CERTAIN RESULTS ON INTEGRALS

D.M. BĂTINEŢU - GIURGIU, MIHÁLY BENCZE, DANIEL SITARU, NECULAI STANCIU

ABSTRACT. In this paper we present some certain definite integrals.

Application 1.

If $a > 0$ then:

$$\int_0^a (x^2 - ax + a^2) \arctan(e^x - 1)dx = \frac{5\pi}{48}a^3$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - ax + a^2, g : \mathbb{R} \to \mathbb{R}, g(x) = \arctan(e^x - 1)$. We note that $f(a - x) = f(x)$. We have:

$$\tan(g(x) + g(a - x)) = \frac{\tan(g(x)) + \tan(g(a - x))}{1 - \tan(g(x))\tan(g(a - x))} =$$

$$= \frac{e^x - 1 + e^{a-x} - 1}{1 - (e^x - 1)(e^{a-x} - 1)} = \frac{e^{2x} + e^a - 2e^x}{e^{2x} + e^a - 2e^x} = 1, \text{ so } g(x) + g(a - x) = \arctan 1 = \frac{\pi}{4}.$$

Therefore, $I = \int_0^a (x^2 - ax + a^2) \arctan(e^x - 1)dx = \int_0^a f(x)g(x)dx$ where we take $x = a - t$ and we obtain

$$I = \int_a^0 f(a - t)g(a - t)(-1)dt = \int_0^a f(t)g(a - t)dt, \text{ so}$$

$$2I = \int_0^a f(x)(g(x) + g(a - x))dx = \frac{\pi}{4}\int_0^a f(x)dx = \frac{\pi}{4}(x^2 - ax + a^2)dx = \frac{5\pi}{24}a^3$$

Hence, $I = \frac{5\pi}{48}a^3$. □

Application 2.

If $a \in [0, \frac{\pi}{4}]$ then,

$$\int_0^a (x^2 - ax + a^2) \ln(1 + \tan x \tan a)dx = \frac{5a^3}{12} \ln(1 + \tan^2 a)$$

Proof. We take $x = a - t$, so

$$I = \int_0^a (x^2 - ax + a^2) \ln(1 + \tan x \tan a)dx =$$

$$= -\int_0^a ((a - t)^2 - a(a - t) + a^2) \ln(1 + \tan(a - t) \tan a)dt =$$

$$= \int_0^a (t^2 - at + a^2) \ln\Big(1 + \frac{\tan a - \tan t}{1 + \tan a \tan x} \tan a\Big)dt =$$

$$= \int_0^a (x^2 - ax + a^2) \ln \frac{1 + \tan a \tan x + \tan^2 a - \tan a \tan x}{1 + \tan a \tan x}dx =$$

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$$= \int_0^a (x^2 - ax + a^2) \ln \frac{1 + \tan^2 a}{1 + \tan a \tan x} dx =$$

$$= \ln(1 + \tan^2 a) \int_0^a (x^2 - ax + a^2) dx - \int_0^a (x^2 - ax + a^2) \ln(1 + \tan x \tan a) dx =$$

$$= \ln(1 + \tan^2 a) \int_0^a (x^2 - ax + a^2) dx - I$$

$$\text{Therefore, } 2I = \ln(1 + \tan^2 a) \Big(\frac{x^3}{3} - \frac{ax^2}{2} + a^2 x \Big) \Big|_0^a = \frac{5a^3}{6} \ln(1 + \tan^2 a)$$

Hence, $I = \frac{5a^3}{12} \ln(1 + \tan^2 a)$. □

Application 3.
Let $a, b \in [0, \pi]$ with $a + b = \pi$, then:

$$\int_a^b \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi}{2} (\arctan(\cos b) - \arctan(\cos a))$$

Proof. We take $a + b - x = t$ and we obtain:

$$I = \int_a^b \frac{x \sin x}{1 + \cos^2 x} dx = \int_a^b \frac{(\pi - t) \sin(\pi - t)}{1 + \cos^2(\pi - t)} \cdot (-1) dt = \int_a^b \frac{(\pi - t) \sin t}{1 + \cos^2 t} dt$$

$$\text{So, } 2I = \pi \int_a^b \frac{\sin x}{1 + \cos^2 x} dx, \text{ which yields to,}$$

$$I = \frac{\pi}{2} (\arctan(\cos b) - \arctan(\cos a))$$

and we are done! □

Application 4.
Let $a, b \in [0, \infty), a < b, P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}, n$ is a positive integer, then:

$$\int_a^b \frac{x^n + n!(\sin x - \cos x)}{e^x + \sin x + P_n(x)} dx = n! \Big(b - a - \ln \Big(\frac{e^b + \sin b + P_n(b)}{e^a + \sin a + P_n(a)} \Big) \Big)$$

Proof. Let $f : \mathbb{R} \to [0, \infty), f(x) = e^x + \sin x + P_n(x)$.
We have $f'(x) = e^x + \cos x + P_{n-1}(x)$, so,

$$f(x) - f'(x) = \sin x - \cos x + \frac{x^n}{n!} = \frac{1}{n!} (x^2 + n!(\sin x - \cos x))$$

$$\text{Therefore, } \int_a^b \frac{x^n + n!(\sin x - \cos x)}{e^x + \sin x + P_n(x)} dx = \int_a^b \frac{n!(f(x) - f'(x))}{f(x)} dx = n! \int_a^b \Big(1 - \frac{f'(x)}{f(x)} \Big) dx =$$

$$= n!(x - \ln f(x)) \Big|_a^b = n!(b - a) - \ln \frac{f(b)}{f(a)} = n! \Big(b - a - \ln \Big(\frac{e^b + \sin b + P_n(b)}{e^a + \sin a + P_n(a)} \Big) \Big)$$

□

Application 5.
Let $u : R \to R$ be a continuous function and $f : (0, \infty) \to (0, \infty)$ be a solution of differential equation $y'(x) - y(x) - u(x) = 0$ for any $x \in (0, \infty)$, then:

$$\int \frac{e^x u(x)}{(e^x + f(x))^2} dx = \int \Big(\frac{e^x}{e^x + f(x)} \Big)' dx = \frac{e^x}{e^x + f(x)} + C$$

Proof. Since,

$$\Big(\frac{e^x}{e^x+f(x)}\Big)' = \frac{e^x(e^x+f(x)) - e^x(e^x+f'(x))}{(e^x+f(x))^2} = \frac{e^x(e^x-f'(x))}{(e^x+f(x))^2} = \frac{e^x u(x)}{(e^x+f(x))^2}$$

$$\text{then, } \int \frac{e^x u(x)}{(e^x+f(x))^2}dx = \int \Big(\frac{e^x}{e^x+f(x)}\Big)' dx = \frac{e^x}{e^x+f(x)} + C$$

□

Application 6.
If $f : [a, b] \to (0, \infty)$ is a continuous function, such that $f(a+b-x)+f(x) = c$, for any $x \in [a, b]$ and $a + b = \frac{\pi}{2}$, then:

$$\int_a^b \frac{\sin^n x + f(x) + d}{\sin^2 x + \cos^n x + c + 2d}dx = \frac{b-a}{2}, \text{ where } n \text{ is positive integer and } d \geq 0.$$

