FUNCTIONS THAT GENERATES DERIVABLE SEQUENCES

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ABSTRACT. Denoting by $\mathcal{F}(\mathbb{R}_+^*; R) = \{f : \mathbb{R}_+^* \to \mathbb{R}\}$, we say, by definition, that $g \in \mathcal{F}(\mathbb{R}_+^*; R)$ generates derivable sequences if there exists the limits:

$$\lim_{x\to\infty} g(x) = L(g) \text{ and } D(g) = \lim_{x\to\infty} (x \cdot (g(x) - L(g))) \in \mathbb{R}.$$

Two theorems concerning these functions are proved and examples of such functions are given.
Examples of operations with functions which generate derivable sequences are given and Leibniz's formula for mth derivative of the product of two functions which generate derivable sequences is done.

A sequence $(x_n)_{n\geq 0}$ of real numbers is derivable if it exists:

$$\lim_{n\to\infty} x_n = L(x_n) \in \mathbb{R} \text{ and } D(x_n) = \lim_{n\to\infty} (n \cdot (x_n - L(x_n))) \in \mathbb{R}$$

Let's denote $\mathcal{F}(\mathbb{R}_+^*; \mathbb{R}) = \{f : \mathbb{R}_+^* \to \mathbb{R}\}$. A function $f \in \mathcal{F}(\mathbb{R}_+^*; \mathbb{R})$ generates derivable sequences if exists

$$L(f) = \lim_{x\to\infty} f(x) \in \mathbb{R} \text{ and } D(f) = \lim_{x\to\infty} (x \cdot (f(x) - L(f))) \in \mathbb{R}.$$

Sometimes we will note that $D_0(f) = L(f)$ and so,

$$D(f) = D_1(f) = \lim_{x\to\infty} (x \cdot (f(x) - D_0(f))) \in \mathbb{R}.$$

We will denote
$\mathcal{D}(\mathbb{R}_+^*; \mathbb{R}) = \mathcal{D}_1(\mathbb{R}_+^*; \mathbb{R}) = \{f \in \mathcal{F}(\mathbb{R}_+^*; \mathbb{R} \Big| f \text{ generates derivable sequences}\}$.
A function $f \in \mathcal{D}(R_+^*; \mathbb{R})$ generates double derivable sequences if it exist:

$$D_2(f) = \lim_{x\to\infty} (x \cdot (x \cdot (f(x) - D_0(f)) - D_1(f))) \in \mathbb{R}$$

and let's denote
$\mathcal{D}_2(R_+^*; R) = \{f \in \mathcal{D}_1(R_+^*; R) \Big| f \text{ generates double derivable sequences }\}$
Analogous, we define the sequences:
$\mathcal{D}_m(R_+^*; \mathbb{R}) = \{f \in \mathcal{D}_{m-1}(\mathbb{R}_+^*; R) \Big| f \text{ generates sequences } m \text{ derivable }\}$.
If a function generates sequences m derivable $\forall m \in \mathbb{N}^*$, then we will say that f generates indefinite derivable sequences.
Hence $f \in \mathcal{D}_m(\mathbb{R}_+^*; \mathbb{R}), m \in \mathbb{N}^*$, if it exist:

$$D_m(f) = \lim_{x\to\infty} \underbrace{(x \cdot (x \cdot (\cdots (x}_{\text{"}m\text{" times}} \cdot (f(x) - D_0(f)) - D_1(f)) - \cdots - D_{m-1}(f))) \in \mathbb{R}$$

In this way appears the so called asymptotic development of the function f, namely:

$$f(x) = D_0(f) + \frac{D_1(f)}{x} + \frac{D_2(f)}{x^2} + \ldots + \frac{D_m(f)}{x^m} + \cdots \tag{1}$$

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Theorem 1.
The function $f \in \mathcal{F}(\mathbb{R}_+^*;\mathbb{R})$ has the property that $f \in \mathcal{D}(\mathbb{R}_+^*;\mathbb{R})$ if and only if $D_0(f) \in \mathbb{R}^*$ and it exists:

$$\lim_{x\to\infty}\left(\frac{f(x)}{D_0(f)}\right)^x = a \in \mathbb{R}_+^* \tag{2}$$

