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ABOUT 1070 INEQUALITY IN TRIANGLE

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1) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$AG \cdot AK + BG \cdot BK + CG \cdot CK \geq \frac{4S}{\sqrt{3}}$$

Proposed by Marian Ursărescu – Romania

Solution:

We prove the following lemma:

Lemma:

2) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$\sum AG \cdot AK = \frac{2}{3} \cdot \frac{s^2(s^2 - 6Rr) - r^2(4R + r)^2}{s^2 - r(4R + r)}$$

Proof:

Using $AG = \frac{2}{3}m_a$ and $AK = \frac{2bc}{a^2+b^2+c^2} \cdot m_a$ we obtain:

$$\sum AG \cdot AK = \sum \frac{2}{3}m_a \cdot \frac{2bc}{a^2 + b^2 + c^2} \cdot m_a = \frac{4}{3(a^2 + b^2 + c^2)} \sum bc \cdot m_a^2 =$$

$= \frac{4}{3 \cdot 2[s^2 - r(4R+r)]} \cdot [s^2(s^2 - 6Rr) - r^2(4R + r)^2] = \frac{2}{3} \cdot \frac{s^2(s^2-6Rr)-r^2(4R+r)^2}{s^2-r(4R+r)}$, which follows from the known identity in triangle $\sum bc \cdot m_a^2 = s^4 - 6s^2Rr - r^2(4R + r)^2$

Back to the main problem:

Using Lemma, the inequality can be written:

$$\frac{2}{3} \cdot \frac{s^2(s^2 - 6Rr) - r^2(4R + r)^2}{s^2 - r(4R + r)} \geq \frac{4rs\sqrt{3}}{3}$$

Taking into account Doucet's inequality $4R + r \geq s\sqrt{3}$ it suffices to prove that:

$\frac{2}{3} \cdot \frac{s^2(s^2-6Rr)-r^2(4R+r)^2}{s^2-r(4R+r)} \geq \frac{4r(4R+r)}{3} \Leftrightarrow s^2(s^2 - 2r^2 - 14Rr) + r^2(4R + r)^2 \geq 0$ **(*)**

We distinguish the following cases:

Case 1). If $(s^2 - 2r^2 - 14Rr) \geq 0$, the inequality is obvious.

Case 2). If $(s^2 - 2r^2 - 14Rr) < 0$, the inequality can be rewritten:

$$r^2(4R + r)^2 \geq s^2(14Rr + 2r^2 - s^2)$$

which follows from Blundon-Gerretsen's inequality $16Rr - 5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that:

$$r^2(4R + r)^2 \geq \frac{R(4R + r)^2}{2(2R - r)}(14Rr + 2r^2 - 16Rr + 5r^2) \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0 \Leftrightarrow$$

$(R - 2r)(2R + r) \geq 0$, obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark:

The inequality can be strengthened:

3) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$AG \cdot AK + BG \cdot BK + CG \cdot CK \geq \frac{4r}{3}(4R + r)$$

Solution:

See (*) from the above proof.

Equality holds if and only if the triangle is equilateral.

Remark:

Inequality 3) is stronger than inequality 1).

4) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$AG \cdot AK + BG \cdot BK + CG \cdot CK \geq \frac{4r}{3}(4R + r) \geq \frac{4S}{\sqrt{3}}$$

Solution:

See 3) and Doucet's inequality $4R + r \geq s\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

Remark:

Let's find an inequality having an opposite sense:

5) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$AG \cdot AK + BG \cdot BK + CG \cdot CK \le \frac{R^2}{3r}(4R + r)$$

Solution:

Using Lemma, the inequality can be written:

$$\frac{2}{3} \cdot \frac{s^2(s^2 - 6Rr) - r^2(4R + r)^2}{s^2 - r(4R + r)} \le \frac{R^2}{3r}(4R + r) \Leftrightarrow$$

$\Leftrightarrow s^2(4R^3 + R^2r + 12Rr^2 - 2rs^2) \ge r(4R + r)^2(R^2 - 2r^2)$, which follows from

Gerretsen's inequality: $\frac{r(4R+r)^2}{R+r} \le 16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$\frac{r(4R + r)^2}{R + r}\left(4R^3 + R^2r + 12Rr^2 - 2r(4R^2 + 4Rr + 3r^2)\right) \ge r(4R + r)^2(R^2 - 2r^2) \Leftrightarrow$$

$\Leftrightarrow 3R^3 - 8Rr + 6Rr^2 - 4r^3 \ge 0 \Leftrightarrow (R - 2r)(3R^2 - 2Rr + 2r^2) \ge 0$, obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark:

The double inequality can be written:

6) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$\frac{4r}{3}(4R + r) \le AB \cdot AK + BG \cdot BK + CG \cdot CK \le \frac{R^2}{3r}(4R + r)$$

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Solution:

See inequalities 3) and 5).

Equality holds if and only if the triangle is equilateral.

Remark:

The following inequalities can be written:

7) In ΔABC, G – centroid, K – Lemoine's point. Prove that:

$$12r^2 \le \frac{4S}{\sqrt{3}} \le \frac{4r}{4}(4R+r) \le AG \cdot AK + BG \cdot BK + CG \cdot CK \le \frac{R^2}{3r}(4R+r) \le \frac{3R^3}{2r}$$

Solution:

See inequalities 4), 6), Euler's inequality $R \ge 2r$ and Mitrinovic's inequality $s \ge 3r\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

Reference:

1. Marian Ursărescu, *1070 Inequality in triangle*, Romanian Mathematical Magazine, February 2019.

2. Marin Chirciu, *Algebraic Inequalities, from beginner to performer*, Paralela 45 Publishing House, Pitești, 2014.

3. Marin Chirciu, *Inequalities with important lines in triangle from beginner to performer, Paralela* 45 Publishing House, Pitești, 2018.