ABOUT SOME TRIANGLE INEQUALITIES

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Abstract. In this paper we present some new triangle inequalities.

Next, we will consider any ABC triangle with the usual notations, and we will denote with s the semiperimeter and with F the triangles's area.

Theorem 1.

If $u, v, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then, in any ABC triangle the following inequality holds:

$$(*)\qquad \frac{x}{uy+vz}\cdot a^4 + \frac{y}{uz+vx}\cdot b^4 + \frac{z}{ux+vy}\cdot c^4 \geq \frac{16}{u+v}F^2$$

Proof. We have:

$$\sum_{cyc}\frac{x}{uy+vz}\cdot a^4 = \sum_{cyc}\frac{x^2}{uxy+vxz}\cdot a^4 = \sum_{cyc}\frac{(xa^2)^2}{uxy+vxz} \geq$$

$$\overset{\text{Bergström}}{\geq} \frac{(a^2x+b^2y+c^2z)^2}{\sum_{cyc}(uxy+vxz)} = \frac{(xa^2+yb^2+zc^2)^2}{(u+v)(xy+yz+zx)} \overset{\text{Klamkin}}{\geq}$$

$$\geq \frac{16(xy+yz+zx)F^2}{(u+v)(xy+yz+zx)} = \frac{16}{u+v}F^2$$

If in (*) we take $u = v$ we obtain the following inequality: □

$$(T)\qquad \frac{x}{y+z}a^4 + \frac{y}{z+x}b^4 + \frac{z}{x+y}c^4 \geq 8F^2$$

namely, G. Tsintsifas's inequality, and if in (T), $x = y = z$, it follows:

$$(F.G.)\qquad a^4+b^4+c^4 \geq 16F^2$$

namely, F. Goldner's inequality.

Theorem 2.

If $m \in \mathbb{N}; u, v, x, y, z \in \mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

(**)

$$3m + \left(\frac{x}{uy+vz}a^4\right)^{m+1} + \left(\frac{y}{uz+vx}b^4\right)^{m+1} + \left(\frac{z}{ux+vy}c^4\right)^{m+1} \geq \frac{16(m+1)}{u+v}F^2$$

Proof 1. We have:

$$3m + \sum_{cyc}\left(\frac{x}{uy+vz}a^4\right)^{m+1} \overset{\text{J. Radon}}{\geq} 3m + \frac{1}{3^m}\left(\sum_{cyc}\frac{xa^4}{uy+vz}\right)^{m+1} \overset{\text{AM-GM}}{\geq}$$

$$\geq (m+1)\sqrt[m+1]{\underbrace{3\cdot 3\cdot \ldots \cdot 3\cdot 3}_{\text{"}m\text{" times}}\cdot\frac{1}{3^m}\left(\sum_{cyc}\frac{xa^4}{uy+vz}\right)^{m+1}} =$$

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$$= (m+1)\cdot\sum_{cyc}\frac{xa^4}{uy+vz}\overset{(*)}{\geq}(m+1)\cdot\frac{16}{u+v}F^2=\frac{16(m+1)}{u+v}F^2$$

□

Proof 2. We have:

$$3m+\sum_{cyc}\Big(\frac{xa^4}{uy+vz}\Big)^{m+1}=\sum_{cyc}\Big(m+\Big(\frac{xa^4}{uy+vz}\Big)^{m+1}\Big)\overset{\text{AM-GM}}{\geq}$$

$$\geq\sum_{cyc}(m+1)\sqrt[m+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1\cdot 1}_{"m"\text{ times}}\Big(\frac{xa^4}{uy+vt}\Big)^{m+1}}=(m+1)\cdot\sum_{cyc}\frac{ax^4}{uy+vz}\overset{(*)}{\geq}$$

$$\geq(m+1)\cdot\frac{16}{u+v}F^2=\frac{16(m+1)}{u+v}F^2$$

□

Theorem 3.
If $m,n,x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$(***)\qquad \frac{xa^8}{(my+nz)^3}+\frac{yb^8}{(mz+nx)^3}+\frac{zc^8}{(mx+ny)^3}\geq\frac{256F^4}{(m+n)^3(xy+yz+zx)}$$

Proof. We have:

$$\sum_{cyc}\frac{xa^8}{(my+nz)^3}=\sum_{cyc}\frac{x^4a^8}{(mxy+mxz)^3}=\sum_{cyc}\frac{(xa^2)^4}{(mxy+nxz)^3}\overset{\text{J. Radon}}{\geq}$$

$$\geq\frac{(xa^2+yb^2+zc^2)^4}{(\sum_{cyc}(mxy+nxz))^3}=\frac{(xa^2+yb^2+zc^2)^4}{(m+n)^3(xy+yz+zx)^3}\overset{\text{Bergström}}{\geq}$$

$$\geq\frac{(16(xy+yz+zx)F^2)^2}{(m+n)^3(xy+yz+zx)^3}=\frac{256F^4}{(m+n)^3(xy+yz+zx)}$$

