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ABOUT 996 INEQUALITY IN TRIANGLE

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1) In ΔABC the following relationship holds:

$$12R \le \sum \frac{b^2 + c^2}{h_a} \le \frac{9R^3\sqrt{3}}{S}$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution:

We will prove the following lemma:

Lemma:

2) In ΔABC the following relationship holds:

$$\sum \frac{b^2 + c^2}{h_a} = \frac{s^2 + r^2 - 2Rr}{r}$$

Proof:

Using $h_a = \frac{2S}{a}$ we obtain:

$$\sum \frac{b^2 + c^2}{h_a} = \sum \frac{b^2 + c^2}{\frac{2S}{a}} = \sum \frac{a(b^2 + c^2)}{2S} = \frac{2s(s^2 + r^2 - 2Rr)}{2rs} = \frac{s^2 + r^2 - 2Rr}{r}$$

Using the Lemma, the left-hand inequality can be written:

$\frac{s^2+r^2-2Rr}{r} \ge 12R \Leftrightarrow s^2 \ge 14Rr - r^2$, true from Gerretsen's inequality:

$s^2 \ge 16Rr - 5r^2$ and Euler's inequality $R \ge 2r$.

For right-hand inequality we prove that:

$\sum \frac{b^2+c^2}{h_a} \le \frac{6R^2}{r} \Leftrightarrow \frac{s^2+r^2-2Rr}{r} \le \frac{6R^2}{r} \Leftrightarrow s^2 \le 6R^2 + 2Rr - r^2$, true from Gerretsen's inequality: $s^2 \le 4R^2 + 4Rr + 3r^2$ and Euler's inequality $R \ge 2r$.

Then $\frac{6R^2}{r} \le \frac{9R^3\sqrt{3}}{rs} \Leftrightarrow s \le \frac{3R\sqrt{3}}{2}$ (Mitrinovic's inequality)

Equality hold if and only if the triangle is equilateral.

Remark:

If we replace h_a with r_a we propose:

3) In ΔABC the following relationship holds:

$$12R \leq \sum \frac{b^2 + c^2}{r_a} \leq 4R\left(\frac{2R}{r} - 1\right)$$

Proposed by Marin Chirciu – Romania

Solution:

We prove the following lemma:

Lemma:

4) In ΔABC the following lemma:

$$\sum \frac{b^2 + c^2}{r_a} = \frac{2(s^2 - 3r^2 - 6Rr)}{r}$$

Proof:

Using $h_a = \frac{S}{s-a}$ we obtain:

$$\sum \frac{b^2 + c^2}{r_a} = \sum \frac{b^2 + c^2}{\frac{S}{s-a}} = \sum \frac{(b^2 + c^2)(s-a)}{S} = \frac{2s(s^2 - 3r^2 - 6Rr)}{rs} =$$

$$= \frac{2(s^2 - 3r^2 - 6Rr)}{r}$$

Using Lemma, the left-hand inequality can be written:

$\frac{2(s^2-3r^2-6Rr)}{r} \geq 12R \Leftrightarrow s^2 \geq 12Rr + 3r^2$**, true from Gerretsen's inequality:**

$s^2 \geq 16Rr - 5r^2$ **and Euler's inequality $R \geq 2r$.**

The right-hand inequality can be written:

$$\frac{2(s^2 - 3r^2 - 6Rr)}{r} \leq 4R\left(\frac{2R}{r} - 1\right) \Leftrightarrow \frac{s^2 + r^2 - 2Rr}{r} \leq \frac{6R^2}{r} \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$$

(Gerretsen's inequality).

Equality holds if and only if the triangle is equilateral.

Remark:

Between the sums $\sum\frac{b^2+c^2}{h_a}$ and $\sum\frac{b^2+c^2}{r_a}$ the following relationship holds:

5) In ΔABC the following relationship holds:

$$\sum\frac{b^2+c^2}{h_a}\le\sum\frac{b^2+c^2}{r_a}$$

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Solution:

Using the above Lemmas, we have:

$\sum\frac{b^2+c^2}{h_a}=\frac{s^2+r^2-2Rr}{r}$ and $\sum\frac{b^2+c^2}{r_a}=\frac{2(s^2-3r^2-6Rr)}{r}$

We the write the inequality:

$\frac{s^2+r^2-2Rr}{r}\le\frac{2(s^2-3r^2-6Rr)}{r}\Leftrightarrow s^2\ge 10Rr+7r^2$, true from Gerretsen's inequality:

$s^2\ge 16Rr-5r^2$ and Euler's inequality $R\ge 2r$.

Remark:

We can write the sequence of inequalities:

6) In ΔABC the following inequality holds:

$$12R\le\frac{b^2+c^2}{h_a}\le\sum\frac{b^2+c^2}{r_a}\le 4R\left(\frac{2R}{r}-1\right)$$

Solution:

See inequalities 5), 3) and 1).

Equality holds if and only if the triangle is equilateral.