Proof. We take $x = u(t) = \frac{\pi}{2} - t, u'(t) = -1, u(0) = \frac{\pi}{2}, u(\frac{\pi}{2}) = 0, u(a) = b, u(b) = a$ and we obtain:

$$I = \int_a^b \frac{\sin^n x + f(x) + d}{\sin^n x + \cos^n x + c + 2d}dx = \int_a^b \frac{\sin^n(\frac{\pi}{2}-t) + f(\frac{\pi}{2}-t) + d}{\sin^n(\frac{\pi}{2}-t) + \cos^n(\frac{\pi}{2}-t) + c + 2d}(-1)dt =$$

$$= \int_a^b \frac{\cos^n t + (c - f(t)) + d}{\cos^n t + \sin^n t + c + 2d}dt = \int_a^b \frac{\cos^n x + c - f(x) + d}{\cos^n x + \sin^n x + c + 2d}dx$$

$$\text{So, } 2I = \int_a^b \frac{\sin^n x + f(x) + d + \cos^n x + c - f(x) + d}{\cos^n x + \sin^n x + c + 2d}dx = \int_a^b dx = b - a$$

Hence, $I = \frac{b-a}{2}$. □

Application 7.
If $a \in (0, \frac{\pi}{2}]$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous and odd function, then compute:

$$\int_{-a}^a (x^{1936} + x^{2018} + 82) \cdot \arccos(\sin(f(x))dx = \pi \cdot \left(\frac{a^{1937}}{1937} + \frac{a^{2019}}{2019} + 82a\right)$$

Proof. We take $x = u(t) = -t, u'(t) = -1, u(-a) = a, u(a) = -a$ and we obtain:

$$I = \int_{-a}^a (x^{1936} + x^{2018} + 82) \cdot \arccos(\sin(f(x))dx =$$

$$= \int_a^{-a} ((-t)^{1936} + (-t)^{2018} + 82) \cdot \arccos(\sin f(-t))(-1)dt =$$

$$= \int_{-a}^a (t^{1936} + t^{2018} + 82) \cdot \arccos(\sin(-f(t)))dt =$$

$$= \int_{-a}^a (t^{1936} + t^{2018} + 82) \cdot \arccos(-\sin f(t))dt =$$

$$= \int_{-a}^a (x^{1936} + x^{2018} + 82) \cdot (\pi - \arccos(\sin f(x)))dx =$$

$$= \pi \int_{-a}^a (x^{1936} + x^{2018} + 82)dx - \int_{-a}^a (x^{1936} + x^{2018} + 82) \cdot \arccos(\sin f(x))dx$$

$$\text{So, } 2I = \pi \int_{-a}^a (x^{1936} + x^{2018} + 82)dx = 2\pi \int_0^a (x^{1936} + x^{2018} + 82)dx$$

Hence, $I = \pi \cdot \left(\frac{a^{1937}}{1937} + \frac{a^{2019}}{2019} + 82a\right)$. □

Application 8.
If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function, then:

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx = e\cdot f(e)$$

Proof. We denote:

$I_n = \lim\limits_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx$. By mean value theorem $\exists\xi_n\in\left(\frac{n^2}{\sqrt[n]{n!}},\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}\right)$

(1) such that $I_n = \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)f\left(\frac{\xi_n}{n}\right)$

(2) We denote $a_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}, \forall n\geq 2$, then $I_n = a_n\cdot f\left(\frac{\xi_n}{n}\right)$

(3)

So, $\lim\limits_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx = \lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty} a_n\cdot\lim\limits_{n\to\infty} f\left(\frac{\xi}{n}\right) = \lim\limits_{n\to\infty} a_n\cdot f\left(\lim\limits_{n\to\infty}\frac{\xi_n}{n}\right)$

(4)

We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n^2}{\sqrt[n]{n!}}(u_n-1) = \lim\limits_{n\to\infty}\frac{n^2}{\sqrt[n]{n!}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n = \lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$

where we denote $u_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n^2}, \forall n\geq 2$.

But, $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim\limits_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} = \lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$ so,

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n+1}{n}\cdot\frac{\sqrt[n]{n!}}{n} = e\cdot 1\cdot\frac{1}{e} = 1$, then $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$

We have $\lim\limits_{n\to\infty} u_n^n = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{(n+1)!} = e^2\cdot\lim\limits_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{(n+1)!} = e^2\cdot\frac{1}{e} = e.$

(5) By above and (4) we obtain $\lim\limits_{n\to\infty} a_n = e$

We have $\xi_n\in\left(\frac{n^2}{\sqrt[n]{n!}},\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}\right)\Rightarrow\frac{\xi_n}{n}\in\left(\frac{n}{\sqrt[n]{n!}},\frac{n+1}{n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)\Rightarrow$

(6) $\Rightarrow\lim\limits_{n\to\infty}\frac{\xi_n}{n} = \lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim\limits_{n\to\infty}\frac{n+1}{n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}} = e$

By (3), (5) and (6) we infer that $\lim\limits_{n\to\infty} I_n = e\cdot f(e)$.

□

Application 9.
Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous and odd function and $g:\mathbb{R}_+^*\to\mathbb{R}$ be a continuous function such that $g(\frac{1}{x})=-g(x),\forall x\in\mathbb{R}_+^*$, then:

$$\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1}{(1+x^2)(1+a^{(f\circ g)(x)})}dx=\frac{\pi}{8},\text{ where } a>1.$$

Proof. Let $x=u(t)=\frac{1}{t}, u'(t)=-\frac{1}{t^2}, u(\sqrt{2}-1)=\sqrt{2}+1, u(\sqrt{2}+1)=\sqrt{2}-1$. Therefore,

$$I=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1}{(1+x^2)(1+a^{(f\circ g)(x)})}dx=\int_{\sqrt{2}+1}^{\sqrt{2}-1}\frac{1}{(1+\frac{1}{t^2})(1+a^{(f\circ g)(\frac{1}{t})})}\Big(-\frac{1}{t^2}\Big)dt=$$

$$=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1}{(1+t^2)(1+a^{f(-g(t))})}dt=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1}{(1+x^2)(1+a^{-(f\circ g)(x)})}dx$$

$$\text{So, } 2I=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\Big(\frac{1}{(1+x^2)(1+a^{(f\circ g)(x)})}+\frac{1}{(1+x^2)(1+a^{-(f\circ g)(x)})}\Big)dx=$$

$$=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1+a^{(f\circ g)(x)}}{(1+x^2)(1+a^{(f\circ g)(x)})}dx=$$

$$=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{1}{1+x^2}dx=\arctan x\Big|_{\sqrt{2}-1}^{\sqrt{2}+1}=\arctan(\sqrt{2}+1)-\arctan(\sqrt{2}-1)=$$

$$=\arctan\frac{\sqrt{2}+1-(\sqrt{2}-1)}{1+(\sqrt{2}+1)(\sqrt{2}-1)}=\arctan 1=\frac{\pi}{4}$$

