

which is a generalisation of Nesbitt's inequality.

If $u = xy, v = yz, w = zx$ then it follows:

$$(1) \qquad \sum_{cyc}\frac{xy}{myz+uzn} \geq \frac{3}{m+n}$$

So, $U = 8\Big(\sum_{cyc} x\Big)^9 \cdot \sum_{cyc}\Big(\frac{xy}{myz+uzx}\Big)^3 \overset{\text{Radon}}{\geq} 8\Big(\sum_{cyc} x\Big)^9 \cdot \frac{1}{9}\Big(\sum_{cyc}\frac{xy}{myz+uzx}\Big)^3 \geq$

$$\overset{(1)}{\geq} \frac{1}{9}\cdot 8\cdot (x+y+z)^9 \cdot \Big(\frac{3}{m+u}\Big)^3 = \frac{8\cdot 27}{(m+u)^3}\cdot (x+y+z)^9\cdot \frac{1}{9} \geq$$

$$\overset{\text{AM-GM}}{\geq} \frac{8\cdot 3}{(m+u)^3}\cdot \big(3\sqrt[3]{xyz}\big)^9 = \frac{8\cdot 3^{10}}{(m+u)^3}x^3y^3z^3$$

So, $m = n = 1$ we obtain Problem's PP.227768 result, proposed by Daniel Sitaru in Octogon. □

Application 5. If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\Big(\frac{x}{y}\Big)^{2019} + \Big(\frac{y}{z}\Big)^{2019} + \Big(\frac{z}{x}\Big)^{2019} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

Proof. We have:

$$U = \sum_{cyc}\Big(\frac{x}{y}\Big)^{2019} \overset{\text{Radon}}{\geq} \frac{1}{3^{2015}}\Big(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\Big)^{2019} =$$

$$= \frac{1}{3^{2018}}\Big(3\cdot\sqrt[3]{\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}}\Big)^{2018}\Big(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\Big) = \Big(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\Big)$$

□

Application 6.
If $x, y \in \mathbb{R}_+ = [0, \infty); a, b, c, x, y \in \mathbb{R}_+^* = (0, \infty)$ then:

$$(a^2+b^2+c^2)\Big(\frac{1}{xa^2+yab} + \frac{1}{xb^2+ybc} + \frac{1}{xc^2+yca}\Big) \geq \frac{9}{x+y}$$

Proof. We have:

$$V(x,y) = \Big(\sum_{cyc} a^2\Big)\cdot \sum_{cyc}\frac{1}{a^2x+aby} \overset{\text{Bergström}}{\geq}$$

$$\geq \Big(\sum_{cyc} a^2\Big)\cdot \frac{9}{\sum_{cyc}(a^2x+aby)} = \frac{9\cdot\sum_{cyc}a^2}{x\cdot\sum_{cyc}a^2 + y\sum_{cyc}ab} \geq$$

$$\overset{\sum x^2 \geq \sum xy}{\geq} \frac{9\cdot \sum_{cyc}a^2}{(x+y)\sum_{cyc}a^2} = \frac{9}{x+y}$$

□

Application 7.
If $n \in \mathbb{N}^* - \{1\}, x_h \in \mathbb{R}_+^* = (0,\infty) \forall k = \overline{1,n}$ then:

$$\sum_{k=1}^{n}\frac{x_k}{1+x_h^2} \leq \frac{n}{2}$$

Proof. Let be:

$U_n = \sum_{k=1}^{n}\frac{x_k}{1+x_k^2} \overset{\text{AM-GM}}{\leq} \frac{1}{2}\sum_{h=1}^{n}\frac{x_k}{\sqrt{x_h^2}} = \frac{1}{2}\sum_{h=1}^{n} 1 = \frac{n}{2}$, we have equality $\Leftrightarrow x_h = 1, \forall h = \overline{1,n}$