Proof 1. It is obviously the fact that:

$$\left(\frac{f(x)}{D_0(f)}\right)^x = \left(\left(1+\frac{f(x)-D_0(f)}{D_0(f)}\right)^{\frac{D_0(f)}{f(x)-D_0(f)}}\right)^{\frac{x(f(x)-D_0(f))}{D_0(f)}}, \forall x \in \mathbb{R}_+^* \tag{3}$$

If it exist $a = \lim\limits_{x\to\infty}\left(\dfrac{f(x)}{D_0(f)}\right)^x$, then relationship (3) shows us that:

$$a = e^{\lim_{x\to\infty}\frac{x(f(x)-D_0(f))}{D_0(f)}} = e^{\frac{1}{D_0(f)}\cdot\lim_{x\to\infty}(x\cdot f(x)-D_0(f)))}$$

wherefrom we deduce that it exist:

$$D(f) = \lim_{x\to\infty}(x\cdot(f(x)-D_0(f))) = D_0(f)\cdot\ln a \in \mathbb{R}$$

Reciprocal, if it exist $D(f) = \lim\limits_{x\to\infty}(x\cdot(f(x)-D_0(f))) \in \mathbb{R}$, then in relationship (3) it follows that it exist: $\lim\limits_{x\to\infty}\left(\dfrac{f(x)}{D_0(f)}\right)^x = e^{\frac{1}{D_0(f)}\cdot\lim_{x\to\infty}(x\cdot(f(x)-D_0(f)))} = e^{\frac{D(f)}{D_0(f)}} = a \in \mathbb{R}_+^*$

□

Proof 2. It is obvious that:

$$x\cdot(f(x)-D_0(f)) = x\cdot D_0(f)\cdot\left(\frac{f(x)}{D_0(f)}-1\right) = D_0(f)\cdot x\cdot(u(x)-1) \text{ where}$$

$u(x) = \dfrac{f(x)}{D_0(f)}$ and $\lim\limits_{x\to\infty} u(x) = 1$ which leads us to the fact that $\lim\limits_{x\to\infty}\dfrac{u(x)-1}{\ln u(x)} = 1$

□

Hence we have:

$$x\cdot f(x)-D_0(f)) = x\cdot D_0(f)\cdot(u(x)-1) = x\cdot D_0(f)\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x) =$$

$$= D_0(f)\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln(u(x))^x = D_0(f)\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln\left(\frac{f(x)}{D_0(f)}\right)^x, \forall x \in \mathbb{R}_+^* \tag{4}$$

If it exist $D(f) = \lim_{x\to\infty}(x\cdot(f(x)-D_0(f))) = b \in \mathbb{R}$, then in (4) we obtain that:

$$b = D_0(f)\cdot 1\cdot\ln\left(\lim_{x\to\infty}\left(\frac{f(x)}{D_0(f)}\right)^x\right) \Leftrightarrow \lim_{x\to\infty}\left(\frac{f(x)}{D_0(f)}\right)^x =$$

$$= e^{\frac{b}{D_0(f)}} = a \in \mathbb{R}_+^* \text{ if } \lim_{x\to\infty}\left(\frac{f(x)}{D_0(f)}\right)^x = e^{\frac{D(f)}{D_0(f)}}$$

Reciprocal, if it exist $\lim_{x\to\infty}\left(\frac{f(x)}{D_0(f)}\right)^x = a \in \mathbb{R}_+^*$, then relationship (4) gives us:

$$\lim_{x\to\infty}(x\cdot(f(x)-D_0(f))) = b = D_0(f)\cdot 1\cdot \ln = D_0(f)\cdot \ln a$$

Theorem 2.
The function $f \in \mathcal{F}(\mathbb{R}_+^*;\mathbb{R})$ if and only if it exist:

$$c = \lim_{x\to\infty}(1+f(x)-D_0(f))^x \in \mathbb{R}_+^* \tag{5}$$

Proof. It is obviously that:

$$(1+f(x)-D_0(f))^x = \left((1+f(x)-D_0(f))^{\frac{1}{f(x)-D_0(f)}}\right)^{x(f(x)-D_0(f))}, \forall x \in \mathbb{R}_+^* \tag{6}$$