Hence, $I=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}$. □

Application 10.
If $f,g:\mathbb{R}_+\to\mathbb{R}_+$ are derivable functions with continuous derivatives, and $a,b\in\mathbb{R}_+^*, a<b$, then:

$$\int_a^b\frac{f(x)g'(x)-f'(x)}{f(x)+e^{g(x)}}dx=g(b)-g(a)-\ln\frac{f(b)+e^{g(b)}}{f(a)+e^{g(a)}}.$$

Proof.

$$I=\int_a^b\frac{f(x)g'(x)-f'(x)}{f(x)+e^{g(x)}}dx=\int_a^b\frac{f(x)g'(x)+e^{g(x)}g'(x)-e^{g(x)}g'(x)-f'(x)}{f(x)+e^{g(x)}}dx=$$

$$=\int_a^b\frac{(f(x)+e^{g(x)})g'(x)}{f(x)+e^{g(x)}}dx-\int_a^b\frac{f'(x)+e^{g(x)}g'(x)}{f(x)+e^{g(x)}}dx=\int_a^b g'(x)dx-\int_a^b\frac{(f(x)+e^{g(x)})'}{f(x)+e^{g(x)}}dx=$$

$$=g(x)\Big|_a^b-\ln(f(x)+e^{g(x)})\Big|_a^b=g(b)-g(a)-\ln\frac{f(b)+e^{g(b)}}{f(a)+e^{g(a)}}$$

□

Application 11.
Let $a\in(0,\frac{\pi}{2}], b\in[\frac{\pi}{2},\pi)$ with $a+b=\pi$, then:

$$\int_a^b\frac{x}{\sin x}dx=\frac{a+b}{2}\ln\frac{\tan\frac{b}{2}}{\tan\frac{a}{2}}$$

Proof. We denote:

$I=\int_a^b \frac{x}{\sin x}dx$, where we make $x=u(t)=a+b-t, u'(t)=-1, u(a)=b, u(b)=a$

$$\text{and we obtain } I=\int_a^b \frac{a+b-t}{\sin(a+b-t)}(-1)dt=\int_a^b \frac{a+b-t}{\sin t}dt=$$

$$=(a+b)\int_a^b \frac{1}{\sin x}dx-I\Rightarrow 2I=(a+b)\int_a^b \frac{1}{\sin x}dx\Leftrightarrow$$

$$\Leftrightarrow I=\frac{a+b}{2}\int_a^b \frac{1}{\sin x}dx=\frac{a+b}{2}\int_a^b \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx=\frac{a+b}{4}\int_a^b \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}dx=$$

$$=\frac{a+b}{4}\int_a^b \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}dx+\frac{a+b}{4}\int_a^b \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}dx=-\frac{a+b}{2}\int_a^b \frac{-\frac{1}{2}\sin\frac{x}{2}}{\cos\frac{x}{2}}dx+\frac{a+b}{2}\int_a^b \frac{\frac{1}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}dx=$$

$$=-\frac{a+b}{2}\ln\cos\frac{x}{2}\Big|_a^b+\frac{a+b}{2}\ln\sin\frac{x}{2}\Big|_a^b=\frac{a+b}{2}\ln\tan\frac{x}{2}\Big|_a^b=\frac{a+b}{2}\left(\ln\tan\frac{b}{2}-\ln\tan\frac{a}{2}\right)=$$

$$=\frac{a+b}{2}\ln\frac{\tan\frac{b}{2}}{\tan\frac{a}{2}}$$

□

Application 12.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x}dx=\frac{\pi}{8}\ln 3$$

Proof. We make $x=\frac{\pi}{2}-t$ and we obtain:

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x}dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\frac{\pi}{2}-t}{\sin(\pi-2t)}(-dt)=\frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin 2t}dx-I\Rightarrow 2I=\frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin 2t}dt\Rightarrow$$

$$\Rightarrow I=\frac{\pi}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2\sin t\cos t}dt=\frac{\pi}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 t+\sin^2 t}{\sin t\cos t}dt=\frac{\pi}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t}dt+\frac{\pi}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin t}{\cos t}dt=$$

$$=\frac{\pi}{8}\left(\ln\sin t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\ln\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\right)=\frac{\pi}{8}\ln\tan t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{8}\ln\frac{\tan\frac{\pi}{3}}{\tan\frac{\pi}{6}}=\frac{\pi}{8}\ln 3$$

□

Application 13.
Let $a,b\in\mathbb{R}, a<b, c\in\mathbb{R}_+^*$ and $f:\mathbb{R}\to\mathbb{R}_+^*$ be a continue function, then:

$$\int_a^b \frac{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}+e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}dx=\frac{b-a}{2}$$

Proof. Let:

$I=\int_a^b \frac{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}+e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}dx$, and we make the change:

$x=u(t)=a+b-t, u'(t)=-1, u(a)=b, u(b)=a$ and we deduce that:

$$I=\int_a^b \frac{e^{f(a+b-x-a)}(f(a+b-x-a))^{\frac{1}{c}}}{e^{f(a+b-x-a)}(f(a+b-x-a))^{\frac{1}{c}}+e^{f(b-a-b+x)}(f(b-a-b+x))^{\frac{1}{c}}}dx=$$

$=\int_a^b \frac{e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}{e^{f(b-x)}(f(b-x))^{\frac{1}{c}}+e^{f(x-a)}(f(x-a))^{\frac{1}{c}}}dx$. Therefore:

$$2I = I + I = \int_a^b \frac{e^{f(x-a)}(f(x-a))^{\frac{1}{c}} + e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}} + e^{f(b-x)}(f(b-x))^{\frac{1}{c}}} dx = \int_a^b dx = x\Big|_a^b = b - a$$

$$\text{so, } I = \frac{b-a}{2}$$

□

Application 14.
Let $a \in (0, \frac{\pi}{2})$ and $b, c \in (1, \infty)$, then:

$$\int_{-a}^{a} \ln(b^{\sin^3 x} + c^{\sin^3 x}) \cdot \sin x \cdot dx = \frac{\ln bc}{32}(12a - 8\sin 2a + \sin 4a)$$

Proof. Let:

$$I = \int_{-a}^{a} \ln(b^{\sin^3 x} + c^{\sin^3 x}) \cdot \sin x dx, \text{ in which we make the change:}$$