□

Application 8

If $a, b, x, y, z, t \in \mathbb{R}_+^*$, then:

$$\frac{x^7}{ax^3+byzt}+\frac{y^7}{ay^3+bxyt}+\frac{z^7}{az^3+bxyt}+\frac{t^7}{at^3+bxyz}\geq\frac{4}{a+b}\cdot x\cdot y\cdot z\cdot t$$

Proof. We have:

$$U=\sum_{cyc}\frac{x^7}{ax^3+by+t}=\sum_{cyc}\frac{x^8}{ax^7+bxy+t}\overset{\text{AM-GM}}{\geq}\sum_{cyc}\frac{x^8}{ax^4+\frac{b}{y}\cdot\sum_{cyc}x^4}=$$

$$=4\cdot\sum_{cyc}\frac{(x^4)^2}{4ax^4+b\cdot\sum_{cyc}x^4}\overset{\text{Bergström}}{\geq}4\frac{(\sum_{cyc}x^4)^2}{4a\cdot\sum_{cyc}x^4+yb\sum_{cyc}x^4}=$$

$$=\frac{1}{a+b}\cdot\sum_{cyc}x^4\overset{\text{AM-GM}}{\geq}\frac{4\sqrt[4]{x^4\cdot y^4\cdot z^4\cdot t^4}}{a+b}=\frac{4}{a+b}\cdot xyzt$$

□

Application 9.

If $m\in\mathbb{R}_+=[0,\infty)$ and $a,b,c\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\left(\frac{a}{b}\right)^{m+1}+\left(\frac{b}{c}\right)^{m+1}+\left(\frac{c}{a}\right)^{m+1}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Proof. We have:

$$U(m)=\sum_{cyc}\left(\frac{a}{b}\right)^{m+1}\overset{\text{Radon}}{\geq}\frac{1}{3^m}\left(\sum_{cyc}\frac{a}{b}\right)^{m+1}=$$

$$=\frac{1}{3^m}\left(\sum_{cyc}\frac{a}{b}\right)^m\cdot\left(\sum_{cyc}\frac{a}{b}\right)\overset{\text{AM-GM}}{\geq}\frac{1}{3^m}\left(3\sqrt[3]{\prod_{cyc}\frac{a}{b}}\right)^m\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=$$

$$=\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

□

Application 10.

If $m\in\mathbb{N}, x,y,z,u,v,\in\mathbb{R}_+^*=(0,\infty)$ then:

$$3m+\left(\frac{ux^4}{v+w}\right)^{m+1}+\left(\frac{vy^4}{w+u}\right)^{m+1}+\left(\frac{wz^4}{u+v}\right)^{m+1}\geq$$

$$\geq\frac{m+1}{8}(x+y+z)^4-(m+1)(x^4+y^4+z^4)$$

Proof. We have:

$$3m+\sum_{cyc}\left(\frac{ux^4}{v+w}\right)^{m+1}=\sum_{cyc}\left(m+\left(\frac{ux^4}{v+w}\right)^{m+1}\right)\overset{\text{AM-GM}}{\geq}$$

$$\geq\sum_{cyc}(m+1)\sqrt[m+1]{\underbrace{1\cdot1\cdot\ldots\cdot1\cdot1}_{\text{"}m\text{" times}}\left(\frac{ux^4}{v+w}\right)^{m+1}}=(m+1)\cdot\sum_{cyc}\frac{ux^4}{v+w}=$$

$$=(m+1)\sum_{cyc}\left(\frac{ux^4}{v+w}+x^4\right)=(m+1)\cdot\sum_{cyc}x^4=$$

$$= (m+1)(u+v+w)\sum_{cyc}\frac{x^4}{v+w} - (m+1)\sum_{cyc}x^4 \overset{\text{Bergström}}{\geq}$$

$$\geq (m+1)(u+v+w)\cdot\frac{(x^2+y^2+z^2)^2}{\sum_{cyc}(v+w)} - (m+1)\cdot\sum_{cyc}x^4 =$$

$$= (m+1)(u+v+w)\frac{(x^2+y^2+z^2)^2}{2(u+v+w)} - (m+1)\cdot\sum_{cyc}x^4 \overset{\text{Bergström}}{\geq}$$

$$\geq \frac{m+1}{8}(x+y+z)^4 - (m+1)(x^4+y^4+z^4)$$

□

Application 11.
If $n \in \mathbb{N}$ and $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\Big(3n + (xy)^{n+1} + (yz)^{n+1} + (zx)^{n+1}\Big)\cdot\Big(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\Big) \geq \frac{9}{4}(n+1)$$

