RMM - Inequalities Marathon 301 - 400

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301. If $a,b,c>0, a^2+b^2=1, b^2+c^2=1$ then:

$$a+2b+c+\frac{a+c}{abc}\geq 4+2\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mohamed Alhafi-Aleppo-Syria

Since $a^2+b^2=1, c^2+b^2=1$ *we must have:* $a=c$. *So, our inequality is:*

$2a+2b+\frac{2}{ab}\geq 4+2\sqrt{2}\Leftrightarrow a+b+\frac{1}{ab}\geq 2+\sqrt{2}$. *Let* $s=a+b, p=ab$ *then:*

$s^2=1+2p\Rightarrow\frac{1}{p}=\frac{2}{s^2-1}$ *so, we need to show:* $s+\frac{2}{s^2-1}\geq 2+\sqrt{2}$ *or*

$s^3-(2+\sqrt{2})s^2-s+4+\sqrt{2}\geq 0$. *Let* $f(x)=x^3-(2+\sqrt{2})x^2-x+4+\sqrt{2}$

$$f'(x)=3x^2-(4+2\sqrt{2})x-1=x(3x-4-2\sqrt{2})-1$$

Clearly $f'(x)<0$ *for* $0<x\leq\sqrt{2}$ *so,* f *is decreasing on the interval* $]0,\sqrt{2}]$. *Now, by Titu's inequality we have:* $1=a^2+b^2\geq\frac{(a+b)^2}{2}\Rightarrow\sqrt{2}\geq s$. *So,* $f(s)\geq f(\sqrt{2})=0$ *and we are done.*

Solution 2 by Nguyen Van Nho-Nghe An-Vietnam

From: $a^2+b^2=1$ *and* $b^2+c^2=1\to a^2=c^2\to a=c$

The inequality $\Leftrightarrow a+b+\frac{1}{ab}\geq 2+\sqrt{2}\to$ *(*)*

LHS (*) $=2\sqrt{2}a+2\sqrt{2}b+\frac{1}{ab}+(1-2\sqrt{2})(a+b)$ *(1)*

$$2\sqrt{2}a+2\sqrt{2}b+\frac{1}{ab}\overset{AM-GM}{\geq}3\sqrt[3]{2\sqrt{2}a\cdot 2\sqrt{b}\cdot\frac{1}{ab}}=6\ \textit{(2)}$$

and: $0<a+b\leq\sqrt{2(a^2+b^2)}=2, 1-2\sqrt{2}<0$ *(3)*

From (2) & (3) $\to LHS$ *(*)* $\geq 6+(1-2\sqrt{2})\sqrt{2}=2+\sqrt{2}=RHS$ *(done)*

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c>0|a^2+b^2=1, b^2+c^2=1, a+2b+c+\frac{a+c}{abc}\geq 4+2\sqrt{2}$$

$$a^2+b^2=b^2+c^2(=1)\Rightarrow c=a$$

$\because 0<a,b<1$ *&* $a^2+b^2=1$, *we can let* $a=\cos\theta, b=\sin\theta\ \left(0<\theta<\frac{\pi}{2}\right)$

$\therefore$ ***given inequality becomes:*** $a+b+\frac{1}{ab}\ge 2+\sqrt{2}\Leftrightarrow \cos\theta+\sin\theta+\frac{1}{\cos\theta\sin\theta}\overset{(1)}{\ge} 2+\sqrt{2}$

Let $f(\theta)=\cos\theta+\sin\theta+\frac{1}{\cos\theta\sin\theta}$; $f'(\theta)=\cos\theta-\sin\theta+\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}=0\Leftrightarrow$

$$\Leftrightarrow(\cos\theta-\sin\theta)\left(1-\frac{\cos\theta+\sin\theta}{\cos^2\theta\sin^2\theta}\right)\overset{(2)}{=}0;$$

$$\cos\theta\sin\theta=\sqrt{\cos^2\theta\sin^2\theta}\overset{G\le A}{\le}\frac{\cos^2\theta+\sin^2\theta}{2}=\frac{1}{2}$$

$$\therefore t^2=\cos\theta\sin\theta\le\frac{1}{2}\Rightarrow t^3\le\frac{1}{2\sqrt{2}}<2$$

Now, $\cos\theta+\sin\theta\ge 2\sqrt{\cos\theta\sin\theta}=2t>t^4\,(\because t^3<2)$

$$\therefore\frac{\cos\theta+\sin\theta}{\cos^2\theta\sin^2\theta}>1\Rightarrow 1-\frac{\cos\theta+\sin\theta}{\cos^2\theta\sin^2\theta}\overset{(3)}{<}0,\forall\theta\in\left(0,\frac{\pi}{2}\right)$$

(2), (3) $\Rightarrow f'(\theta)=0\Leftrightarrow\cos\theta=\sin\theta\Leftrightarrow\theta=\frac{\pi}{4}$. ***Also,***

$$f''\left(\frac{\pi}{4}\right)=\left(\frac{2\sin\theta}{\cos^3\theta}+\frac{2\cos\theta}{\sin^3\theta}-\cos\theta-\sin\theta\right)\Big|_{\theta=\frac{\pi}{4}}>0$$

$\therefore f(\theta)_{\min}=f\left(\frac{\pi}{4}\right)=2+\sqrt{2}\Rightarrow$ ***(1) is true (proved)***

Solution 4 by Tran Hong-Vietnam

$$\begin{cases}a^2+b^2=1\\ b^2+c^2=1\end{cases}\Rightarrow a=c$$

$$b=ta,t>0\Rightarrow a^2+t^2a^2=1\Rightarrow a^2=\frac{1}{1+t^2}$$

$$a=\frac{1}{\sqrt{1+t^2}}\Rightarrow b=\frac{t}{\sqrt{1+t^2}}$$

$$LHS: a+2b+c+\frac{a+c}{abc}=\frac{1+t}{\sqrt{1+t^2}}+\frac{t^2+1}{t}=2\sqrt{2}\cdot\frac{1}{\sqrt{1+t^2}}+2\sqrt{2}\frac{t}{\sqrt{1+t^2}}+\frac{t^2+1}{t}t+$$

$$+(1-2\sqrt{2})\left(\frac{1+t}{\sqrt{1+t^2}}\right)\quad (*)$$

$$2\sqrt{2}\frac{1}{\sqrt{1+t^2}}+2\sqrt{2}\frac{t}{\sqrt{1+t^2}}+\frac{t^2+1}{t}\overset{Cauchy}{\ge}3\sqrt[3]{2\sqrt{2}\frac{1}{\sqrt{1+t^2}}\cdot2\sqrt{2}\frac{t}{\sqrt{1+t^2}}\cdot\frac{t^2+1}{t}}=$$

$=3\cdot2=6$; $\frac{1+t}{\sqrt{1+t^2}}\le\sqrt{2}\Leftrightarrow 1+t\le\sqrt{2(1+t^2)}\Leftrightarrow 0\le(t-1)^2$ ***(True)***

(*) $\ge 6+\sqrt{2}(1-2\sqrt{2})=6-4+\sqrt{2}=2+\sqrt{2}$ ***(proved)***

$$„=" \Leftrightarrow t=1 \Leftrightarrow \begin{cases} a=b=c \\ a^2+b^2=1 \end{cases} \Leftrightarrow a=b=c=\frac{1}{\sqrt{2}}$$

