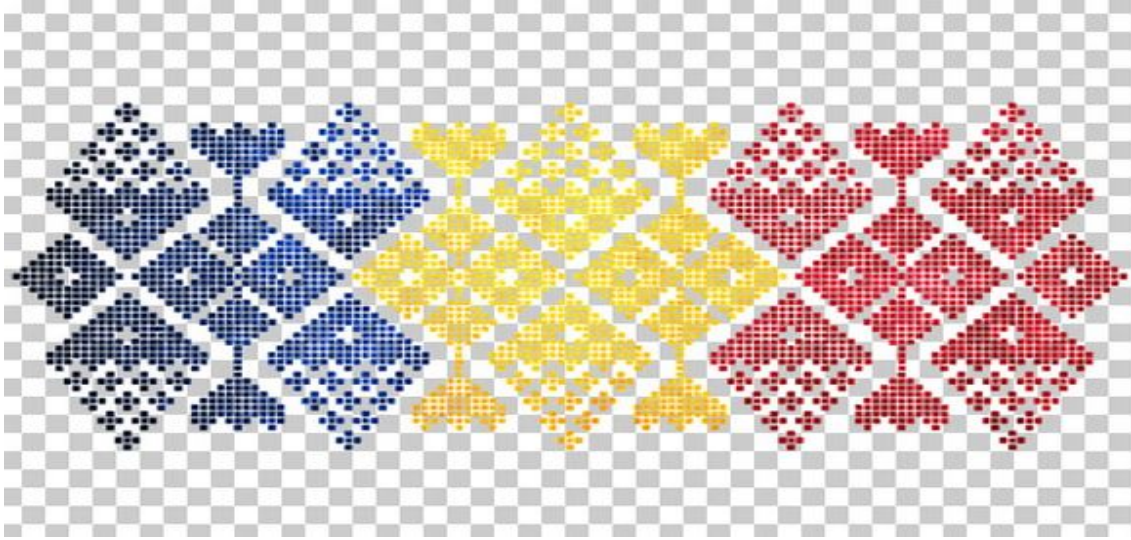


R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro



If $0 \leq x, y, z \leq \frac{\pi}{4}$ then:

$$\pi(\sin x + \sin y + \sin z) \geq 6\sqrt[6]{8x^2y^2z^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Andrew Okukura-Romania, Solution 2 by Adrian Popa-Romania

Solution 1 by Andrew Okukura-Romania

$$6\sqrt[6]{8x^2y^2z^2} = 6\sqrt{2} \cdot \sqrt[3]{xyz} \stackrel{AM \leq GM}{\leq} \frac{6\sqrt{2}}{3}(x + y + z) = 2\sqrt{2}(x + y + z) \quad (1)$$

We will define the function $f(x) = \pi \sin x - 2\sqrt{2} \cdot x$, $f: \left[0, \frac{\pi}{4}\right] \rightarrow \mathbb{R}$

We observe that $f'(x) = \pi \cos x - 2\sqrt{2} \leq \pi - 2\sqrt{2} < 0 \Rightarrow f$ is strictly decreasing

$$\text{That means } \pi \sin x - 2\sqrt{2} \cdot x \geq \pi \sin \frac{\pi}{4} - 2\sqrt{2} \frac{\pi}{4} = 0$$

As such:

$$\pi(\sin x + \sin y + \sin z) \geq 2\sqrt{2}(x + y + z) \quad (2)$$

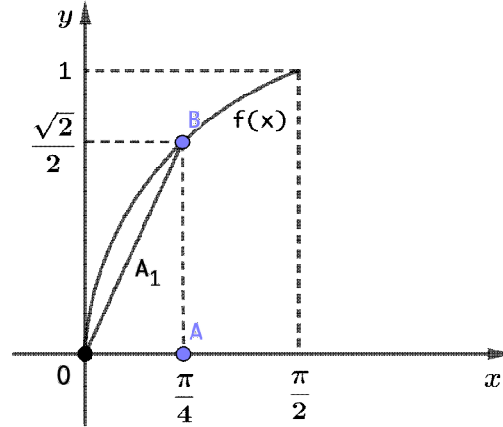
From (1) and (2) we obtain the inequality we wanted to prove.

Solution 2 by Adrian Popa-Romania

Let be $f(x) = \sin x$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro



$$S_{A_1} > S_{OAB} \Rightarrow \int_0^{\frac{\pi}{4}} \sin x \, dx > \int_0^{\frac{\pi}{4}} \frac{2\sqrt{2}}{\pi} x \, dx \quad (1)$$

$$\therefore \text{The equation of the line } OB: \left. \begin{array}{l} O(0, 0) \\ B\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \end{array} \right\} \Rightarrow \frac{x}{\frac{\pi}{4}} = \frac{y}{\frac{\sqrt{2}}{2}} \Rightarrow y = \frac{\sqrt{2}x}{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} \cdot \frac{4}{\pi} x \Rightarrow y = \frac{2\sqrt{2}}{\pi} \therefore$$

$$\text{From (1)} \Rightarrow \sin x \geq \frac{2\sqrt{2}x}{\pi} \quad (\forall) x \in \left[0, \frac{\pi}{4}\right] \Rightarrow \pi \sin x \geq 2\sqrt{2}x$$

$$\begin{aligned} \text{So, } \pi(\sin x + \sin y + \sin z) &\geq 2\sqrt{2}(x + y + z) \Rightarrow \\ \Rightarrow \pi(\sin x + \sin y + \sin z) &\geq \sqrt{2}x + \sqrt{2}x + \sqrt{2}y + \sqrt{2}y + \sqrt{2}z + \sqrt{2}z \stackrel{MA \geq MG}{\geq} \\ &\geq 6\sqrt[6]{\sqrt{2}^6 x^2 y^2 z^2} = 6\sqrt[6]{8x^2 y^2 z^2} \end{aligned}$$