If it exist $c = \lim_{x\to\infty}(1+f(x)-D_0(f))^x \in \mathbb{R}_+^*$, then relationship (6) lead us to the fact that:

$$c = e^{\lim_{x\to\infty}(x\cdot(f(x)-D_0(f)))}, \text{ wherefrom we deduce that it exists}$$

$$b = D(f) = \lim_{x\to\infty}(x\cdot f(x)-D_0(f))) = \ln c \in \mathbb{R}$$

Reciprocal, if it exist $b = \lim_{x\to\infty}(x\cdot(f(x)-D_0(f))) = D(f) \in \mathbb{R}$, then relationship (6) show that it exist:

$$c = \lim_{x\to\infty}(1+f(x)-D_0(f))^x = e^{D(f)} = e^b \in \mathbb{R}_+^*$$

□

Examples of functions that generates derivable sequences

1. $f(x) = (\Gamma(x+2))^{\frac{1}{x+1}} - (\Gamma(x+1))^{\frac{1}{x}}$. It is known that $\lim_{x\to\infty}(fx) = e^{-1}$ and we have to calculate:

$$D(f) = \lim_{x\to\infty}((f(x)-e^{-1})\cdot x) \tag{7}$$

For this we make use of the asymptotical development of function f (see [3] and OCTOGON collection).

$$f(x) = (\Gamma(x+2))^{\frac{1}{x+1}} - (\Gamma(x+1))^{\frac{1}{x}} = \frac{1}{e}\cdot\Big(1+\frac{1}{2x}-\frac{\ln^2 x}{8x^2}-\frac{(2A-1)\cdot\ln x}{4x^2}-$$

$$-\frac{3A^2-3A+2}{6x^2}+\cdots\Big), \text{ where } A=\sqrt{2\pi}=0,918938\ldots. \tag{8}$$

From relationship (8), by passing to the limit with $x\to\infty$ it follows that:

$$D(f) = \lim_{x\to\infty}\Big(x\cdot(f(x)-e^{-1})\Big) = \frac{1}{2e} \tag{9}$$

and also, from (8) it follows that:

$$x\cdot\Big(x\cdot(f(x)-e^{-1})-\frac{1}{2e}\Big) = -\frac{\ln^2 x}{8e}-\frac{(2A-1)\ln x}{4e}-\frac{3A^2-3A+2}{6e}+\ldots,$$

wherefrom by passing to the limit with $x\to\infty$ it follows that $D_2(f) = -\infty$. So, the function f doesn't generates sequences twice derivable.

2. The function $f:\mathbb{R}_+^*\to\mathbb{R}, f(x) = \Big(1+\frac{p}{x}\Big)^x$, generates derivable functions. Indeed,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\Big(1+\frac{p}{x}\Big)^x = \lim_{x\to\infty}\left(\Big(1+\frac{p}{x}\Big)^{\frac{x}{p}}\right)^p = D_0(f) = e^p$$

The sequence generated by f is $e_n(p) = \left(1 + \frac{p}{n}\right)^n$ with $\lim_{n\to\infty} e_n(p) = e^p$ and this sequence is derivable $\forall p \in \mathbb{R}_+^*$.
Indeed, according to Bencze-Toth inequalities we have:

$$\frac{e^p}{a \cdot n + b} < e^p - e_n(p) < \frac{e^p}{a \cdot n + c}, \forall n \in \mathbb{N}^* \tag{10}$$

where $a = \frac{2}{p^2}$ and $b, c \geq 0$. From (10), we obtain:

$$\frac{n \cdot e^p}{a \cdot n + b} < n \cdot (e^p - e_n(p)) < \frac{n \cdot e^p}{a \cdot n + b}, \forall n \in \mathbb{N},$$

wherefrom, by passing to the limit it follows

$$D(e_n(p)) = -\frac{p^2 \cdot e^p}{2}$$

which shows that f generates derivable sequences.