$x = u(t) = -t, u'(t) = -1, u(a) = -a, u(-a) = a$ and we deduce that:

$$I = -\int_a^{-a} \ln(b^{\sin^3(-t)} + c^{\sin^3(-t)}) \cdot \sin(-t) dt = \int_{-a}^{a} \ln(b^{-\sin^3 t} + c^{-\sin^3 t}) \cdot (-\sin t) dt =$$

$$= -\int_{-a}^{a} \ln \frac{b^{\sin^3 t} + c^{\sin^3 t}}{(bc)^{\sin^3 t}} \cdot \sin t dt = -I + \int_{-a}^{a} \ln(bc)^{\sin^3 t} \cdot \sin t dt = -I + \ln(bc) \cdot \int_{-a}^{a} \sin^4 t dt$$

$$\Leftrightarrow 2I = \ln(bc) \cdot \int_{-a}^{a} \sin^4 x dx = \ln(bc) \cdot \int_{-a}^{a} (\sin^2 x)^2 dx = \ln(bc) \cdot \int_{-a}^{a} \frac{1}{4}(2\sin^2 x)^2 dx =$$

$$= \frac{\ln(bc)}{4} \cdot \int_{-a}^{a} (1 - \cos 2x)^2 dx = \frac{\ln(bc)}{4}\left(\int_{-a}^{a} dx - 2\int_{-a}^{a} \cos 2x dx + \int_{-a}^{a} \cos^2 2x dx\right) =$$

$$= \frac{\ln bc}{4}\left(2a - \sin 2x\Big|_{-a}^{a} + \frac{1}{2}\int_{-a}^{a}(1 + \cos 4x) dx\right) = \frac{\ln bc}{4}\left(2a - 2\sin 2a + a + \frac{1}{2}\int_{-a}^{a} \cos 4x dx\right) =$$

$$= \frac{\ln bc}{4}\left(3a - 2\sin 2a + \frac{1}{8}\sin 4x\Big|_{-a}^{a}\right) = \frac{\ln bc}{4}\left(3a - 2\sin 2a + \frac{1}{4}\sin 4a\right) =$$

$$= \frac{\ln bc}{16}(12a - 8\sin 2a + \sin 4a)$$

$$\text{Therefore, } I = \frac{\ln bc}{32}(12a - 8\sin 2a + \sin 4a)$$

□

Application 15.
All continue functions $f : (0, \infty) \to (-\infty, \infty)$ such that:

$$\frac{1}{2} \cdot \frac{u}{v} \int_a^b f^2(e^x) dx + \frac{1}{6} \cdot \frac{v}{u}(b^3 - a^3) \le \int_a^b x f(e^x) dx$$

for any $a, b \in (-\infty, +\infty)$ with $a < b$ and for any $u, v \in (0, \infty)$ are $f : \mathbb{R}_+^* \to \mathbb{R}$, $f(x) = \frac{v}{u} \ln x$.

Proof. The inequality from the statement is equivalent with:

$$u^2\int_a^b f^2(e^x)dx+\frac{v^2}{3}(b^3-a^3)-2uv\int_a^b xf(e^x)dx\le 0 \Leftrightarrow \int_a^b (uf(e^x)-vx)^2dx\le 0$$

$$\text{But, } \int_a^b (uf(e^x)-vx)^2dx\ge 0$$

$$\text{So, } \int_a^b (uf(e^x)-vx)^2dx=0 \Leftrightarrow (uf(x)-vx)^2=0 \Leftrightarrow f(e^x)=\frac{v}{u}x$$

We take, $x=\ln t, t>0$ and we obtain that: $f(t)=\frac{v}{u}\ln t$. Therefore, $f:\mathbb{R}_+^*\to\mathbb{R}$, $f(x)=\frac{v}{u}\ln x$. □

Application 16. Let $a>0, b,c>1$ and $f,g:\mathbb{R}\to\mathbb{R}$ be continuous and odd functions, then:

$$\int_{-a}^a f(x)\ln(b^{g(x)}+c^{g(x)})dx=(\ln(bc))\int_0^a f(x)g(x)dx$$

Proof. Let:

$$I=\int_{-a}^a f(x)\ln(b^{g(x)}+c^{g(x)})dx, \text{ where we make the changes of variable}$$

$$x=u(t)=-t \text{ with } u'(t)=-1, u(a)=-a, u(-a)=a, \text{ so:}$$

$$I=\int_a^{-a} -f(t)\ln(b^{-g(t)}+c^{-g(t)})(-1)dt=-\int_{-a}^a f(x)\ln\frac{b^{g(x)}+c^{g(x)}}{(bc)^{g(x)}}dx=$$

$$=-I+\int_{-a}^a f(x)\ln(bc)^{g(x)}dx=-I+(\ln(bc))\int_{-a}^a f(x)g(x)dx \tag{1}$$

Because f,g are odd then:
$(fg)(-x)=f(-x)g(-x)=-f(x)(-g(x))=(fg)(x)$, i.e. $fg:\mathbb{R}\to\mathbb{R}$, is even function, so, by (1), we obtain:

$$2I=\ln(bc)\int_{-a}^a (fg)(x)dx=2(\ln(bc))\int_0^a f(x)g(x)dx, \text{ which yields that:}$$

$$I=(\ln(bc))\int_0^a f(x)g(x)dx$$

□

Application 17.
Let $f:\mathbb{R}\to\mathbb{R}$ be an even function and derivable with its derivable continue, then:

$$\int_{-a}^a\Big(\frac{f(x)}{1+e^x}+f'(x)\ln(1+e^x)\Big)dx=af(a), \text{ for any } a\in\mathbb{R}_+$$

Proof. We know that if f is a derivable even function then the function $f':\mathbb{R}\to\mathbb{R}$ is an odd function. Denoting

$$I=\int_{-a}^a\Big(\frac{f(x)}{1+e^x}+f'(x)\ln(1+e^x)\Big)dx,$$

where we make the following change of variable

$$x=u(t)=-t, \text{ with } u'(t)=-1, u(a)=-a, u(-a)=a$$

and we obtain that:

$$I = \int_a^{-a} \Big(\frac{f(-t)}{1+e^{-t}} + f'(-t)\ln(1+e^{-t})\Big)(-1)dt = \int_{-a}^{a} \Big(\frac{f(t)e^t}{1+e^t} - f'(t)\ln\frac{1+e^t}{e^t}\Big)dt =$$

$$= \int_{-a}^{a} \Big(\frac{f(x)e^x}{1+e^x} + xf'(x) - f'(x)\ln(1+e^x)\Big)dx$$

So,

$$2I = \int_{-a}^{a} \Big(\frac{f(x)}{1+e^x} + \frac{f(x)e^x}{1+e^x} + xf'(x) + f'(x)\ln(1+e^x) - f'(x)\ln(1+e^x)\Big)dx =$$

$$= \int_{-a}^{a} (f(x) + xf'(x))dx = \int_{-a}^{a} (xf(x))'dx = xf(x)\Big|_{-a}^{a} = af(a) - (-af(-a)) =$$

$$= af(a) + af(a) = 2af(a)$$

Hence, $I = af(a)$. □

Application 18. Let $a, b \in \mathbb{R}, a < b$ and continue functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that: $f(a+b-x) = -f(x), g(a+b-x) = g(x), h(a+b-x) = -h(x), \forall x \in \mathbb{R}$, then:

$$\int_a^b f(x)(\arctan(x))\ln(1+e^{h(x)})dx = \frac{1}{2}\int_a^b f(x)h(x)\arctan(x)dx$$