Proof. We have:

$$3n + \sum_{cyc}(xy)^{n+1} \overset{\text{Radon}}{\geq} 3n + \frac{1}{3^n}\cdot(xy+yz+zx)^{n+1} \overset{\text{AM-GM}}{\geq}$$

$$\geq (n+1)\cdot\sqrt[n+1]{\underbrace{3\cdot 3\cdot\ldots\cdot 3\cdot 3}_{\text{"}n\text{" times}}\cdot\frac{1}{3^n}(xy+yz+zx)^{n+1}} = (n+1)(xy+yz+zx)$$

Hence:

$$\Big(3n+\sum_{cyc}(xy)^{n+1}\Big)\sum_{cyc}\frac{1}{(x+y)^2} \geq (n+1)\cdot(xy+yz+zx)\cdot\Big(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\Big) \geq$$

$$\geq \frac{9}{4}(n+1)$$

Above, we have presented Ji Chen's inequality from Problem 1970 from Crux Mathematicorum, Vol. 20, Nr. 4, pp. 108, 1995. □

Application 12.
If $n \in \mathbb{N}$, and $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then:

$$(xy+yz+zx)\cdot\Big(\frac{1}{(x+y)^{2(n+1)}} + \frac{1}{(y+z)^{2(n+1)}} + \frac{1}{(x+y)^{2(n+1)}} + 3n\Big) \geq \frac{9}{4}(n+1)$$

Proof. We have:

$$3n + \sum_{cyc}\frac{1}{(x+y)^{2(n+1)}} \overset{\text{Radon}}{\geq} 3n + \frac{1}{3^n}\Big(\sum_{cyc}\frac{1}{(x+y)^2}\Big)^{n+1} \geq$$

$$\overset{\text{AM-GM}}{\geq} (n+1)\sqrt[n+1]{\underbrace{3\cdot 3\cdot\ldots\cdot 3\cdot 3}_{\text{"}n\text{" times}}\cdot\frac{1}{3^n}\Big(\sum_{cyc}\frac{1}{(x+y)^2}\Big)^{n+1}} = (n+1)\sum_{cyc}\frac{1}{(x+y)^2}$$

and then:

$$(xy+yz+zx)\Big(3n+\sum_{cyc}\frac{1}{(x+y)^{2(n+1)}}\Big) \geq (n+1)(xy+yz+zx)\cdot\Big(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\Big)^2 \geq$$

$$\overset{\text{Ji Chen}}{\geq} (n+1)\cdot\frac{9}{4} = \frac{9(n+1)}{4}$$

Ji Chen's inequality is

$$\text{(J.C)} \qquad \left(\sum_{cyc} xy\right) \cdot \sum_{cyc} \frac{1}{(x+y)^2} \geq \frac{9}{4}$$

This inequality was established by Ji Chen, in Problem 1940, from Crux Mathematicorum, Vol. 20, Nr. 4., pp. 108, 1995 and then was used at Iran's Mathematical Olympiad. □

Application 13.
If $n \in \mathbb{N}^* - \{1\}$ and $a_h, b_h \in \mathbb{R}_+^*, \forall k \in \overline{1,n}$, and $\sum_{h=1}^n a_k = 2019 \cdot \sum_{h=1}^n b_h$, then:

$$\sum_{k=1}^n \frac{a_h^2}{a_h + b_k} \geq \frac{2019^2}{2020} \cdot \sum_{h=1}^n b_k$$