302. If $a,b,c>0$ then:

$$\left(\frac{a^4}{4}+\frac{b^8}{8}+\frac{5\sqrt[5]{c^8}}{8}\right)\left(\frac{5\sqrt[5]{a^8}}{8}+\frac{b^8}{8}+\frac{c^4}{4}\right) \ge \frac{27(abc)^4}{(ab+bc+ca)^3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{a^4}{4}+\frac{b^8}{8}+\frac{5\sqrt[5]{c^8}}{8}\right)\left(\frac{5\sqrt[5]{a^8}}{8}+\frac{b^8}{8}+\frac{c^4}{4}\right) \overset{(1)}{\ge} \frac{27(abc)^4}{(ab+bc+ca)^3}$$

$$\frac{a^4}{4}+\frac{b^8}{8}+\frac{5c^{\frac{8}{5}}}{8}=\frac{a^4}{8}+\frac{a^4}{8}+\frac{b^8}{8}+\frac{c^{\frac{8}{5}}}{8}+\frac{c^{\frac{8}{5}}}{8}+\frac{c^{\frac{8}{5}}}{8}+\frac{c^{\frac{8}{5}}}{8}+\frac{c^{\frac{8}{5}}}{8} \underset{(i)}{\overset{A-G}{\ge}} 8\sqrt[8]{\frac{a^4\cdot a^4\cdot b^8\left(c^{\frac{8}{5}}\right)^5}{8^8}}=abc$$

$$\text{Also, } \frac{5a^{\frac{8}{5}}}{8}+\frac{b^8}{8}+\frac{c^4}{4}=\frac{a^{\frac{8}{5}}}{8}+\frac{a^{\frac{8}{5}}}{8}+\frac{a^{\frac{8}{5}}}{8}+\frac{a^{\frac{8}{5}}}{8}+\frac{a^{\frac{8}{5}}}{8}+\frac{b^8}{8}+\frac{c^4}{8}+\frac{c^4}{8} \underset{(ii)}{\overset{A-G}{\ge}} 8\sqrt[8]{\frac{\left(a^{\frac{8}{5}}\right)^5 b^8c^4c^4}{8^8}}=abc$$

$$(i),(ii) \Rightarrow \text{LHS of (1)} \ge a^2b^2c^2 \overset{?}{\ge} \frac{27(abc)^4}{(\sum ab)^3} \Leftrightarrow \left(\sum ab\right)^3 \overset{?}{\ge} 27a^2b^2c^2 \Leftrightarrow \sum ab \overset{?}{\ge} 3\sqrt[3]{ab\cdot bc\cdot ca}$$

$\rightarrow$ ***true by A-G (proved)***

303. If $a\ge b\ge c$ then:

$$\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(a+b+c)\ge\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sarah El-Kenitra-Morocco

$$\left(\sqrt{a^2-b^2}+\sqrt{2}b\right)^2=a^2+b^2+2b\sqrt{2(a^2-b^2)}\ge a^2+b^2 \text{ hence}$$

$\sqrt{a^2-b^2}+\sqrt{2b}\ge\sqrt{a^2+b^2}$. *Using the same method, we get*

$\sqrt{b^2-c^2}+\sqrt{2}c\ge\sqrt{b^2+c^2}$ *and* $\sqrt{a^2-c^2}+\sqrt{2}c\ge\sqrt{a^2+c^2}$. *After the sum we get*

$$\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(b+2c)\ge\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

But we have $a \geq c$ *therefore* $\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(a+b+c) \geq$

$$\geq \sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{a^2-b^2}+\frac{\sqrt{2}}{2}(a+b) \geq \sqrt{a^2+b^2} \Leftrightarrow \frac{a+b}{\sqrt{2}} \overset{(1)}{\geq} \sqrt{a^2+b^2}-\sqrt{a^2-b^2}$$

$$\because \sqrt{a^2+b^2}-\sqrt{a^2-b^2} > 0 \text{ as } a^2+b^2 > a^2-b^2$$

$$\therefore (1) \Leftrightarrow \frac{(a+b)^2}{2} \geq a^2+b^2+a^2-b^2-\sqrt{a^4-b^4} = 2a^2-2\sqrt{a^4-b^4}$$

$$\Leftrightarrow a^2+2ab+b^2 \geq 4a^2-4\sqrt{a^4-b^4} \Leftrightarrow 4\sqrt{a^4-b^4} \geq (3a+b)(a-b)$$

$$\Leftrightarrow 16(a^2+b^2)(a+b)(a-b) \geq (3a+b)^2(a-b)^2$$

$$\Leftrightarrow (a-b)(7a^3+19a^2b+21ab^2+17b^3) \geq 0 \rightarrow \text{true} \therefore a \geq b \geq 0$$

$$\therefore \sqrt{a^2-b^2}+\frac{\sqrt{2}}{2}(a+b) \overset{(a)}{\geq} \sqrt{a^2+b^2}$$

Similarly, $\sqrt{b^2-c^2}+\frac{\sqrt{2}}{2}(b+c) \overset{(b)}{\geq} \sqrt{b^2+c^2}$ *and,* $\sqrt{a^2-c^2}+\frac{\sqrt{2}}{2}(a+c) \overset{(c)}{\geq} \sqrt{a^2+c^2}$

(a)+(b)+(c)$\Rightarrow \sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(a+b+c) \geq$

$$\geq \sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2} \quad \textit{(Proved)}$$

304. **If** $0 < a \leq b \leq c \leq d \leq e$ **then:**

$$2\sqrt{ab}+3\sqrt[3]{abc}+4\sqrt[4]{abcd} \leq 9\sqrt[5]{abcde}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $P = 2\sqrt{ab}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}$. *Now, we have* $\sqrt{ab} \leq \sqrt[3]{abc}$.

Because, $(ab)^3 \leq (abc)^2 \Rightarrow ab \leq c^2$ *(1)*

Now, we have $a \leq c, b \leq c \Rightarrow ab \leq c^2$. *So, (1) is true* $\Rightarrow$ *hence* $\sqrt{ab} \leq \sqrt[3]{abc}$

$$P \leq 2\sqrt[3]{abc}+3\sqrt[3]{abc}+4\sqrt[4]{abcd};\ P \leq 5\sqrt[3]{abc}+4\sqrt[4]{abcd}$$

Also, we have $\sqrt[3]{abc} \leq \sqrt[4]{abcd}$. *Because,* $(abc)^4 \leq (abcd)^3 \Rightarrow abc \leq d^3$ *(3)*

$$\because a \leq d, b \leq d, c \leq d \Rightarrow abc \leq d^3 \rightarrow \textit{True}$$

And hence $\sqrt[3]{abc} \le \sqrt[4]{abcd}$; $P \le 5\sqrt[4]{abcd} + 4\sqrt[4]{abcd} \le 9\sqrt[4]{abcd}$

Also, we have $\sqrt[4]{abcd} = \sqrt[5]{abcde} \Rightarrow (abcd)^5 \le (abcde)^4 \Rightarrow abcd \le e^4$

$\because a \le e, b \le e, c \le e, d \le e \Rightarrow abcd \le e^4$ *and hence* $\sqrt[4]{abcd} \le \sqrt[5]{abcde} \Rightarrow$

$\Rightarrow P \le 9\sqrt[4]{abcde}$ *(proved)*

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $a^{\frac{1}{60}} = x, b^{\frac{1}{60}} = y, c^{\frac{1}{60}} = z, d^{\frac{1}{60}} = u, e^{\frac{1}{60}} = v$

$\therefore a = x^{60}, b = y^{60}, c = z^{60}, d = u^{60}, e = v^{60}$ *&* $x \le y \le z \le u \le v$

$\therefore$ *given inequality becomes:*

$$2x^{30}y^{30} + 3x^{20}y^{20}z^{20} + 4x^{15}y^{15}z^{15}u^{15} \overset{(1)}{\le} 9x^{12}y^{12}z^{12}u^{12}v^{12}$$

$$v \ge u, z, y, x$$

Now, $4x^{12}y^{12}z^{12}y^{12}(v^{12}) \underset{(a)}{\overset{v\ge u,z,y,x}{\ge}} 4x^{12}y^{12}z^{12}y^{12}(y^3z^3y^3x^3) = 4x^{15}y^{15}z^{15}y^{15}$

Again, $3x^{12}y^{12}z^{12}(u^{12}v^{12}) \overset{\substack{u\ge y\\ u\ge z}}{\ge} 3x^{12}y^{12}z^{12}(z^8y^8y^4v^4)$

$$\underset{(b)}{\overset{\substack{u\ge x\\ v\ge x}}{\ge}} 3x^{12}y^{20}z^{20}(x^4x^4) = 3x^{20}y^{20}z^{20}$$

Also, $2x^{12}y^{12}(z^{12}y^{12}v^{12}) \overset{\substack{u\ge x\\ v\ge y}}{\ge} 2x^{12}y^{12}(z^{12}x^{12}y^{12})$

$$\underset{(c)}{\overset{\substack{z\ge x\\ v\ge y}}{\ge}} 2x^{12}y^{12}(x^6y^6x^{12}y^{12}) = 2x^{30}y^{30}$$

(a)+(b)+(c)$\Rightarrow$ (1) is true (proved)

305. If $a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 3$ then:

$$|a + (a + c)b + c| \le 4$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Vietnam

We have: $0 \le a^2, b^2, c^2 \le 3$ *then:* $|a + (a + c)b + c|^2 = |(a + c)(1 + b)|^2$

$$= (a + c)^2(1 + b)^2 \le [2(a^2 + c^2)][2(1 + b^2)] = 4(3 - b^2)(1 + b^2)$$

$$\overset{(Cauchy)}{\leq} (3-b^2+1+b^2)^2 = 4^2 = 16 \Rightarrow |a+(a+c)+b| \leq 4 \text{. Proved.}$$