Proof. Let:

$$I = \int_a^b f(x)(\arctan(x))\ln(1+e^{h(x)})dx, \text{ where we make the changes}$$

$$x = u(t) = a+b-t, u'(t) = -1, u(a) = b, u(b) = a \text{ and we get:}$$

$$I = -\int_a^b f(a+b-t)(\arctan(a+b-t))\ln(1+e^{h(a+b-t)})dt =$$

$$= -\int_a^b f(x)(\arctan(x))\ln(1+e^{-h(x)})dx = -\int_a^b f(x)(\arctan(x))\ln\frac{1+e^{h(x)}}{e^{h(x)}}dx =$$

$$= -\int_a^b f(x)(\arctan(x))\ln(1+e^{h(x)})dx + \int_a^b f(x)(\arctan(x))h(x)dx$$

$$\text{So, } 2I = \int_a^b f(x)h(x)\arctan(x)dx, \text{ and we are done.}$$

□

Application 19.
Let $a, b \in \mathbb{R}_+^*$ and $f : \mathbb{R} \to \mathbb{R}$ be an even continuous function on $\mathbb{R}$, then:

$$\int_{-a}^{a} \frac{f(x)}{b^2 + \arctan x + \sqrt{b^4 + \arctan^2 x}}dx = \frac{1}{b^2}\int_0^a f(x)dx$$

Proof. We make the change of variable $x = u(t) = -t$, with $u'(t) = -1, u(a) = -a, u(-a) = a$ so:

$$I = \int_{-a}^{a} \frac{f(x)}{b^2 + \arctan x + \sqrt{b^2 + \arctan^2 x}}dx =$$

$$= \int_a^{-a} \frac{f(-t)}{b^2 - \arctan t + \sqrt{b^2 + \arctan^2 t}}(-1)dt =$$

$$= \int_{-a}^{a} \frac{f(t)}{b^2 - \arctan t + \sqrt{b^2 + \arctan^2 t}} dt$$

Therefore,

$$2I = I + I = \int_{-a}^{a} f(x)\Big(\frac{1}{b^2 + \arctan x + \sqrt{b^2 + \arctan^2 x}} + \frac{1}{b^2 - \arctan x + \sqrt{b^2 + \arctan^2 x}}\Big) dx =$$

$$= \int_{-a}^{a} f(x) \cdot \frac{2(b^2 + \sqrt{b^2 + \arctan^2 x})}{(b^2 + \sqrt{b^2 + \arctan^2 x})^2 - \arctan^2 x} dx =$$

$$= \int_{-a}^{a} f(x) \cdot \frac{2(b^2 + \sqrt{b^2 + \arctan^2 x})}{2b^2(b^2 + \sqrt{b^2 + \arctan^2 x})} dx = \frac{1}{b^2} \int_{-a}^{a} f(x)dx = \frac{2}{b^2} \int_{0}^{a} f(x)dx$$

which yields: $I = \frac{1}{b^2} \int_{0}^{a} f(x)dx$

□

Application 20.
Let $a > 0$ and $f, g : [-a, a] \to \mathbb{R}$ integrable functions such that f is even and g is odd, then:

$$\int_{-a}^{a} \frac{f(x)}{b^2 - g(x) + \sqrt{b^2 + g^2(x)}} dx = \frac{1}{b^2} \int_{0}^{a} f(x)dx$$

Proof. We make the changes of variable $x = u(t) = -t$, with $u'(t) = -1, u(a) = -a$, $u(-a) = a$, so:

$$I = \int_{-a}^{a} \frac{f(x)}{b^2 - g(x) + \sqrt{b^2 + g^2(x)}} dx = \int_{a}^{-a} \frac{f(-t)}{b^2 - g(-t) + \sqrt{b^2 + g^2(-t)t}} (-1)dt =$$

$$= \int_{-a}^{a} \frac{f(t)}{b^2 + g(t) + \sqrt{b^2 + g^2(t)}} dt$$

Therefore, $2I = I + I = \int_{-a}^{a} f(x)\Big(\frac{1}{b^2 + g(x) + \sqrt{b^2 + g^2(x)}} + \frac{1}{b^2 - g(x) + \sqrt{b^2 + g^2(x)}}\Big) dx =$

$$= \int_{-a}^{a} f(x) \cdot \frac{2(b^2 + \sqrt{b^2 + g^2(x)})}{(b^2 + \sqrt{b^2 + g^2(x)})^2 - g^2(x)} dx =$$

$$= \int_{-a}^{a} f(x) \cdot \frac{2(b^2 + \sqrt{b^2 + g^2(x)}))}{2b^2(b^2 + \sqrt{b^2 + g^2(x)})} dx = \frac{1}{b^2} \int_{-a}^{a} f(x)dx = \frac{2}{b^2} \int_{0}^{a} f(x)dx$$

which yields that: $I = \frac{1}{b^2} \int_{a}^{b} f(x)dx.$

□

Application 21.
Let $a > 0$, and $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuos and odd functions, $k : \mathbb{R} \to (1, \infty)$ be an even and continuous function, then:

$$\int_{-a}^{a} f(x) \ln((k(x)^{g(x)} + (k(x))^{h(x)})dx = \int_{0}^{a} f(x)(g(x) + h(x)) \ln k(x)dx$$

Proof. We make the change of variable $x = u(t) = -t$, with $u'(t) = -1, u(a) = -a$, $u(-a) = a$, so:

$$I = \int_{-a}^{a} f(x)\ln((k(x)^{g(x)} + (k(x))^{h(x)})dx =$$

$$= \int_{a}^{-a} f(-t)\ln((k(-t)^{g(-t)} + (k(-t))^{h(-t)})(-1)dt =$$

$$= -\int_{-a}^{a} f(x)\ln((k(x)^{-g(x)} + (k(x))^{-h(x)})dx = -\int_{-a}^{a} f(x)\ln\frac{(h(x))^{g(x)} + (k(x))^{h(x)}}{(k(x))^{g(x)+h(x)}}dx =$$

$$= -I + \int_{-a}^{a} f(x)\ln(k(x))^{g(x)+h(x)}dx = -I + \int_{-a}^{a} f(x)(g(x)+h(x))\ln(k(x))dx$$

Hence: $2I = \int_{-a}^{a} f(x)(g(x) + (h(x))\ln(k(x))dx$ and because

$$f(-x)(g(-x)+h(-x))\ln(k(-x)) = -f(x)(-g(x)-h(x))\ln(k(x)) = f(x)(g(x)+h(x))\ln(k(x))$$

i.e. $f(x)(g(x) + h(x))\ln(k(x))$ is even function, we have that:

$2I = 2\int_{0}^{a} f(x)(g(x)+h(x))\ln(k(x))dx$, namely $I = \int_{0}^{a} f(x)(g(x)+h(x))\ln(k(x))dx$

□

Application 22.
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = f(1-x), \forall x \in \mathbb{R}$, then:

$$\int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}} f(x)dx = \sqrt{2} \cdot \int_0^1 f(x)dx$$

Proof. We make the change of variables $x = u(t) = 1 - t$, with: $u'(t) = -1$, $u(0) = 1, u(1) = 0$, so:

$$I = \int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}} f(x)dx = -\int_1^0 \frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}} f(1-t)dt = \int_0^1 \frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}} f(t)dt$$