Proposition 1.
If $f, g \in \mathcal{D}(\mathbb{R}_+^*; \mathbb{R})$ the function $f \cdot g \in \mathcal{D}(\mathbb{R}_+^*; \mathbb{R})$ and

$$D(f \cdot g) = D(f) \cdot D_0(g) + D_0(f) \cdot D(g) \tag{11}$$

Proof. We have:

$$D_0(f \cdot g) = \lim_{x\to\infty} (f \cdot g)(x) = \lim_{x\to\infty} (f(x) \cdot g(x)) = D_0(f) \cdot D_0(f) \in \mathbb{R} \text{ and}$$

$$\lim_{x\to\infty} (((f \cdot g)(x) - D_0(f \cdot g))x) = \lim_{x\to\infty} (x \cdot (f(x) \cdot g(x) - f(x) \cdot D_0(g) +$$
$$+ f(x) \cdot D_0(g) - D_0(f) \cdot D_0(g))) = \lim_{x\to\infty} (x \cdot f(x)(g(x) - D_0(g))) +$$
$$+ \lim_{x\to\infty} (x \cdot D_0(g) \cdot (f(x) - D_0(f))) = D_0(f) \cdot D(g) + D(f) \cdot D_0(g)$$

□

Proposition 2.
If the functions $f, g \in \mathcal{F}(\mathbb{R}_+^*; \mathbb{R})$ generates sequences indefinite derivable, then the function $f \cdot g$ generates indefinite derivable functions and

$$D_m(f \cdot g) = \sum_{k=0}^{m} C_m^k \cdot D_{m-k}(f) \cdot D_k(g), \forall m \in \mathbb{N}^* \tag{12}$$

Proof. This is actually a Leibniz type formula, for the derivations of the product of two functions. We will prove by mathematical induction after $m \in \mathbb{N}^*$.
For $m = 1$ we have already proved that:

$$D_1(f \cdot g) = D(f \cdot g) = D_0(f) \cdot D_1(f) + D_1(f) \cdot D_0(f) \tag{13}$$

We assume that the formula is true for $m \in \mathbb{N}^*$, namely

$$D_m(f \cdot g) = D(f \cdot g) = D_0(f) \cdot D_1(g) + D_1(f) \cdot D_0(g) \tag{14}$$

and let's prove that the formula is also true for $m + 1$, namely:

$$D_{m+1}(f \cdot g) = \sum_{k=0}^{m} C_m^k \cdot D_{m-k}(f) \cdot D_k(g) \tag{15}$$

We have:

$$D_{m+1}(f \cdot g) = D(D_m(f \cdot g)) = D\Big(\sum_{k=0}^{m} C_m^k \cdot D_{m-k}(f) \cdot D_k(g)\Big) =$$

$$= \sum_{k=0}^{m} C_m^k \cdot D(D_{m-k}(f) \cdot D_k(g)) = \sum_{k=0}^{m} C_m^k \cdot (D_{m-k+1}(f) \cdot D_k(g) + D_{m-k}(f) \cdot D_{k+1}(g)) =$$

$$= C_m^0 \cdot D_{m+1}(f) \cdot D_0(g) + \sum_{k=1}^{m} C_m^k \cdot D_{m-k+1}(f) \cdot D_k(g) + \sum_{k=0}^{m-1} C_m^k \cdot D_{m-k}(f) \cdot D_{k+1}(g) +$$

$$+ C_m^m \cdot D_0(f) \cdot D_{m+1}(g) = C_{m+1}^0 \cdot D_{m+1}(f) \cdot D_0(g) + C_{m+1}^{m+1} \cdot D_0(f) \cdot D_{m+1}(g) +$$

$$+ \sum_{k=1}^{m} C_m^k \cdot D_{m-k+1}(f) \cdot D_k(g) + \sum_{j=1}^{m} C_m^{j-1} \cdot D_{m-j+1}(f) \cdot D_j(g) = C_{m+1}^0 \cdot D_{m+1}(f) \cdot D_0(g) +$$

$$+ C_{m+1}^{m+1} \cdot D_0(f) \cdot D_{m+1}(g) + \sum_{k=1}^{m} (C_m^k + C_m^{k-1}) \cdot D_{m-k+1}(f) \cdot D_k(g) =$$

$$= \sum_{k=0}^{m+1} C_{m+1}^k \cdot D_{m+1-k}(f) \cdot D_k(g)$$

which proves relationship (15). According to mathematical induction principle it follows that the relationship from enunciation is true for any $m \in \mathbb{N}^*$. □

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