306. **If $0 \leq x \leq \frac{\sqrt{6}}{3}$ then:**

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2} \geq 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

We use the fundamental inequality

$$\sqrt{x^2+y^2}+\sqrt{a^2+b^2} \geq \sqrt{(x+a)^2+(y+b)^2} \quad (1)$$

Equality holds when $\frac{x}{a}=\frac{y}{b}$

$$2x^2+(\sqrt{2}-\sqrt{6})x+2 = 2\left(x^2+\left(\frac{\sqrt{2}-\sqrt{6}}{2}\right)x+1\right)$$

$$= 2\left(\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2+1-\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2\right) = 2\left[\left(x+\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2\right]$$

$$\Rightarrow \sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2} = \sqrt{2}\sqrt{\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2} \quad (2)$$

Similarly, $\sqrt{2x^2-(\sqrt{2}-\sqrt{6})x+2} = \sqrt{2}\sqrt{\left(x-\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}$

$$= \sqrt{2}\sqrt{\left(\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2} \quad (3)$$

Adding (2) & (3)

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2} =$$

$$= \sqrt{2}\left[\sqrt{\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2}+\sqrt{\left(\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}\right]$$

$$\overset{from\ (i)}{\geq} \sqrt{2}\sqrt{\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}+\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}$$

$$\geq\sqrt{2}\sqrt{\left(\frac{\sqrt{2}-\sqrt{6}+\sqrt{2}+\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+1+\sqrt{3}-1}{2\sqrt{2}}\right)^2}\geq\sqrt{2}\sqrt{\frac{1}{2}+\frac{3}{2}}\geq\sqrt{2}\times\sqrt{2}\geq 2$$

$$\therefore\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}-\sqrt{6})x+2}\geq 2\ \textit{(proved)}$$

Equality occurs when: $\frac{x+\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)}{\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)-x}=\frac{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}$ ***. From (i), equality holds when*** $\frac{x}{a}=\frac{y}{b}$

Solving, we get $x=\frac{\sqrt{6}}{3}$ $\therefore$ ***Equality holds when*** $x=\frac{\sqrt{6}}{3}$

Solution 2 by Serban George Florin-Romania

$$f:\left[0,\frac{\sqrt{6}}{3}\right]\to\mathbb{R}, f(x)=\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}$$

$$-2, f'(x)=\frac{4x+\sqrt{2}-\sqrt{6}}{2\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}}+\frac{4x-\sqrt{2}-\sqrt{6}}{2\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}}$$

$$f'(x)=0\Rightarrow(4x+\sqrt{2}-\sqrt{6})\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}=(\sqrt{6}+\sqrt{2}-4x)\cdot$$

$$\cdot\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}\ |^2, (4x+\sqrt{2}-\sqrt{6})^2(2x^2-(\sqrt{2}+\sqrt{6})x+2)=$$

$$=(\sqrt{6}+\sqrt{2}-4x)^2(2x^2+(\sqrt{2}-\sqrt{6})x+2)$$

$$[16x^2+8x(\sqrt{2}-\sqrt{6})+(8-4\sqrt{3})]\cdot[2x^2-(\sqrt{2}+\sqrt{6})x+2]=$$

$$=[16x^2-8x(\sqrt{6}+\sqrt{2})+8+4\sqrt{3}]\cdot(2x^2+(\sqrt{2}-\sqrt{6})x+2)$$

$$\Rightarrow 32x^4-16(\sqrt{2}+\sqrt{6})x^3+32x^2+16x^3(\sqrt{2}-\sqrt{6})-8x^2(-4)+$$

$$+16x(\sqrt{2}-\sqrt{6})+(16-8\sqrt{3})x^2-(4\sqrt{6}-4\sqrt{2})+16-8\sqrt{3}=$$

$$=32x^4+(16\sqrt{2}-16\sqrt{6})x^3+32x^2-16x^3(\sqrt{6}+\sqrt{2})-8x^2(2-6)$$

$$-16x(\sqrt{6}+\sqrt{2})+(16+8\sqrt{3})x^2+(-4\sqrt{2}-4\sqrt{6})x+16+8\sqrt{3},$$

$$\Rightarrow -16\sqrt{2}x^3-16\sqrt{6}x^3+16\sqrt{2}x^3-16\sqrt{6}x^3+16\sqrt{2}x-16\sqrt{6}x+16x^2$$

$$-8\sqrt{3}x^2-4\sqrt{6}x+4\sqrt{2}x-8\sqrt{3}=16\sqrt{2}x^3-16\sqrt{6}x^3-16\sqrt{6}x^3$$

$$-16\sqrt{2}x^3-16\sqrt{6}x-16\sqrt{2}x+16x^2+8\sqrt{3}x^2-4\sqrt{2}x-4\sqrt{6}x+8\sqrt{3}\Rightarrow$$

$$\Rightarrow -16\sqrt{3}x^2+32\sqrt{2}x+8\sqrt{2}x-16\sqrt{3}=0|:(-8)$$

$$2\sqrt{3}x^2-4\sqrt{2}x-\sqrt{2}x+2\sqrt{3}=0;\ 2\sqrt{3}x^2-5\sqrt{2}x+2\sqrt{3}=0,\Delta=2$$

$$x_1=\frac{5\sqrt{2}+\sqrt{2}}{4\sqrt{3}}=\frac{6\sqrt{2}}{4\sqrt{3}}=\frac{3\sqrt{2}}{2\sqrt{3}}=\frac{3\sqrt{6}}{6}=\frac{\sqrt{6}}{2};\ x_2=\frac{5\sqrt{2}-\sqrt{2}}{4\sqrt{3}}=\frac{4\sqrt{2}}{4\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$$

x	$-\infty$	0	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$	∞
$f'(x)$	$-$ $-$ $-$ $-$ $-$ $-$	$-$ $-$ $-$ $-$ $-$	0	0	
$f(x)$	∞ $\searrow$		$2(\sqrt{2}-1)$ $\searrow$	0	

$$f'(0)=-\sqrt{3}<0, f(0)=2\sqrt{2}-2=2(\sqrt{2}-1)>0$$

$$f\left(\frac{\sqrt{6}}{3}\right)=0\Rightarrow f:\left[0,\frac{\sqrt{6}}{3}\right]\to\mathbb{R}, f\searrow\Rightarrow f(x)\geq 0,(\forall)x\in\left[0,\frac{\sqrt{6}}{3}\right]$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}\overset{(1)}{\geq}2$$

$$\textbf{(1)}\Leftrightarrow 2x^2-(\sqrt{6}-\sqrt{2})x+2+2x^2-(\sqrt{6}+\sqrt{2})x+2+$$

$$+2\sqrt{(2x^2-(\sqrt{6}-\sqrt{2})x+2)(2x^2-(\sqrt{6}+\sqrt{2})x+2)}\geq 4\ \textit{(upon squaring)}$$

$$\Leftrightarrow 2\sqrt{\{2x^2-(\sqrt{6}-\sqrt{2})x+2\}\{2x^2-(\sqrt{6}+\sqrt{2})x+2\}}\overset{(2)}{\geq}2(\sqrt{6}x-2x^2)$$

$$\because x\leq\frac{\sqrt{6}}{3}<\frac{\sqrt{6}}{2}\therefore\sqrt{6}-2x>0\Rightarrow\sqrt{6}x-2x^2>0$$

$$\therefore\textbf{(2)}\Leftrightarrow(2x^2+2)^2-(2x^2+2)(\sqrt{6}+\sqrt{2})x-(2x^2+2)(\sqrt{6}-\sqrt{2})x+$$

$$+(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})x^2\geq 6x^2+4x^4-4\sqrt{6}x^3\ \textit{(upon squaring)}$$

$$\Leftrightarrow 4x^4+4+8x^2-(2x^2+2)(2\sqrt{6}x)+4x^2\geq 6x^2+4x^4-4\sqrt{6}x^3$$

$$\Leftrightarrow 6x^2+4-4\sqrt{6}x^3-4\sqrt{6}x\geq -4\sqrt{6}x^3$$

$$\Leftrightarrow 3x^2+2-2\sqrt{6}x\geq 0\Leftrightarrow(\sqrt{3}x-\sqrt{2})^2\geq 0\to\textit{true}\therefore\textit{(1) is true, with equality at}$$

$$x=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}\ \textit{(proved)}$$

307. If $a, b, c, d, e, f \geq 1$ then:

$$a + b + 2c + 2d + e + f \leq abc^2d^2ef + 7$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Omran Kouba-Damascus-Syria