Therefore, $2I = \int_0^1 \Big(\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2x}} + \frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}\Big) f(x)dx =$

$$= \int_0^1 (\sqrt{x}+\sqrt{1-x}) f(x) \Big(\frac{1}{1+\sqrt{2x}} + \frac{1}{1+\sqrt{2(1-x)}}\Big)dx =$$

$$= \int_0^1 (\sqrt{x}+\sqrt{1-x}) f(x) \cdot \frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{1+\sqrt{2x}+\sqrt{2(1-x)}+2\sqrt{x(1-x)}}dx =$$

$$= \int_0^1 \frac{(\sqrt{x}+\sqrt{1-x}) f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+1+2\sqrt{x}(1-x)}dx =$$

$$= \int_0^1 \frac{(\sqrt{x}+\sqrt{1-x}) f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+(\sqrt{x}+\sqrt{1-x})^2}dx =$$

$$= \int_0^1 \frac{\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x}) f(x)}{\sqrt{2}+\sqrt{x}+\sqrt{1-x}}dx = \sqrt{2} \cdot \int_0^1 f(x)dx$$

□

Application 23.
Let $a,b\in\mathbb{R}, c\in\mathbb{R}-\{1\}$ and the continuous functions $f,g:\mathbb{R}\to\mathbb{R}$, such that: $f(a+b-x)=cf(x), g(a+b-x)=-g(x), \forall x\in\mathbb{R}$, then:

$$\int_a^b f(x)\ln(1+e^{g(x)})dx=\frac{c}{c-1}\int_a^b f(x)g(x)dx$$

Proof. Let

$$I=\int_a^b f(x)\ln(1+e^{g(x)})dx, \text{ where we make the changes}$$

$$x=u(t)=a+b-t, \text{ with } u(a)=b, u(b)=a, u'(t)=-1, \text{ and we obtain:}$$

$$I=\int_b^a f(a+b-t)\ln(1+e^{g(a+b-t)})(-1)dt=\int_a^b cf(t)\ln(1+e^{-g(t)})dt=$$

$$=c\cdot\int_a^b f(t)\ln\frac{1+e^{g(t)}}{e^{g(t)}}dt=c\cdot\int_a^b f(t)\ln(1+e^{g(t)})dt-c\cdot\int_a^b f(t)\ln e^{g(t)}dt=$$

$$=cI-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow(1-c)I=-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow$$

$$\Leftrightarrow(c-1)I=c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow I=\frac{c}{c-1}\int_a^b f(x)g(x)dx$$

□

Application 24.
Let $f:\mathbb{R}\to\mathbb{R}$ be a continue and odd function, then:

$$\int_{\frac{1}{e}}^e\frac{1}{(x^2+1)(1+xe^{f(\ln x)})}dx=\frac{1}{2}\arctan\frac{e^2-1}{2e}.$$

Proof. Let

$$I=\int_{\frac{1}{e}}^e\frac{1}{(x^2+1)(1+xe^{f(\ln x)})}dx, \text{ where we make the change}$$

$$x=u(t)=\frac{1}{t}, \text{ with } u(e)=\frac{1}{e}, u\Big(\frac{1}{e}\Big)=e, u'(t)=-\frac{1}{t^2}, \text{ and we deduce that:}$$

$$I=\int_e^{\frac{1}{e}}\frac{1}{(\frac{1}{t^2}+1)(1+\frac{1}{t}e^{f(\ln\frac{1}{t})})}\cdot\frac{-1}{t^2}dt=\int_{\frac{1}{e}}^e\frac{t}{(1+t^2)(t+e^{f(-\ln t)})}dt=$$

$$=\int_{\frac{1}{e}}^e\frac{t}{(1+t^2)(t+e^{-f(\ln t)})}dt=\int_{\frac{1}{e}}^e\frac{te^{f(\ln t)}}{(1+t^2)(te^{f(\ln t)}+1)}dt$$

So, $2I=\int_{\frac{1}{e}}^e\Big(\frac{1}{(x^2+1)(1+xe^{f(\ln x)})}+\frac{xe^{f(\ln x)}}{(x^2+1)(1+xe^{f(\ln x)})}\Big)dx=\int_{\frac{1}{e}}^e\frac{1}{1+x^2}dx=$

$$=\arctan x\Big|_{\frac{1}{e}}^e=\arctan e-\arctan\frac{1}{e}=\arctan\frac{e-\frac{1}{e}}{2}=\arctan\frac{e^2-1}{2e}.$$

Hence, $I=\frac{1}{2}\arctan\frac{e^2-1}{2e}$. □

Application 25.
Let $n\in\mathbb{N}$, then:

$$I_n=\int_0^{\frac{\pi}{2}}\sin^2 x\Big(\cos x\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)+\sin x\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\Big)dx=\frac{2}{\pi}\cdot\frac{(2n)!!}{(2n+1)!!}$$

Proof. We make the change: $t = u(x) = \frac{\pi}{2} - x, u'(x) = -1, u(0) = \frac{\pi}{2}, u\Big(\frac{\pi}{2}\Big) = 0$ and we have:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^2 x\Big(\sin x \cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big) + \cos x \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\Big)dx, \text{ therefore}$$

$$2I_n = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\Big(\sin x \cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big) + \cos x \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\Big)dx =$$

$$= \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin\Big)dx + \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx = A_n + B_n, \text{ where}$$

$$A_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx, B_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin x\Big)dx$$

We, have: $A_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\Big(\frac{\pi}{2}\cos x\Big)dx =$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx$, and analogous $B_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin x\Big)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx$

So, $2I_n = A_n + B_n = 2\cdot\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx \Leftrightarrow I_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx \Leftrightarrow I_n = \frac{2}{\pi}J_n$, where

$J_n = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx$, which by integrating by parts we obtain the recurrence:

$$(2n+1)J_n = 2nJ_{n-1}, \forall n \in \mathbb{N}^*, \text{ where } J_0 = \int_0^{\frac{\pi}{2}} \cos x dx = 1.$$

By $(2k+1)J_k = 2kJ_{k-1}, \forall k \in \mathbb{N}^*$, we deduce that: $\prod_{k=1}^{n}(2k+1)J_k = \prod_{k=1}^{n}(2k)J_{k-1}, \forall k \in \mathbb{N}^*$

$\Rightarrow (2n+1)!!J_n = (2n)!!J_0 = (2n)!! \Leftrightarrow J_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}, \forall n \in \mathbb{N}^*$

Hence, $I_n = \frac{2}{\pi}J_n = \frac{2}{\pi}\cdot\frac{(2n)!!}{(2n+1)!!}$ □

Application 26.

$$\int_0^{\frac{\pi}{2}} \cos^2 x\Big(\sin x \sin^2\Big(\frac{\pi}{2}\cos x\Big) + \cos x \sin^2\Big(\frac{\pi}{2}\sin x\Big)\Big)dx = \frac{1}{2}$$

Proof. We make the change: $t = u(x) = \frac{\pi}{2} - x, u'(x) = -1, u(0) = \frac{\pi}{2}, u(\frac{\pi}{2}) = 0$ and we have:

