CERTAIN IONESCU-WEITZENBÖCK TYPE INEQUALITIES

D.M. BĂTINEŢU - GIURGIU, MIHÁLY BENCZE, CLAUDIA NĂNUŢI, NECULAI STANCIU

Abstract. In this paper we present some certain Ionescu-Weitzenböck type inequalities.

Theorem 1.
If ABC is a triangle with area S and the usual notations, then:

$$\sqrt{(a^4+1)(b^4+1)}+\sqrt{(b^4+1)(c^4+1)}+\sqrt{(c^4+1)(a^4+1)}\geq 8\sqrt{3}S$$

Proof. First, we note that:

$$(*)\qquad \sqrt{(x^4+1)(y^4+1)}\geq x^2+y^2, \forall x,y\in\mathbb{R}_+^*$$

Indeed, $\sqrt{(x^4+1)(y^4+1)}\geq x^2+y^2 \Leftrightarrow (x^2y^2-1)^2\geq 0$, which is true with equality iff $xy=1$. So,

$$(1)\qquad \sum_{cyc}\sqrt{(a^4+1)(b^4+1)}\overset{(*)}{\geq}\sum_{cyc}(a^2+b^2)=2\sum_{cyc}a^2=2(a^2+b^2+c^2)$$

By Ionescu-Weitzenböck inequality we have:

$$(2)\qquad a^2+b^2+c^2\geq 4\sqrt{3}S$$

From (1) and (2) we get the conclusion and we are done.
We have equality iff $a=b=c=1$. □

Theorem 2.
If $A_1A_2\dots A_n, n\geq 3$ is a convex polygon with area S and sides $[A_kA_{k+1}]$ with lengths:

$a_k, k\in\overline{1,n}, a_{n+1}=a_1$, then $\displaystyle\sum_{cyc}((a_k^4+1)(a_{k+1}^4+1))^{\frac{m+1}{2}}\geq\frac{2^{3(m+1)}S^{m+1}}{n^m}\tan^{m+1}\frac{\pi}{n}$

Proof. First, we note that

$$(*)\qquad \sqrt{(x^4+1)(y^4+1)}\geq x^2+y^2, \forall x,y\in\mathbb{R}_+^*$$

Indeed, $\sqrt{(x^4+1)(y^4+1)}\geq x^2+y^2 \Leftrightarrow (x^2y^2-1)^2\geq 0$, which is true with equality iff $xy=1$. By Radon's inequality and (*) we infer:

$$\sum_{cyc}((a_k^4+1)(a_{k+1}^4+1))^{\frac{m+1}{2}}\overset{\text{Radon}}{\geq}\frac{1}{n^m}\Big(\sum_{cyc}\sqrt{(a_k^4+1)(a_{k+1}^4+1}\Big)^{m+1}\overset{(*)}{\geq}$$

$$(1)\qquad \geq\frac{1}{n^m}\Big(\sum_{cyc}(a_k^2+a_{k+1}^2)\Big)^{m+1}=\frac{2^{m+1}}{n^m}\Big(\sum_{k=1}^n a_k^2\Big)^{m+1}$$

Key words and phrases. Inequalities, Ionescu-Weitzenböck's inequality, Radon's inequality, E. Just, N. Schaumberger.

By E. Just and N. Schaumberger (Problem 1634 from AMM, 70 (1963)) we have:

$$\sum_{k=1}^{n} a_k^2 \geq 4S \cdot \tan\frac{\pi}{n} \tag{2}$$

By (1) and (2), we obtain:

$$\sum_{cyc}((a_k^4+1)(a_{k+1}^4+1))^{\frac{m+1}{2}} \geq \frac{2^{m+1}}{n^m}4^{m+1}S^{m+1}\tan^{m+1}\frac{\pi}{n} = \frac{2^{3(m+1)}S^{m+1}}{n^m}\tan^{m+1}\frac{\pi}{n}$$

and we are done. We have equality iff the polygon is regular with the lengths sides equal with 1. □

Theorem 3.
If $A_1A_2\ldots A_n, n \geq 3$ is a convex polygon with the area S and the sides $[A_kA_{k+1}]$ with the lengths:

$$a_k, k = \overline{1,n}, a_{n+1} = a_1, \text{ then } \sum_{cyc}\sqrt{(a_k^4+1)(a_{k+1}^4+1)} \geq 8S \cdot \tan\frac{\pi}{n}$$

Proof. First we note that:

$$\sqrt{(x^4+1)(y^4+1)} \geq x^2+y^2, \forall x,y \in \mathbb{R}_+^* \tag{*}$$

Indeed, $\sqrt{(x^4+1)(y^4+1)} \geq x^2+y^2 \Leftrightarrow (x^2y^2-1)^2 \geq 0$, which is true with equality iff $xy = 1$. So, by (*), we infer:

$$\sum_{cyc}\sqrt{(a_k^4+1)(a_{k+1}^4+1)} \geq \sum_{cyc}(a_k^2+a_{k+1}^2) = 2\sum_{k=1}^{n}a_k^2 \tag{1}$$