□

Theorem 4.
If $m\in\mathbb{R}_+=[0,\infty); u,v,x,y,z\in\mathbb{R}_+^*$, then in any ABC triangle the following inequality holds:

$$(1)\quad \frac{x\cdot a^{2m+2}}{(uy+vz)^m}+\frac{y\cdot b^{2m+2}}{(uz+vx)^m}+\frac{z\cdot c^{2m+2}}{(ux+vy)^m}\geq\frac{4^{m+1}}{(u+v)^m}\cdot(xy+yz+zx)^{\frac{1-m}{2}}\cdot F^{m+1}$$

Proof. We have:

$$\sum_{cyc}\frac{xa^{2m+2}}{(uy+vz)^m}=\sum_{cyc}\frac{(xa^2)^{m+1}}{(uxy+vxz)^m}\overset{\text{J.Radon}}{\geq}\frac{(xa^2+yb^2+zc^2)^{m+1}}{(\sum_{cyc}(uxy+vxz))^m}=$$

$$=\frac{(xa^2+yb^2+zc^2)^{m+1}}{(u+v)^m(xy+yz+zx)^m}\overset{\text{Klamkin}}{\geq}\frac{(4\sqrt{xy+yz+zx}\cdot F)^{m+1}}{(u+v)^m\cdot(xy+yz+zx)^m}=$$

$$=\frac{4^{m+1}}{(u+v)^m}\cdot(xy+yz+zx)^{\frac{m+1}{2}-m}\cdot F^{m+1}=$$

$$=\frac{4^{m+1}}{(u+v)^m}\cdot(xy+yz+zx)^{\frac{1-m}{2}}\cdot F^{m+1}$$

If $m=3$ from inequality (1) we obtain equality (***). □

Theorem 5.
If $m \in \mathbb{N}, n \in \mathbb{N}^* - \{1, 2\}$ and $A_1A_2 \ldots A_n$ is a convex poligon with the area F and the sides having the lengths $a_k = A_kA_{k+1}, A_{n+1} = A_1$, then:

$$m \cdot n + \sum_{k=1}^{n} a_k^{2m+2} \geq 4(m+1) \cdot F \cdot \tan \frac{\pi}{n} \tag{2}$$

Proof. We have:

$$m \cdot n + \sum_{k=1}^{n} a_k^{2(m+1)} = \sum_{k=1}^{n} (m + a_k^{2(m+1)}) \overset{\text{AM-GM}}{\geq}$$

$$\geq \sum_{k+1}^{n} (m+1) \sqrt[m+1]{\underbrace{1 \cdot 1 \cdot \ldots \cdot 1 \cdot 1}_{\text{"}m\text{" times}} a^{2(m+1)}} = (m+1) \cdot \sum_{k=1}^{n} a_k^2 \tag{3}$$

In problem 1634 from AMM, 70(1963), E. Just and N. Schaumberger proved that in any convex polygon $A_1A_2 \ldots A_n, n \geq 3$ having the area F the following inequality holds:

$$\sum_{k=1}^{n} a_k^2 \geq 4F \cdot \tan \frac{\pi}{n} \tag{J-S}$$

From (3) and (J-S) we deduce that:

$$m \cdot n + \sum_{k=1}^{n} a_k^{2m+2} \geq (m+1) \cdot 4 \cdot F \cdot \tan \frac{\pi}{n} = 4(m+1) \cdot F \cdot \tan \frac{\pi}{n}$$

□

Let's finish with a demonstration of D.S. Mitrinovič's inequality. In Mathematical Gazette, Vol. XLVIII (1942-1943) it is proved that in any ABC triangle the following inequalities hold:

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \tag{i}$$

$$ab + bc + ca = s^2 + r^2 + 4Rr \tag{ii}$$

So,

$$4s^2 = (a+b+c)^2 \geq 3(ab+bc+ca) \overset{\text{(ii)}}{=} 3(s^2 + r^2 + 4Rr) \Rightarrow$$

$$\Rightarrow s^2 \geq 3r(4R + r) \overset{\text{Euler}}{\geq} 3r(4 \cdot 2r + r) \Leftrightarrow s^2 \geq 27r^2 \Rightarrow$$

$$\Rightarrow s \geq 3\sqrt{3} \cdot r, \text{ we've obtained Mitrinovič's inequality.}$$

Or, taking into account (i), (ii) and the fact that $x^2 + y^2 + z^2 \geq xy + yz + zx$, $\forall x, y, z \in \mathbb{R}_+$ we have:

$$a^2 + b^2 + c^2 \geq ab + bc + ca \Leftrightarrow 2(s^2 - r^2 - 4Rr) \geq s^2 + r^2 + 4Rr \Leftrightarrow$$

$$\Leftrightarrow s^2 \geq 3r(4R + r) \Leftrightarrow s^2 \overset{\text{Euler}}{\geq} 27r^2 \Rightarrow s \geq 3\sqrt{3}r$$

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