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x\Big(\cos x \sin^2\Big(\frac{\pi}{2}\sin x\Big) + \sin x \sin^2\Big(\frac{\pi}{2}\cos x\Big)\Big)dx, \text{ therefore}$$

$$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\Big(\cos x \sin^2\Big(\frac{\pi}{2}\sin x\Big) + \sin x \sin^2\Big(\frac{\pi}{2}\cos x\Big)\Big)dx =$$

$$= \int_0^{\frac{\pi}{2}} \Big(\sin^2\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx + \int_0^{\frac{\pi}{2}} \Big(\sin^2\Big(\frac{\pi}{2}\cos x\Big)\sin x\Big)dx = A + B, \text{ where}$$

$A = \int_0^{\frac{\pi}{2}} \Big(\sin^2\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx, B = \int_0^{\frac{\pi}{2}} \Big(\sin^2\Big(\frac{\pi}{2}\cos x\Big)\sin x\Big)dx$. We easily deduce that:

$$A = \int_0^{\frac{\pi}{2}} \Big(\sin^2\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2\Big(\frac{\pi}{2}\sin x\Big)\Big(\frac{\pi}{2}\cos x\Big)dx =$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2 t dt = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(1-\cos 2t)dt = \frac{1}{2}$$

and analogous $B = \int_0^{\frac{\pi}{2}}\left(\sin^2\left(\frac{\pi}{2}\cos x\right)\sin x\right)dx = \frac{1}{2}$.

Hence, $2I = A + B = 1 \Leftrightarrow I = \frac{1}{2}$. □

Application 27.
Let $n \in \mathbb{N}$, then:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^2 x\left(\cos x \sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)+\sin x \sin^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx = \frac{2}{\pi}\cdot\frac{(2n)!!}{(2n+1)!!}$$

Proof. We make the change: $t = u(x) = \frac{\pi}{2} - x, u'(x) = -1, u(0) = \frac{\pi}{2}, u(\frac{\pi}{2}) = 0$ and we have:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x\left(\sin x \sin^{2n+1}\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)\right)dx, \text{ then:}$$

$$2I_n = \int_0^{\frac{\pi}{2}}(\sin^2 x + \cos^2 x)\left(\cos x \sin^{2n+1}\left(\frac{\pi}{2}\sin x\right) + \sin x \sin^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx =$$

$$= \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)\cos x\right)dx + \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\cos x\right)\sin x\right)dx = A_n + B_n, \text{ where}$$

$$A_n = \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)\cos x\right)dx, B_n = \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\cos x\right)\sin x\right)dx. \text{ Therefore:}$$

$$A_n = \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)\cos x\right)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\left(\frac{\pi}{2}\sin x\right)\left(\frac{\pi}{2}\cos x\right)dx =$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x dx, \text{ and analogous.}$$

$$B_n = \int_0^{\frac{\pi}{2}}\left(\sin^{2n+1}\left(\frac{\pi}{2}\cos x\right)\sin x\right)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = A_n$$

$$2I_n = A_n + B_n = 2\cdot\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \Leftrightarrow I_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \Leftrightarrow I_n = \frac{2}{\pi}J_n, \text{ where}$$

$$J_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx, \text{ which easily by integrating by parts we obtain the}$$

well-known recurrence: $(2n+1)J_n = 2nJ_{n-1}, \forall n \in \mathbb{N}^*$, where $J_0 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$.

By $(2k + 1)J_k = 2kJ_{k-1}, \forall k \in \mathbb{N}^*$, we deduce that:

$$\prod_{k=1}^{n}(2k + 1)J_k = \prod_{k=1}^{n}(2k)J_{k-1}, \forall k \in \mathbb{N}^*$$

$$\Rightarrow (2n+1)!!J_n = (2n)!!J_0 = (2n)!! \Leftrightarrow J_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}, \forall n \in \mathbb{N}^*$$

Hence, $I_n = \frac{2}{\pi}J_n = \frac{2}{\pi}\cdot\frac{(2n)!!}{(2n+1)!!}$ □

Application 28. Let $n \in \mathbb{N}$, then:

$$I_n = \int_0^{\frac{\pi}{2}} x\Big(\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x + \cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)x\sin x\Big)dx$$
$$= \frac{(2n)!!}{(2n+1)!!} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}, \forall n \in \mathbb{N}^*$$

Proof. We make the change:

$$t = u(x) = \frac{\pi}{2} - x, u'(x) = -1, u(0) = \frac{\pi}{2}, u\Big(\frac{\pi}{2}\Big) = 0 \text{ and we deduce that:}$$
$$I_n = \int_0^{\frac{\pi}{2}} \Big(\frac{\pi}{2} - x\Big)\Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin x + \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx =$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin x + \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx - I_n \Leftrightarrow$$
$$\Leftrightarrow I_n = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\cos x + \cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx = \frac{\pi}{4}(A_n + B_n)$$

where $A_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\cos x\Big)\sin x\Big)dx$ and $B_n = \int_0^{\frac{\pi}{2}} \Big(\cos^{2n+1}\Big(\frac{\pi}{2}\sin x\Big)\cos x\Big)dx$

In A_n we make the change: $t = w(x) = \frac{\pi}{2}\cos x, w'(x) = -\frac{\pi}{2}\sin x, w(0) = \frac{\pi}{2}$, $w(\frac{\pi}{2}) = 0$ and easily we obtain that:

$$A_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt.$$

In B_n we make the change: $t = v(x) = \frac{\pi}{2}\sin x, v'(x) = \frac{\pi}{2}\cos x, v(0) = \frac{\pi}{2}, v(\frac{\pi}{2}) = 0$ and easily we obtain that:

$$B_n = A_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt.$$

$$\text{So, } I_n = \frac{\pi}{4}\Big(\frac{2}{\pi} + \frac{2}{\pi}\Big)\int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt = \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt$$

For $I_n = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx$, we easily (integrating by parts) obtain the recurrence:

$$(2n+1)I_n = 2nI_{n-1}, \forall n \in \mathbb{N}^*, \text{ with } I_0 = \int_0^{\frac{\pi}{2}} \cos x dx = 1.$$

$$\text{So, } (2k+1)I_k = 2kI_{k-1}, \forall k \in \mathbb{N}^* \text{ and yields:}$$
$$\prod_{k=1}^{n}(2k+1)I_k = \prod_{k=1}^{n}(2k)I_{k-1}, \forall k \in \mathbb{N}^*$$
$$\Rightarrow (2n+1)!!I_n = (2n)!!I_0 = (2n)!! \Leftrightarrow$$
$$\Leftrightarrow I_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)}, \forall n \in \mathbb{N}^*$$

□

Application 29.

$$\int_0^{\frac{\pi}{2}} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}dx = \frac{12\pi + 7\ln\frac{4}{3}}{25} \text{ and } 14\ln\frac{4}{3} > \pi$$