Let $\mathcal{P}_n$ be the following property: $\forall (x_1, \dots, x_n) \in [1, +\infty)^n, \sum_{n=1}^{n} x_k \leq \prod_{k=1}^{n} x_k + n - 1$

We will prove that $\mathcal{P}_n$ holds true for every positive integer n by induction. Clearly, $\mathcal{P}_1$ is trivially true, and $\mathcal{P}_2$ follows from $(x_1 - 1)(x_2 - 1) \geq 0$. Now, suppose we have proved $\mathcal{P}_n$ and consider $(x_1, \dots, x_{n+1}) \in [1, +\infty)^{n+1}$.

$$\sum_{k=1}^{n+1} x_k = x_{n+1} + \sum_{k=1}^{n} x_k$$

$$\leq x_{n+1} + \prod_{k=1}^{n} x_k + n - 1 \quad \text{using } \mathcal{P}_n$$

$$\leq x_{n+1} \cdot \prod_{k=1}^{n} x_k + 1 + n - 1 \quad \text{using } \mathcal{P}_2$$

$$= \prod_{k=1}^{n+1} x_k + n$$

So, $\mathcal{P}_{n+1}$ is also true, and this completes the proof of $\mathcal{P}_n$ by induction for all $n \geq 1$.

Choosing some of the x_k's equal yields the following generalization:

Corollary. Let $x_1, \dots, x_n$ be real numbers greater or equal to 1, and let $m_1, \dots, m_n$ be positive integers, then:

$$\sum_{k=1}^{n} m_k x_k \leq \prod_{k=1}^{n} x_k^{m_k} + \sum_{k=1}^{n} m_k - 1$$

For example, with $(x_1, \dots, x_6) = (a, b, c, d, e, f)$ and $(m_1, \dots, m_6) = (1, 1, 2, 2, 1, 1)$ we get $a + b + 2c + 2d + e + f \leq abc^2d^2ef + 7$ for all $a, b, c, d, e, f \geq 1$

Solution 2 by Marian Ursărescu-Romania

Inequality $\Leftrightarrow$

$$abc^2d^2ef \geq a - 1 + b - 1 + 2(c - 1) + 2(d - 1) + e - 1 + f - 1 + 1 \quad (1)$$

Let $x_1 = a - 1, x_2 = b - 1, x_3 = c - 1, x_4 = d - 1, x_5 = e - 1, x_6 = f - 1$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

(1) becomes: $(x_1 + 1)(x_2 + 1)(x_3 + 1)^2(x_4 + 1)^2(x_5 + 1)(x_6 + 1) \geq$

$$\geq x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + 1 \Leftrightarrow$$

$$\Leftrightarrow (x_1 + 1)(x_2 + 1)(x_3^2 + 2x_3 + 1)(x_4^2 + 2x_4 + 1)(x_5 + 1)(x_6 + 1) \geq$$

$$\geq x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + 1$$

By brute force $\Rightarrow x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + 1 + E(x_1, x_2, x_3, x_4, x_5) \geq$

$\geq x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + 1 \Leftrightarrow E(x_1, x_2, x_3, x_4, x_5, x_6) \geq 0$ ***with equality for***

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 \Leftrightarrow a = b = c = d = e = f = 1$$

Observation: scientific method by induction.

308. **If** $x, y, z > 0, xyz(3x + 2y + 36z) = 6$ **then:**

$$\left(\frac{x^2y^2}{36} + 1\right)(4y^2z^2 + 1)(9z^2x^2 + 1) \geq 64x^4y^4z^4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

Let $a = 3x, b = 2y, c = 36z \Rightarrow x = \frac{a}{3}, y = \frac{b}{2}, z = \frac{c}{36} \Rightarrow abc(a + b + c) = 36^2$

Inequality $\Leftrightarrow (a^2b^2 + 36^2)(b^2c^2 + 36^2)(c^2a^2 + 36^2) \geq 64(abc)^4$

$\Leftrightarrow (ab + bc + ca + a^2)(ab + bc + ca + b^2)(ab + bc + ca + c^2) \geq 64(abc)^2$ ***(*)***

(Because: $36^2 = abc(a + b + c)$***)***

$ab + bc + ca + a^2 \overset{(Cauchy)}{\geq} 4\sqrt[4]{ab \cdot bc \cdot ca \cdot a^2} = 4a\sqrt[4]{(bc)^2}$ ***(1)***

Similarly: $ab + bc + ca + b^2 \geq 4b\sqrt[4]{(ac)^2}$ ***(2)***

$ab + bc + ca + c^2 \geq 4c\sqrt[4]{(ab)^2}$ ***(3)***

$\overset{(1).(2).(3)}{\Rightarrow} LHS_{(*)} \geq 4^3abc\sqrt[4]{(abc)^4} = 64(abc)^2 \Rightarrow$ ***Proved.***

Solution 2 by Michael Sterghiou-Greece

$xyz(3x + 2y + 36z) = 6$ ***(c), then:*** $\left(\frac{x^2y^2}{36} + 1\right)(4y^2z^2 + 1)(9z^2x^2 + 1) \geq 64x^4y^4z^4$ ***(1)***

Let $x = 2a, y = 3b, z = \frac{c}{6}: xyz = abc$***. From (c)*** $6 \overset{AM-GM}{\geq} xyz \cdot 3\sqrt[3]{6 \cdot 36xyz}$

or $xyz = abc \leq \left(\frac{1}{3}\right)^{\frac{3}{4}}$***. Also, (c)***$\rightarrow abc(a + b + c) = 1$ ***(2). Let***

$$(p,q,r) = \left(\sum_{cyc} a;\sum_{cyc} ab, abc\right).$$

We have $p\cdot r = 1, r \le \left(\frac{1}{3}\right)^{\frac{3}{4}}, q^2 \ge 3pr = 3 \to q \ge \sqrt{3}$

Now, (2) reduces to:

$\prod_{cyc}(a^2b^2+1) \ge 64r^4$ *or* $a^4b^4c^4 + r^2\left(\sum_{cyc} a^2\right) + \left(\sum_{cyc} a^2b^2\right)$ *(3)*

$+1 \ge 64r^2$*. But* $\sum_{cyc} a^2 = p^2 - 2q, \sum_{cyc} a^2b^2 = q^2 - 2pr$ *hence (3) reduces to*

$q^2 - 2qr^2 - 63r^4 \ge 0$ *because* $pr = 1$*. It suffices to prove the stronger inequality*

$q^2 - 2\cdot q\cdot\left(\frac{1}{3}\right)^{\frac{6}{4}} - 63\cdot\left(\frac{1}{3}\right)^{\frac{12}{4}} \ge 0$ *or* $q^2 - 2q\left(\frac{1}{3}\right)^{\frac{3}{2}} - \frac{63}{27} \ge 0$ *for* $q \ge \sqrt{3}$ *or*

$\frac{1}{9}(q-\sqrt{3})(9q+7\sqrt{3}) \ge 0$ *which holds. Done!*

Solution 3 by Le Van-Hanoi-Vietnam

Let $a = 3x, b = 2y; c = 36z$ *and* $k = 1296$ *the problem becomes:*

Given $a,b,c > 0$ *such that* $abc(a+b+c) = k$*, prove:*

$$\left(\frac{a^2b^2}{k}+1\right)\left(\frac{b^2c^2}{k}+1\right)\left(\frac{c^2a^2}{k}+1\right) \ge \frac{64}{k^3}a^4b^4c^4 \quad (*)$$

Indeed, LHS (*) $= \prod\left(\frac{a^2b^2+abc(a+b+c)}{k}\right) = \frac{1}{k^3}\prod[ab(b+c)(c+a)]$

$= \frac{1}{k^3}a^2b^2c^2[(a+b)(b+c)(c+a)]^2$*. Accordingly, that is enough to prove that:*

$$[(a+b)(b+c)(c+a)]^2 \ge 64a^2b^2c^2 \Rightarrow (a+b)(b+c)(c+a) \ge 8abc \quad (**)$$

Indeed, LHS (**) – RHS (**)$= a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \ge 0$

Q.E.D. Equality holds when $a = b = c = 2\sqrt[4]{27}$ *in other words,*

$$(x,y,z) = \left(\frac{2}{\sqrt[4]{3}}, \sqrt[4]{27}, \frac{1}{6\sqrt[4]{3}}\right)$$