Proof. We have:

$$V(n) = \sum_{k=1}^n \frac{a_h^2}{a_h + b_k} \overset{\text{Bergström}}{\geq} \frac{(\sum_{h=1}^n a_h)^2}{\sum_{h=1}^n (a_h + b_k)} =$$

$$= \frac{2019^2 \cdot (\sum_{h=1}^n b_n)^2}{2019 \cdot \sum_{h=1}^h b_k + \sum_{h=1}^h b_k} = \frac{2019^2}{2020} \cdot \sum_{h=1}^h b_k$$

□

Application 14.
If $a, b, c, x, y \in \mathbb{R}_+^* = (0, \infty)$, then:

$$a(bx+cy)^2 + b(cx+ay)^2 + c(ax+by)^2 \geq 12abcxy$$

Proof. We have:

$$V = \sum_{cyc} a(bx+cy)^2 \overset{\text{AM-GM}}{\geq} \sum_{cyc} a(2\sqrt{bcxy})^2 =$$

$$= y \cdot \sum_{cyc} abcxy = 12abcxy$$

□

Application 15.
If $m \in R_+ = [0, \infty)$, and $a, b, c, x, y \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\frac{a}{(bx+cy)^{m+1}} + \frac{b}{(cx+ay)^{m+1}} + \frac{c}{(ax+by)^{m+1}} \geq \frac{3^{m+1}}{(x+y)^{m+1}(a+b+c)^m}$$

Proof. We have:

$$V(m) = \sum_{cyc} \frac{a}{(bx+cy)^{m+1}} = \sum_{cyc} \frac{(\frac{a}{bx+cy})^{m+1}}{a^m} \overset{\text{J. Radon}}{\geq}$$

$$\geq \frac{(\sum_{cyc} \frac{a}{bx+cy})^{m+1}}{(\sum_{cyc} a)^m} \geq \frac{(\frac{3}{x+y})^{m+1}}{(a+b+c)^m} = \frac{3^{m+1}}{(x+y)^{m+1}(a+b+c)^m}$$

Above we have used the fact that:

$$(*) \qquad \sum \frac{a}{bx+cy} \geq \frac{3}{x+y}, \forall a, b, c, x, y \in \mathbb{R}_+^*$$

Indeed, $\displaystyle\sum_{cyc}\frac{a}{bx+cy}=\sum_{cyc}\frac{a^2}{abx+acy}\overset{\text{Bergström}}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}(abx+acy)}=$

$$=\frac{(a+b+c)^2}{(ab+bc+ca)(x+y)}\geq\frac{3(ab+bc+ca)}{(ab+bc+ca)(x+y))}=\frac{3}{x+y}$$

□

Application 16.
If $m,n\in\mathbb{N}, a,b,c\in\mathbb{R}_+^*=(0,\infty), t\in\mathbb{R}_+=[0,\infty)$, then:

$$\Big(m+(a+b)^{m+1}\Big)\cdot\Big(n+\frac{1}{(c+t)^{n+1}}\Big)+\Big(m+(b+c)^{m+1}\Big)\cdot\Big(n+\frac{1}{(a+t)^{m+1}}\Big)+$$

$$+\Big(m+(c+a)^{m+1}\Big)(n+\frac{1}{(b+t)^{m+1}}\Big)\geq\frac{6(m+1)(n+1)(a+b+c)}{a+b+c+3t}$$

Proof. We have:

$$V(m,n)=\sum_{cyc}\Big(m+(a+b)^{m+1}\Big)\Big(n+\frac{1}{(c+t)^{n+1}}\Big)\overset{\text{AM-GM}}{\geq}$$

$$\geq\sum_{cyc}(m+1)\sqrt[m+1]{\underbrace{1\cdot1\cdot\ldots\cdot1\cdot1}_{\text{"}m\text{" times}}(a+b)^{m+1}}\cdot(n+1)\sqrt[n+1]{\underbrace{1\cdot1\cdot\ldots\cdot1\cdot1}_{\text{"}n\text{" times}}\frac{1}{(c+t)^{n+1}}}=$$