By E. Just and N. Schaumberger (Problem 1634 from AMM, 70 (1963)), we have:

$$\sum_{k=1}^{n} a_k^2 \geq 4S \cdot \tan\frac{\pi}{n} \tag{2}$$

By (1) and (2), we obtain:

$$\sum_{cyc}\sqrt{(a_k^4+1)(a_{k+1}^4+1)} \geq 8S \cdot \tan\frac{\pi}{n}$$

and we are done. We have equality iff the polygon is regular with the lengths sides equal with 1. □

Theorem 4.
In any ABC triangle with $BC = a, CA = b, AB = c$ and the area F the following inequality is true:

$$(b^2+c^2)\sin\frac{A}{2} + (c^2+a^2)\sin\frac{B}{2} + (a^2+b^2)\sin\frac{C}{2} \geq 4\sqrt{3}F$$

Proof. We have:

$$\sum_{cyc}(b^2+c^2)\sin\frac{A}{2} \overset{\text{AM-GM}}{\geq} 2\sum_{cyc}bc\sin\frac{A}{2} = 2\sum_{cyc}\frac{bc\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}} =$$

$$= \sum_{cyc}\frac{bc\sin A}{\cos\frac{A}{2}} = 2F\sum_{cyc}\frac{1}{\cos\frac{A}{2}} \tag{1}$$

The function $f:(0,\frac{\pi}{2})\to\mathbb{R}, f(x)=\frac{1}{\cos x}$ is convex on $(0,\frac{\pi}{2})$; indeed
$f'(x)=\frac{\sin x}{\cos^2 x}$

$$f''(x)=\frac{\cos^3 x+2\sin^2 x\cos x}{\cos^4 x}=\frac{\cos^2 x+2\sin^2 x}{\cos^3 x}=\frac{1+\sin^2 x}{\cos^3 x}>0,\forall x\in\Big(0,\frac{\pi}{2}\Big)$$

Therefore by Jensen's inequality we have:
(2)

$$\sum_{cyc} f\Big(\frac{A}{2}\Big)\geq 3f\Big(\frac{A+B+C}{3}\Big)\Leftrightarrow\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\geq 3\cdot\frac{1}{\cos\frac{A+B+C}{6}}=3\cdot\frac{1}{\cos\frac{\pi}{6}}=2\sqrt{3}$$

By (1) and (2) we obtain the desired inequality. □

Theorem 5.
In any ABC triangle with $BC=a, CA=b, AB=c$ and the area F the following inequality is true:

$$ab\Big(1+\sin^2\frac{C}{2}\Big)+bc\Big(1+\sin^2\frac{A}{2}\Big)+ca\Big(1+\sin^2\frac{B}{2}\Big)\geq 4\sqrt{3}F$$

Proof. We have:

$$\sum_{cyc} ab\Big(1+\sin^2\frac{C}{2}\Big)\overset{\text{AM-GM}}{\geq} 2\sum_{cyc} ab\sin\frac{C}{2}=2\sum_{cyc}\frac{ab\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{C}{2}}$$

$$=\sum_{cyc}\frac{ab\sin C}{\cos\frac{C}{2}}=2F\sum_{cyc}\frac{1}{\cos\frac{C}{2}} \tag{1}$$

The function $f:(0,\frac{\pi}{2})\to\mathbb{R}, f(x)=\frac{1}{\cos x}$ is convex on $(0,\frac{\pi}{2})$; indeed
$f'(x)=\frac{\sin x}{\cos^2 x}$

$$f''(x)=\frac{\cos^3 x+2\sin^2 x\cos x}{\cos^4 x}=\frac{\cos^2 x+2\sin^2 x}{\cos^3 x}=\frac{1+\sin^2 x}{\cos^3 x}>0,\forall x\in\Big(0,\frac{\pi}{2}\Big)$$

Therefore, by Jensen's inequality we have:

$$\sum_{cyc} f\Big(\frac{A}{2}\Big)\geq 3f\Big(\frac{A+B+C}{3}\Big)\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\geq 3\cdot\frac{1}{\cos\frac{A+B+C}{6}}=3\cdot\frac{1}{\cos\frac{\pi}{6}}=2\sqrt{3} \tag{2}$$

By (1) and (2), we obtain the desired inequality. □

References

[1] Daniel Sitaru, Mihály Bencze, *699 Olympic Mathematical Challenges.* Studis Publishing House, Iaşi, 2017.
[2] Mihály Bencze, Daniel Sitaru, *Quantum Mathematical Power.* Studis Publishing House, Iaşi, 2018.
[3] Mihály Bencze, Daniel Sitaru, Marian Ursărescu, *Olympic Mathematical Energy.* Studis Publishing House, Iaşi, 2018.
[4] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, ROMANIA.
Email address: dansitaru63@yahoo.com