309. **If $x,y \in \mathbb{R}$ then:**

$$(x^3+2y^3-3xy^2)^2 \le (x^2+2y^2)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Van Nho-Nghe An-Vietnam

Case 1: $y = 0$ *then* $LHS = RHS = x^6 \to$ *true*

Case 2: $y \neq 0$***, put*** $x = ty, t \in \mathbb{R}$***. The inequality becomes:***

$$\left(y^3(t^3+2-3t)\right)^2 \leq \left(y^2(t^2+2)\right)^3 \leftrightarrow (t^3+2-3t^2) \leq (t^2+2)^3$$

$$\leftrightarrow 12t^4-4t^3+3t^2+12t+8 \geq 0$$

$$\leftrightarrow (2t^2-t)^2+6t^2+2t^4+2t^2+8+12t \geq 0 \rightarrow (*)$$

Using Cauchy's inequality: $2t^4+2t^2+8 \geq 12\sqrt[12]{t^{12}} = 12|t|$

and $|t|+t \geq 0, \forall t$ ***so, (*) is true. Done.***

Solution 2 by Tran Hong-Vietnam

$$(x^3+2y^3-3xy^2)^2 = x^6+4y^6+9x^2y^4+4x^3y^3-6y^2x^4-12xy^5$$

$$(x^2+2y^2)^3 = x^6+8y^6+6x^4y^2+12x^2y^4.$$ ***Must show that:***

$$x^6+4y^6+9x^2y^4+4x^3y^3-6y^2x^4-12xy^5 \leq x^6+8y^6+6x^4y^2+12x^2y^4$$

$$\Leftrightarrow 2y^6+12x^4y^2+3x^2y^4+12xy^5-4x^3y^3 \geq 0$$

$$\Leftrightarrow y^2(2y^4+12x^4+3x^2y^2+12xy^3-4yx^3) \geq 0$$

$$\Leftrightarrow y^2(2x+y)^2(3x^2-4xy+4y^2) \geq 0$$

It is true, because: $y^2 \geq 0;\ (2x+y)^2 \geq 0;\ 3x^2-4xy+4y^2 = 2x^2+(2y-x)^2 \geq 0$

Solution 3 by Jamal Gadirov-Azerbaijan

$$x^6+4y^6+9x^2y^4+4x^3y^3-6x^4y^2-12y^5x \leq x^6+8y^6+3x^2\cdot 2y^2(x^2+2y^2)$$

$$= x^6+8y^6+6x^4y^2+12x^2y^4+4y^6+12x^4y^2+3x^2y^4$$

$$4x^3y^3 \leq 12y^5x+4y^6+12x^4y^2+3x^2y^4$$

$$4 \leq 12\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)^3+12\left(\frac{x}{y}\right)+3\left(\frac{y}{x}\right);\ 4 \leq 12\omega^2+4\omega^3+\frac{12}{\omega}+3\omega$$

$$4\omega \leq 12\omega^3+4\omega^4+12+3\omega^2 l\ 4\omega^4+12\omega^3+3\omega^2-4\omega+12 \overset{?}{\geq} 0$$

Factorizing, gives: $(\omega+2)^2(4\omega^2-4\omega+3) \geq 0$

Here $\omega = \frac{y}{x}$ ***and W.L.O.G we have assumed*** $x \neq 0$***.***

310. $a, b, c, d \geq 0, p \geq q \geq r \geq 0$

$$x = \frac{a+b+c+d}{4} - \sqrt[4]{abcd},\ y = \frac{a+b+c}{3} - \sqrt[3]{abc},\ z = \frac{a+b}{2} - \sqrt{ab}$$

Prove that:

$$3(4px + 3qy + 2rz) \ge (4x + 3y + 2z)(p + q + r)$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Vietnam

We prove that: $4x \ge 3y \ge 2z \ge 0$

$$\because 4x \ge 3y \Leftrightarrow a + b + c + d - \sqrt[4]{abcd} \ge a + b + c - \sqrt[3]{abc} \Leftrightarrow d + 3\sqrt[3]{abc} \ge 4\sqrt[4]{abcd}$$

It is true because:

$$d + 3\sqrt[3]{abc} = d + \sqrt[3]{abc} + \sqrt[3]{abc} + \sqrt[3]{abc} \overset{AM-GM}{\ge} 4\sqrt[4]{d\sqrt[3]{[abc]^3}} = 4\sqrt[4]{abcd}$$

$$\because 3y \ge 2z \Leftrightarrow a + b + c - 3\sqrt[3]{abc} \ge a + b - 2\sqrt{ab} \Leftrightarrow c + 2\sqrt{ab} \ge 3\sqrt[3]{abc}$$

It is true because: $c + 2\sqrt{ab} = c + \sqrt{ab} + \sqrt{ab} \ge 3\sqrt[3]{c\sqrt{(ab)^2}} = 3\sqrt[3]{abc}$

$\because 2z \ge 0 \Leftrightarrow z \ge 0 \Leftrightarrow a + b \ge 2\sqrt{ab}$ ***(true). Similarly:*** $3y, 4x \ge 0$.

Hence: $4x \ge 3y \ge 2z \ge 0$. ***More,*** $p \ge q \ge r \ge 0$ ***then using Chebyshev's inequality:***

$$4xp + 3yq + 2zr \ge \frac{1}{3}(4x + 3y + 2z)(p + q + r)$$

$$\Leftrightarrow 3(4px + 3qy + 2rz) \ge (4x + 3y + 2z)(p + q + r).$$ ***Proved***

311. GENERALIZATION FOR HUNG NGUYEN VIET'S INEQUALITY

If $a, b, c, x, y, z > 0$, $a^3x + b^3y + c^3z = xyz$ then:

$$x + y + z \ge (a + b + c)\sqrt{a + b + c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum x \ge \sum a\sqrt{\sum a}$$

$$a^3x + b^3y + c^3z = xyz\left(\frac{a^3}{yz} + \frac{b^3}{zx} + \frac{c^3}{xy}\right)$$

$$\overset{Holder}{\ge} xyz\frac{(\sum a)^3}{3\sum xy} \ge xyz\frac{(\sum a)^3}{(\sum x)^2} \quad \left(\because 3\sum x \le \left(\sum x\right)^2\right)$$

$$\Rightarrow xyz \geq xyz\frac{(\sum a)^3}{(\sum x)^2} \ (\because xyz = a^3x + b^3y + c^3z)$$

$$\Rightarrow (\sum x)^2 \geq (\sum a)^3 \Rightarrow \sum x \geq (\sum a)\sqrt{\sum a} \text{ (proved)}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$x + y + z \overset{?}{\geq} (a + b + c)\sqrt{a + b + c}$. *Suppose:* $T_1 = \frac{x+y+z}{3}$ *and* $T_2 = \frac{a+b+c}{3}$. *So:*

$$3T_1 \overset{?}{\geq} 3 \cdot T_2 \cdot \sqrt{3T_2};\ T_1 \overset{?}{\geq} T_2 \cdot \sqrt{3} \cdot \sqrt{T_2};\ T_1^2 \overset{?}{\geq} 3T_2^2 \cdot T_2;\ T_1^2 \overset{?}{\geq} 3T_2^3$$

$$3T_2^3 - T_1^2 \leq 0;\ 3\left(\frac{(a + b + c)^3}{27}\right) \overset{?}{\leq} \frac{(x + y + z)^2}{2}$$

Let us prove: $(a + b + c)^3 \overset{?}{\leq} (x + y + z)^2$

From assumption: $a^3x + b^3y + c^3z = xyz$ *divided by* x

$\frac{a^3}{yz} + \frac{b^3}{xz} + \frac{c^3}{xy} = 1$ *by using Hölder's inequality:*

$$1 \geq \frac{(a + b + c)^3}{(x + y + z)^2} \Rightarrow (x + y + z)^2 \geq (a + b + c)^3$$