$$=(m+1)(n+1)\cdot\sum_{cyc}\frac{a+b}{c+t}=(m+1)(n+1)\cdot\Big(\sum\frac{a+b}{c+t}+3-3\Big)=$$

$$=(m+1)(n+1)\cdot\Big(\sum_{cyc}\Big(\frac{a+b}{c+t}\Big)-3\Big)=$$

$$=(m+1)(n+1)\Big((a+b+c+t)\cdot\sum_{cyc}\frac{1}{c+t}-3\Big)\overset{\text{Bergström}}{\geq}$$

$$\geq(m+1)(n+1)\Big((a+b+c+t)\cdot\frac{9}{\sum_{cyc}(c+t)}-3\Big)=$$

$$=(m+1)(n+1)\Big((a+b+c+t)\frac{9}{a+b+c+3t}-3\Big)=$$

$$=(m+1)(n+1)\cdot\frac{9(a+b+c)+9t-3(a+b+c)-9t}{a+b+c+3t}=$$

$$=\frac{6(m+1)(n+1)(a+b+c)}{a+b+c+3t}$$

□

Application 17.
If $u,v,x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\frac{x^2+u^2y^2+v^2z^2+2uvyz}{uxy+vxz}+\frac{y^2+u^2z^2+v^2x^2+2zxuv}{uyz+vyx}+\frac{z^2+u^2x^2+v^2y^2+2xyuv}{uzx+vzy}\geq$$

$$\geq\frac{3}{u+v}((1+u+v)^2-2(u+v))$$

Proof. We have:

$$\sum_{cyc}\frac{x^2+u^2y^2+v^2z^2+2uvyt}{uxy+vxz}=\sum_{cyc}\frac{x^2+u^2y^2+v^2z^2+2uvyz+2uxy+2vxz}{uxy+vxz}=$$

$$=\sum_{cyc}\frac{(x+uy+vz)^2}{uxy+vxz}-6\overset{\text{Bergström}}{\geq}\frac{(x+y+z)+(u+v)(u+y+z)^2}{\sum_{cyc}(uxy+vxz)}-6=$$

$$=\frac{(x+y+z)^2(1+u+v)^2}{(u+v)(xy+yz+zx)}-6\geq\frac{3(xy+yz+zx)(1+u+v+yc)^2}{(u+v)(xy+yz+zx)}-6=$$

$$=\frac{3(1+u+v)^2}{u+v}-6=\frac{3(1+u+v)^2-6(u+v)}{u+v}=\frac{3}{u+v}((1+u+v)^2-2(u+v))$$

□

Application 18. If $m\in\mathbb{R}=[0,\infty)$ and $a,b,c,x,y\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\frac{a+c^{m+1}}{(bx+cy)^m}+\frac{b+a^{m+1}}{(cx+ay)^m}+\frac{c+b^{m+1}}{(ax+by)^m}\geq\frac{a+b+c}{(x+y)^n}\left(\left(\frac{3}{a+b+c}\right)^m+1\right)$$

Proof. We have:

$$\sum_{cyc}\frac{a+c^{m+1}}{(bx+cy)^m}=\sum_{cyc}\frac{a^{m+1}}{(abx+acy)^m}+\sum_{cyc}\frac{c^{m+1}}{(bx+cy)^m}\overset{\text{Radon}}{\geq}$$

$$\geq\frac{(a+b+c)^{m+1}}{\left(\sum_{cyc}(abx+acy)\right)^m}+\frac{(a+b+c)^{m+1}}{\left(\sum_{cyc}(bx+cy)\right)^m}=\frac{(a+b+c)^{m+1}}{(x+y)^m(ab+bc+ca)^m}+$$

$$+\frac{(a+b+c)^{m+1}}{(x+y)^m(a+b+c)^m}\geq\frac{(a+b+c)^{m+1}}{(x+y)^m\left(\frac{(a+b+c)^2}{3}\right)^3}+\frac{a+b+c}{(x+y)^m}=$$

$$=\frac{3^m(a+b+c)^{m+1}}{(x+y)^m(a+b+c)^m}+\frac{a+b+c}{(x+y)^m}=$$

$$=\frac{3^m\cdot(a+b+c)^{1-m}+(a+b+c)}{(x+y)^m}=\frac{a+b+c}{(x+y)^m}\cdot\left(\left(\frac{3}{a+b+c}\right)^m+1\right)$$