Proof. Indeed, let

$$f:\Big[0,\frac{\pi}{2}\Big]\to\mathbb{R}, f(x)=\frac{3\sin x+4\cos x}{3\cos x+4\sin x}, I=\int_0^{\frac{\pi}{2}} f(x)dx, \text{ and}$$

$$u(x)=3\sin x+4\cos x, v(x)=3\cos x+4\sin x. \text{ We have:}$$

$$u(x)=av(x)+bv'(x)\Leftrightarrow 3\sin x+4\cos x=a(3\cos x+4\sin x)+b(-3\sin x+4\cos x)$$

$$\Leftrightarrow 4a-3b=3, 3a+4b=4, \text{ which yields that } a=\frac{24}{25}, b=\frac{7}{25}.$$

$$\text{So, } f(x)=\frac{u(x)}{v(x)}=\frac{1}{25}\Big(24+\frac{7v'(x)}{v(x)}\Big), \text{ then:}$$

$$I=\int_0^{\frac{\pi}{2}} f(x)dx=\frac{1}{25}\int_0^{\frac{\pi}{2}}\Big(24+7\cdot\frac{v'(x)}{v(x)}\Big)=\frac{1}{25}\cdot 24x\Big|_0^{\frac{\pi}{2}}+\frac{7}{25}\int_0^{\frac{\pi}{2}}\frac{v'(x)}{v(x)}dx=$$

$$(1)\qquad =\frac{12\pi}{25}+\frac{7}{25}\cdot\ln v(x)\Big|_0^{\frac{\pi}{2}}=\frac{12\pi+7\ln\frac{4}{3}}{25}$$

Using the change of variable $x=\frac{\pi}{2}-t=w(t), w'(t)=-1, w(0)=\frac{\pi}{2}, w(\frac{\pi}{2})=0$

$$\text{we obtain } I=\int_0^{\frac{\pi}{2}} f(x)dx=\int_0^{\frac{\pi}{2}}\frac{1}{f(x)}dx, \text{ so}$$

$$2I=\int_0^{\frac{\pi}{2}}\Big(f(x)+\frac{1}{f(x)}\Big)dx\overset{\text{AM-GM}}{\geq}\int_0^{\frac{\pi}{2}}2\sqrt{f(x)\cdot\frac{1}{f(x0}}dx=2\int_0^{\frac{\pi}{2}}dx=\pi, \text{ i.e.}$$

$$(2)\qquad I>\frac{\pi}{2}$$

By (1) and (2) we get:

$$\frac{12\pi+7\ln\frac{4}{3}}{25}>\frac{\pi}{2}\Leftrightarrow 24\pi+14\ln\frac{4}{3}>25\pi\Leftrightarrow 14\ln\frac{3}{2}>\pi, \text{ and we are done.}$$

□

Application 30.

If $a\in\mathbb{R}_+^*, f,g,h:\mathbb{R}\to\mathbb{R}$ are continuous with f and g are odd and h is even, then:

$$\int_{-a}^{a} f(x)\cdot\ln(1+e^{g(x)})\cdot\arctan(h(x))dx=\int_0^a f(x)g(x)\arctan(h(x))dx.$$

Proof.

$$I=\int_{-a}^{a} f(x)\ln(1+e^{g(x)})\arctan(h(x))dx, \text{ where we changes the variable:}$$

$$x=u(t)=-t, u'(t)=-1, u(a)=-a, u(-a)=a, \text{ then:}$$

$$I=\int_{-a}^{a} f(-x)\ln(1+e^{g(-x)})\arctan(h(-x))dx=-\int_{-a}^{a} f(x)\ln(1+e^{-g(x)})\arctan(h(x))dx=$$

$$=-\int_{-a}^{a} f(x)\ln\Big(\frac{1+e^{g(x)}}{e^{g(x)}}\Big)\arctan(h(x))dx=-\int_{-a}^{a} f(x)\ln(1+e^{g(x)})\arctan(h(x))dx+$$

$$+\int_{-a}^{a} f(x)g(x)\arctan(h(x))dx=-I+2\cdot\int_0^a f(x)g(x)\arctan(h(x))dx, \text{ and we get}$$

$$I=\int_0^a f(x)g(x)\arctan(h(x))dx$$

□

Application 31.
If $a, b \in \mathbb{R}, a < b$ and $f, g : \mathbb{R} \to \mathbb{R}$ are continue functions such that:
$f(x)f(a+b-x) = 1, g(x) = g(a+b-x), \forall x \in \mathbb{R}$, then:

$$\int_a^b \frac{g(x)}{1+f(x)}dx = \frac{1}{2} \cdot \int_a^b g(x)dx$$

Proof.

$I = \displaystyle\int_a^b \frac{g(x)}{1+f(x)}dx$ where we put $x = u(t) = a+b-t, u'(t) = -1, u(a) = b, u(b) = a$

$$\text{then: } I = \int_a^b \frac{g(a+b-t)}{1+f(a+b-t)}dt = \int_a^b \frac{g(x)}{1+\frac{1}{f(x)}}dx = \int_a^b \frac{f(x)g(x)}{1+f(x)}dx, \text{ so}$$

$2I = I+I = \displaystyle\int_a^b \frac{g(x)}{1+f(x)}dx + \int_a^b \frac{f(x)g(x)}{1+f(x)} = \int_a^b \frac{(1+f(x))g(x)}{1+f(x)}dx = \int_a^b g(x)dx$, then

$$I = \frac{1}{2} \cdot \int_a^b g(x)dx.$$

□

Application 32.
If $a, b \in \mathbb{R}, a < b$ and $f : \mathbb{R} \to \mathbb{R}$ is continue with derivative also continue and $g : \mathbb{R} \to \mathbb{R}_+$ such that $f(a+b-x) = f(x), g(a+b-x)g(x) = 1, \forall x \in \mathbb{R}$, then:

$$\int_a^b \Big(\frac{f(x)}{1+g(x)} + f'(x)\ln(1+g(x))\Big)dx = \frac{1}{2}\int_a^b (f(x) + f'(x)\ln g(h))dx$$

Proof. Since $f(a+b-x) = f(x), \forall x \in \mathbb{R}$ we have that $f'(a+b-x) = -f'(x)$, $\forall x \in \mathbb{R}$.

$$I = \int_a^b \Big(\frac{f(x)}{1+g(x)} + f'(x)\ln(1+g(x))\Big)dx, \text{ and if we are putting:}$$

$x = u(t) = a+b-t$, with $u'(t) = -1, u(a) = b, u(b) = a$, we obtain:

$$I = -\int_b^a \Big(\frac{f(a+b-t)}{1+g(a+b-t)} + f'(a+b-t)\ln(1+g(a+b-t))\Big)dt =$$

$$= \int_a^b \left(\frac{f(x)}{1+\frac{1}{g(x)}} - f'(x)\ln\Big(1+\frac{1}{g(x)}\Big)\right)dt =$$

$$= \int_a^b \Big(\frac{f(x)g(x)}{1+g(x)} - f'(x)\ln(1+g(x)) + f'(x)\ln g(x)\Big)dt$$

Hence:

$$2I = \int_a^b \Big(\frac{(1+g(x))f(x)}{1+g(x)} + f'(x)\ln g(x)\Big)dx = \int_a^b (f(x) + f'(x)\ln g(x))dx$$

which is the conclusion. □

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Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.
Email address: dansitaru63@yahoo.com