Solution 3 by Tran Hong-Vietnam

Let $z = kx, y = mx\ (k, m > 0)$

$$\Rightarrow a^3x + b^3(mx) + c^3(kx) = x(mx)(kx) \Leftrightarrow a^3 + mb^3 + kc^3 = mkx^2$$

$$\Rightarrow x = \frac{1}{\sqrt{mk}}\sqrt{a^3 + mb^3 + kc^3};\ y = \sqrt{\frac{m}{k}} \cdot \sqrt{a^3 + mb^3 + kc^3}$$

$$z = \sqrt{\frac{k}{m}} \cdot \sqrt{a^3 + mb^3 + kc^3};\ x + y + z \geq (a + b + c)\sqrt{a + b + c}$$

$$\Leftrightarrow \left(\frac{1}{\sqrt{mk}} + \sqrt{\frac{m}{k}} + \sqrt{\frac{k}{m}}\right)\sqrt{a^3 + mb^3 + kc^3} \geq (a + b + c)\sqrt{a + b + c}$$

$$\Leftrightarrow \left(\frac{1 + m + k}{\sqrt{mk}}\right)^2 (a^3 + mb^3 + kc^3) \geq (a + b + c)^3$$

$$\Leftrightarrow a^3 + mb^3 + kc^3 \geq \frac{mk(a + b + c)^3}{(1 + m + k)^2}$$

$$\frac{a^3}{1}+\frac{b^3}{\frac{1}{m}}+\frac{c^3}{\frac{1}{k}}\overset{Holder}{\geq}\frac{1}{3}\cdot\frac{(a+b+c)^3}{1+\frac{1}{m}+\frac{1}{k}}=\frac{mk}{3}\cdot\frac{(a+b+c)^3}{mk+m+k}$$

Must show that: $\frac{1}{3(mk+m+k)}\geq\frac{1}{(1+m+k)^2}$

$$\Leftrightarrow(1+m+k)^2\geq 3(mk+k+m)\Leftrightarrow 1+m^2+k^2\geq m+k+mk$$

It is true. Proved.

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c,x,y,z>0$ *and* $a^3x+b^2y+c^3z=xyz$

We have $1=\frac{a^3}{yz}+\frac{b^3}{zx}+\frac{c^3}{xy}\geq\frac{(a+b+c)^3}{3(xy+yz+zx)}$

Hence $3(xy+yz+zx)\geq(a+b+c)^3$. *Hence* $(x+y+z)^2\geq(a+b+c)^3$

Hence $x+y+z\geq(a+b+c)\sqrt{(a+b+c)}$

312. If $a,b,c,d\in\mathbb{R}$ then:

$$(2a+3b+4c+5d)^2\geq 8(3ab+5ad+6bc+10cd)$$

Proposed by Marian Ursărescu – Romania

Solution by Lahiru Samarakoon-Sri Lanka

$$LHS-RHS$$

$$4a^2+9b^2+16c^2+25d^2+12ab+16ac+2ad+24bc+30bd+40cd$$

$$-(24ab+40ad+48bc+80cd)$$

$$\Rightarrow 4a^2+9b^2+16c^2+25d^2+12ab+16ac-20ad-24bc+30bd-40cd$$

$$\underbrace{(2a-3b+4c-5d)^2}_{\geq 0}$$

$LHS-RHS\geq 0$; $LHS>RHS$; *(proved)*

313. If $a,b,c>0, a+b+c+2=abc$ then:

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\geq 2$$

Proposed by Vadim Mitrofanov-Kiev-Ukraine

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{a^2}{ab+a} \overset{Bergström}{\geq} \frac{(\sum a)^2}{\sum ab + \sum a} \geq \frac{(\sum a)^2}{\frac{(\sum a)^2}{3} + \sum a} \left(\because \sum ab \leq \frac{(\sum a)^2}{3}\right)$$

$$= \frac{3x}{x+3}\left(x = \sum a\right) \overset{?}{\geq} 2 \Leftrightarrow x \underset{(1)}{\overset{?}{\geq}} 6. \text{ Now, } x = \sum \underset{(2)}{\overset{A-G}{\geq}} 3\sqrt[3]{abc} = 3\sqrt[3]{t} \ (t = abc)$$

(1), (2) $\Rightarrow$ *it suffices to prove:* $3\sqrt[3]{t} \geq 6 \Leftrightarrow t \overset{(3)}{\geq} 8$

Now, $abc - 2 = \sum a \overset{A-G}{\geq} 3\sqrt[3]{abc} \Rightarrow t - 2 \geq 3\sqrt[3]{t} \Rightarrow (t-2)^3 - 27t \geq 0$

$\Rightarrow (t-8)(t+1)^2 \geq 0 \Rightarrow t \geq 8 \Rightarrow$ *(3) is true (proved)*

Solution 2 by Marian Ursărescu-Romania

$a + 1 = x, b + 1 = y, c + 1 = z.$ *Then* $a + b + c + 2 = abc \Rightarrow$

$$x - 1 + y - 1 + z - 1 + 2 = (x-1)(y-1)(z-1) \Rightarrow$$

$$x + y + z - 1 = xyz + x + y + z - xy - xz - yt - 1 \Rightarrow$$

$$xyz = xy + xz + yz \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \quad (1)$$

Inequalities becomes: $\frac{x-1}{y} + \frac{y-1}{z} + \frac{z-1}{x} \geq 2 \Leftrightarrow \frac{x}{y} - \frac{1}{y} + \frac{y}{z} - \frac{1}{z} + \frac{z}{x} - \frac{1}{x} \geq 2 \Leftrightarrow$ *use (1)* $\Leftrightarrow$

$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ *(true because:* $\frac{x}{y} + \frac{y}{z} + \frac{t}{x} \geq 3\sqrt[3]{\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = 3)$

314. If $a, b, c \geq 0$ then:

$$12 + \sum (a^8 + 1)\left(\frac{1}{b^4+1} + \frac{1}{c^4+1}\right) \geq 12\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow \sum (a^8 + 1)\left(\frac{1}{b^4+1} + \frac{1}{c^4+1}\right) \overset{(1)}{\geq} 12(\sqrt{2} - 1)$

LHS of (1) $= \sum_{cyc} \frac{a^8+1}{b^4+1} + \sum_{cyc} \frac{b^8+1}{a^4+1} = \sum_{cyc}\left(\frac{a^8+1}{b^4+1} + \frac{b^8+1}{a^4+1}\right) \underset{(2)}{\overset{A-G}{\geq}} 2\sum_{cyc} \sqrt{\frac{a^8+1}{a^4+1} \cdot \frac{b^8+1}{b^4+1}}$

Let $f(x)=\frac{x^8+1}{x^4+1}\ \forall x\ge 0$. *We have* $f'(x)=\frac{4x^3(x^8+2x^4-1)}{(x^4+1)^2}$

and $f''(x)=\frac{4x^2(3x^{12}+9x^8+19x^4-3)}{(x^4+1)^3}$, $f'(x)=0$ *iff* $x=0$ *or* $x=\sqrt[4]{\sqrt{2}-1}$

$f''(0)=0$ *with* $f(0)=1$ *and* $f''\left(\sqrt[4]{\sqrt{2}-1}\right)>0$ *with* $f\left(\sqrt[4]{\sqrt{2}-1}\right)=2(\sqrt{2}-1)$

$\therefore f(x)\forall x\ge 0$ *attains its minimum at* $x=\sqrt[4]{\sqrt{2}-1}$ *and* $f_{\min}\overset{(3)}{=}2(\sqrt{2}-1)$

(2), (3) $\Rightarrow LHS\ge 6\sqrt{\left(2(\sqrt{2}-1)\right)^2}=12(\sqrt{2}-1)\Rightarrow$ *(1) is true (proved)*

315. **For $a,b,c\in(0;+\infty)\wedge abc=1$. Prove:**

$$\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}\le 2(a^4+b^4+c^4).$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Christos Eythimiou-Greece

$$a,b,c>0\wedge abc=1\Rightarrow\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}\le 2(a^4+b^4+c^4)$$

$$a,b,c>0\wedge abc=1\Rightarrow\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}=\frac{a^2+b^2}{\frac{1}{ab}}+\frac{b^2+c^2}{\frac{1}{bc}}+\frac{c^2+a^2}{\frac{1}{ca}}$$

$$=\sqrt[4]{(a^4)^3b^4}+\sqrt[4]{a^4(b^4)^3}+\sqrt[4]{(b^4)^3c^4}+\sqrt[4]{b^4(c^4)^3}+\sqrt[4]{(c^4)^3a^4}+\sqrt[4]{c^4(a^4)^3}\le$$

$$\frac{3a^4+b^4}{4}+\frac{a^4+3b^4}{4}+\frac{3b^4+c^4}{4}+\frac{b^4+3c^4}{4}+\frac{3c^4+a^4}{4}+\frac{c^4+3a^4}{4}=2(a^4+b^4+c^4)$$