□

Application 19.
If $x,y,a,b,c\in\mathbb{R}_+^*=(0,\infty)$, then:

$$a^3+b^3+c^3+\frac{x^4b^4+y^4c^4}{a}+\frac{x^4c^4+y^4a^4}{b}+\frac{x^4a^4+y^4b^4}{c}\geq 6xy\cdot abc$$

Proof. We have:

$$V(x,y)=\sum_{cyc}a^3+\sum_{cyc}\frac{x^4b^4+y^4c^4}{a}\overset{\text{AM-GM}}{\geq}3\cdot\sqrt[3]{a^3b^3c^3}+2x^2y^2\cdot\sum_{cyc}\frac{b^2c^2}{a}=3abc+2x^2y^2\cdot\sum_{cyc}\frac{b^2c^2}{a}\geq$$

$$\overset{\text{AM-GM}}{\geq}3abc+2x^2y^2\cdot 3\sqrt[3]{\prod_{cyc}\frac{b^2c^2}{a}}=3abc+6x^2y^2\sqrt[3]{\frac{a^4b^4c^4}{abc}}$$

$$=3abc+6x^2y^2abc\cdot 3(1+x^2y^2)abc\overset{\text{AM-GM}}{\geq}3\cdot 2\sqrt{1\cdot x^2y^2abc}=6xyabc$$

□

Application 20.

If $m \in \mathbb{N}^* - \{1\}$ and $a, b, c, d \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\sqrt[m+1]{a^{m+1} + b^{m+1} + c^{m+1}} + \sqrt[m+1]{b^{m+1} + c^{m+1} + d^{m+1}} + \sqrt[m+1]{c^{m+1} + d^{m+1} + a^{m+1}} + $$
$$+ \sqrt[m+1]{d^{m+1} + a^{m+1} + b^{m+1}} \geq \sqrt[m+1]{3}(a + b + c + d)$$

Proof. We have:

$$V_m = \sum_{cyc} \sqrt[m+1]{a^{m+1} + b^{m+1} + c^{m+1}} \overset{\text{Radon}}{\geq} \sum_{cyc} \sqrt[m+1]{\frac{(a+b+c)^{m+1}}{8^m}} =$$
$$= \left(\frac{1}{3}\right)^{\frac{m}{m+1}} \cdot \sum_{cyc} (a + b + c) = \left(\frac{1}{3}\right)^{\frac{m}{m+1}} \cdot 3(a + b + c + d) =$$
$$= 3^{1 - \frac{m}{m+1}}(a + b + c + d) = \sqrt[m+1]{3}(a + b + c + d)$$

□

Application 21.

If $m \in \mathbb{N}^* - \{1, 2\}$ and $a, b, c, d, e \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\sqrt[m]{a^m + b^m + c^m + d^m} + \sqrt[m]{b^m + c^m + d^m + e^m} + \sqrt[m]{b^m + d^m + e^m + a^m} +$$
$$+ \sqrt[m]{d^m + e^m + a^m + b^m} + \sqrt[m]{e^m + a^m + b^m + c^m} \geq \sqrt[4]{4}(a + b + c + d + e)$$

Proof. We have:

$$U(m) = \sum_{cyc} \sqrt[m]{a^m + b^m + c^m + d^m} \overset{\text{Radon}}{\geq} \sum_{cyc} \sqrt[m]{\frac{(a+b+c+d)^m}{4^{m-1}}} =$$
$$= \sqrt[m]{\left(\frac{1}{4}\right)^{m-1}} \cdot \sum_{cyc} (a + b + c + d) = 4^{\frac{1-m}{m}} \cdot 4(a + b + c + d + e) =$$
$$= \sqrt[m]{4}(a + b + c + d + e)$$

□

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