Solution 2 by Catinca Alexandru-Romania

$$LHS=a^2\left(\frac{1}{c}+\frac{1}{b}\right)+b^2\left(\frac{1}{a}+\frac{1}{c}\right)+c^2\left(\frac{1}{a}+\frac{1}{b}\right)=$$

$$=a^2\cdot\frac{b+c}{bc}+b^2\cdot\frac{a+c}{ac}+c^2\cdot\frac{a+b}{ab}\overset{(abc=1)}{=}$$

$$=a^3(b+c)+b^3(a+c)+c^3(a+b)=$$

$$=(a^3b+b^3c+c^3a)+(a^3c+c^3b+b^3a)\overset{Rearangements(Muirhead)}{\le}$$

$$\le(a^4+b^4+c^4)+(a^4+b^4+c^4)=2(a^4+b^4+c^4)=RHS$$

Solution 3 by Marian Ursărescu-Romania

$abc=1$ ***we show this:*** $(a^2+b^2)ab+(b^2+c^2)bc+(c^2+a^2)ca\le 2(a^4+b^4+c^4)$

But $a^4+b^4\ge ab(a^2+b^2)$ ***(1)*** $\Leftrightarrow a^4-a^3b+b^4-ab^3\ge 0\Leftrightarrow$

$a^3(a-b)+b^3(b-a)\ge 0\Leftrightarrow (a-b)(a^3-b^3)\ge 0\Leftrightarrow (a-b)^2(a^2+ab+b^2)\ge 0$

true. From (1) $\Rightarrow \left.\begin{matrix} a^4+b^4\ge ab(a^2+b^2)\\ b^2+c^4\ge bc(b^2+c^2)\\ c^4+a^4\ge ac(a^2+c^2)\end{matrix}\right\}\Rightarrow$

$2(a^4+b^4+c^4)\ge ab(a^2+b^2)+bc(b^2+c^2)+ac(a^2+c^2)$

316. **If** $a,b,c>0, a+b+c=3$ **then:**

$$5(a^4+b^4+c^4)\ge 12+a^5+b^5+c^5$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=5x^4-4-x^4$ ***for all*** $x\in(0,3)$

$f'(x)=20x^3-5x^4, f''(x)=60x^2-20x^3=20x^2(3-x)\ge 0 \because x\in(0,3)$

f ***is convex hence*** $\sum_{cyc} f(a)\ge 3f\left(\frac{a+b+c}{3}\right)=0\Rightarrow 5\sum_{cyc}a^4\ge 12+\sum_{cyc}a^5$ ***(proved)***

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a,b,c>0: a+b+c=3$ ***then*** $5(a^4+b^4+c^4)\ge 12+a^5+b^5+c^5$

We have $5a^4-a^5\ge 15a-11\ \forall 0<a<3$ ***(1)***

It is true since (1) $\Rightarrow (a-1)^2(a^3-3a^2-7a-11)\le 0\Rightarrow$

$\Rightarrow (a-1)^2[a^2(a-3)-7a-11]\le 0$

(True since $(a-1)^2\ge 0$ ***and*** $a^2(a-3)-7a-11<-11<0$)

Similalry, we have $5b^4-b^5\ge 15b-11\ \forall 0<b<3$ ***(2) and***

$5c^4-c^5\ge 15c-11\ \forall 0<c<3$ ***(3)***

(1), (2) and (3) $\Rightarrow 5(a^4+b^4+c^4)-(a^5+b^5+c^5)\ge 15(a+b+c)-33\Rightarrow$

$\Rightarrow 5(a^4+b^4+c^4)-(a^5+b^5+c^5)\ge 12\Rightarrow 5(a^4+b^4+c^4)\ge 12+a^5+b^5+c^5$

The equality occurs when $a=b=c=1$.

317. If $a,b,c>0$ then:

$$\left(7+a^3+b^3+c^3\right)\left(7+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\geq 100$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Madan Mastermind-Varanasi-India

$$\left(7+a^3+b^3+c^3\right)\cdot\left(7+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\geq 100$$

$\frac{a^3+b^3+c^3}{3}\geq abc$ so, $a^3+b^3+c^3=3abc$; $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{3}{abc}$, put $abc=t$

$(7+3t)\cdot\left(7+\frac{3}{t}\right)=f(t)$. Solving $f(t)=\frac{21t^2+58t+21}{t}$ finding $\min f(t)$

$f'(t)=2\left|t^2-2\right|=0; t^2=1$, so $t=\pm 1$; $f''(t)=42t \quad t=1$

$f''(1)>0$ so at $t=1$ $f(t)$ is min. $f(1)=100$ so, $f(t)\geq 100$.

Solution 2 by Abdallah El Farissi-Bechar-Algerie

$$\left(7+a^3+b^3+c^3\right)\left(7+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=$$

$$=49+7\left(a^3+\frac{1}{a^3}+b^3+\frac{1}{b^3}+c^3+\frac{1}{c^3}\right)+\left(a^3+b^3+c^3\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

$$\geq 49+42+9=100$$

318. For $a,b,c>0 \wedge ab+bc+ca=3abc$. Prove:

$$\frac{\sqrt{ab}}{\left(\sqrt{a}+\sqrt{b}\right)^4}+\frac{\sqrt{bc}}{\left(\sqrt{b}+\sqrt{c}\right)^3}+\frac{\sqrt{ca}}{\left(\sqrt{c}+\sqrt{a}\right)^4}\leq\frac{3}{16}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Marian Ursarescu-Romania

$$ab+bc+ca=3abc \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3.$$

Let $\frac{1}{\sqrt{a}}=x, \frac{1}{\sqrt{b}}=y, \frac{1}{\sqrt{c}}=z$ with $x,y,z>0 \wedge x^2+y^2+z^2=3$

With this notation the inequality becomes: $\sum\frac{\frac{1}{xy}}{\left(\frac{1}{x}+\frac{1}{y}\right)^4}\leq\frac{3}{16} \Leftrightarrow \sum\frac{x^3y^3}{(x+y)^4}\leq\frac{3}{16}$ (1)

But $x+y\ge 2\sqrt{xy}\Rightarrow (x+y)^4\ge 16x^2y^2\Rightarrow \frac{1}{(x+y)^4}\le\frac{1}{16x^2y^2}\Rightarrow\frac{x^3y^3}{(x+y)^4}\le\frac{1}{16}xy$ *(2)*

From (1) + (2) the inequality becomes: $\frac{1}{16}\sum xy\le\frac{3}{16}\Leftrightarrow\sum xy\le 3$ *(3)*

But $\sum xy\le\sum x^2=3$ *(4) (from hypothesis) .From (3) + (4) = 1 the inequality is true.*

319. If a,b,c are positive real numbers such that $a+b+c=3$, then:

$$\frac{ab^2}{\sqrt{b^2+bc+c^2}}+\frac{bc^2}{\sqrt{c^2+ca+a^2}}+\frac{ca^2}{\sqrt{a^2+ab+b^2}}+\frac{\sqrt{3}}{4}(a^2+b^2+c^2)\ge\frac{7\sqrt{3}}{4}$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

Solution 1 by Christos Eythimiou-Greece

$$a,b,c>0\wedge a+b+c=3\Rightarrow\frac{ab^2}{\sqrt{b^2+bc+c^2}}+\frac{bc^2}{\sqrt{c^2+ca+a^2}}+\frac{ca^2}{\sqrt{a^2+ab+b^2}}+\frac{\sqrt{3}}{4}(a^2+b^2+c^2)=$$

$$\frac{a^2b^2}{a\sqrt{b^2+bc+c^2}}+\frac{b^2c^2}{b\sqrt{c^2+ca+a^2}}+\frac{c^2a^2}{c\sqrt{a^2+ab+b^2}}+\frac{\sqrt{3}}{4}\left((a+b+c)^2-2(ab+bc+ca)\right)\ge$$

$$\frac{(ab+bc+ca)^2}{\sqrt{a}\sqrt{ab^2+abc+ac^2}+\sqrt{b}\sqrt{bc^2+bca+ba^2}+\sqrt{c}\sqrt{ca^2+cab+cb^2}}+\frac{\sqrt{3}}{4}\left(3^2-2(ab+bc+ca)\right)\ge$$

$$\frac{(ab+bc+ca)^2}{\sqrt{a+b+c}\sqrt{ab^2+abc+ac^2+bc^2+bca+ba^2+ca^2+cab+cb^2}}+\frac{\sqrt{3}}{4}\left(9-2(ab+bc+ca)\right)=$$

$$\frac{(ab+bc+ca)^2}{\sqrt{3}\sqrt{(a+b+c)(ab+bc+ca)}}+\frac{9\sqrt{3}}{4}-\frac{\sqrt{3}}{2}(ab+bc+ca)=\frac{\left(\sqrt{ab+bc+ca}\right)^4}{\sqrt{3}\sqrt{3(ab+bc+ca)}}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{9\sqrt{3}}{4}=$$

$$\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}+\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{7\sqrt{3}}{4}\ge$$

$$3\sqrt[3]{\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}\cdot\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}\cdot\frac{\sqrt{3}}{2}}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{7\sqrt{3}}{4}=\frac{7\sqrt{3}}{4}$$

Solution 2 by Nguyen Duc Viet-Vietnam

By the CBS inequality, we have

$$\sum\frac{ab^2}{\sqrt{b^2+bc+c^2}}=\sum\frac{a^2b^2}{a\sqrt{b^2+bc+c^2}}\ge\frac{(ab+bc+ca)^2}{a\sqrt{b^2+bc+c^2}+b\sqrt{c^2+ca+a^2}+c\sqrt{a^2+ab+b^2}}$$

$$\ge\frac{(ab+bc+ca)^2}{\sqrt{(a+b+c)[a(b^2+bc+c^2)+b(c^2+ca+a^2)+c(a^2+ab+b^2)]}}=\sqrt{\frac{(ab+bc+ca)^3}{9}}$$

Let $\sqrt{ab+bc+ca}=x$ *then* $a^2+b^2+c^2=9-2x^2$

We will prove that: $\frac{x^3}{3}+\frac{\sqrt{3}}{4}(9-2x^2)\ge\frac{7\sqrt{3}}{4}\Leftrightarrow\left(x-\sqrt{3}\right)^2\left(x+\frac{2\sqrt{3}}{4}\right)\ge 0$ *(true)*

The equality holds for $a=b=c=1$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{ab^2}{\sqrt{b^2+bc+c^2}}=\sum\frac{a^2b^2}{a\sqrt{b^2+bc+c^2}}\overset{Bergström}{\ge}\frac{(\sum ab)^2}{\sum a\sqrt{b^2+bc+c^2}}$$

$$=\frac{(\sum ab)^2}{\sum\sqrt{a}\sqrt{ab^2+abc+ac^2}}\overset{CBS}{\ge}\frac{(\sum ab)^2}{\sqrt{\sum a}\sqrt{\sum a^2b+\sum ab^2+3abc}}$$

$$=\frac{(\sum ab)^2}{\sqrt{\sum a}\sqrt{(\sum a)(\sum ab)}}=\frac{(\sum ab)^2}{(\sum a)\sqrt{ab}}\Rightarrow LHS\ge\frac{(\sum ab)^2}{(\sum a)\sqrt{\sum ab}}+\frac{\sqrt{3}}{4}\sum a^2$$

$$\overset{?}{\ge}\frac{7\sqrt{3}}{4}=\frac{7\sqrt{3}}{36}\left(\sum a^2+2\sum ab\right)\Leftrightarrow\frac{y^2}{\sqrt{3y(x+2y)}}+\frac{x}{4}\overset{?}{\ge}\frac{7}{36}(x+2y)$$ *(where* $x=\sum a^2, y=\sum ab$*)*

$$\Leftrightarrow\frac{y^2}{\sqrt{3y(x+2y)}}\overset{?}{\ge}\frac{7y-x}{18}\rightarrow (1)$$

If $7y\le x$, *RHS of (1)* ≤ 0 *and* $\because LHS>0\ \therefore LHS$ *of (1)* $>RHS$ *of (1)*

Now, let us consider the case when $7y>x$, *i.e.,* $\frac{y}{x}\in\left(\frac{1}{7},1\right]$

Then, $\frac{y^2}{\sqrt{3y(x+2y)}}\overset{?}{\ge}\frac{7y-x}{18}\Leftrightarrow\frac{y^4}{3y(x+2y)}\overset{?}{\ge}\frac{(7y-x)^2}{324}\Leftrightarrow 10t^3-21t^2+12t-1\overset{?}{\ge}0$

(where $t=\frac{y}{x}\in\left(\frac{1}{7},1\right]$*)* $\Leftrightarrow(10t-1)(t-1)^2\overset{?}{\ge}0\Leftrightarrow\left(t-\frac{1}{10}\right)(t-1)^2\overset{?}{\ge}0\rightarrow$ *true*

$\because t>\frac{1}{7}>\frac{1}{10}$. *Hence, (1) is true with equality occuring when* $t=1\Leftrightarrow\sum a^2=\sum ab$

$\Leftrightarrow a=b=c=1$ *(proved)*

320. If $a,b,c>0, abc=1$ then:

$$\sqrt{\frac{a^5+b^5}{a^2+b^2}}+\sqrt{\frac{b^5+c^5}{b^2+c^2}}+\sqrt{\frac{c^5+a^5}{c^2+a^2}}\ge 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Abdallah El Farisi-Bechar-Algerie

We have $f(x)=x^3$ *is convexe function then*

$$\sqrt{\frac{a^2a^3+b^2b^3}{a^2+b^2}}\geq\left(\frac{a^3+b^3}{a^2+b^2}\right)^{\frac{3}{2}}=\left((a+b)\left(1-\frac{ab}{a^2+b^2}\right)\right)^{\frac{3}{2}}\geq\left(\frac{a+b}{2}\right)^{\frac{3}{2}}\geq(ab)^{\frac{3}{4}}$$

$$\text{then}\sum\sqrt{\frac{a^5+b^5}{a^2+b^2}}\geq\sum(ab)^{\frac{3}{4}}\geq 3(abc)^{\frac{1}{2}}=3$$

Solution 2 by Boris Colakovic-Belgrade-Serbia

Assure $a\geq b\geq c$

$$a^5+b^5=a^2\cdot a^3+b^2\cdot b^3\overset{Chebishev}{\geq}\frac{1}{2}(a^2+b^2)(a^3+b^3)\Rightarrow\frac{a^5+b^5}{a^2+b^2}\geq\frac{1}{2}(a^3+b^3)\Leftrightarrow$$

$$\Leftrightarrow\sqrt{\frac{a^5+b^5}{a^2+b^2}}\geq\frac{1}{\sqrt{2}}\cdot\sqrt{a^3+b^3}\overset{AM-GM}{\geq}\frac{1}{2}\cdot\sqrt{2}\sqrt[4]{(ab)^3}=\sqrt[4]{(ab)^3}$$

$$\text{Similalry}\sqrt{\frac{b^5+c^5}{b^2+c^2}}\geq\sqrt[4]{(bc)^3};\sqrt{\frac{c^5+a^5}{c^2+a^2}}\geq\sqrt[4]{(ca)^3}$$

$$LHS\geq\sqrt[4]{(ab)^3}+\sqrt[4]{(bc)^3}+\sqrt[4]{(ca)^3}\geq 3\sqrt[12]{(abc)^6}=3\sqrt{\underbrace{abc}_{=1}}=3$$

Sign „=" holds for $a=b=c=1.$

Solution 3 by Lazaros Zachariades-Thessaloniki-Greece

$$\left(\frac{a^5+b^5}{2}\right)^{\frac{1}{5}}\geq\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}\Leftrightarrow\frac{(a^5+b^5)^2}{2^2}\geq\frac{(a^2+b^2)^5}{2^2\cdot 2^3}\Leftrightarrow\left(\frac{a^5+b^5}{a^2+b^2}\right)^2\geq\left(\frac{a^2+b^2}{2}\right)^3$$

$$\text{Likewise}\left(\frac{b^5+c^5}{b^2+c^2}\right)^2\geq\left(\frac{b^2+c^2}{2}\right)^3,\left(\frac{c^5+a^5}{c^2+a^2}\right)^2\geq\left(\frac{c^2+a^2}{2}\right)^3$$

$$LHS\geq\left(\frac{a^2+b^2}{2}\right)^{\frac{3}{4}}+\left(\frac{b^2+c^2}{2}\right)^{\frac{3}{4}}+\left(\frac{c^2+a^2}{2}\right)^{\frac{3}{4}}\geq 3\sqrt[3]{\frac{(a^2+b^2)^{\frac{3}{4}}(b^2+c^2)^{\frac{3}{4}}(c^2+a^2)^{\frac{3}{4}}}{2^{\frac{3}{4}}\cdot 2^{\frac{3}{4}}\cdot 2^{\frac{3}{4}}}}$$

$$\geq 3\cdot\frac{2^{\frac{1}{4}}(ab)^{\frac{1}{4}}\cdot 2^{\frac{1}{4}}(bc)^{\frac{1}{4}}\cdot 2^{\frac{1}{4}}(ca)^{\frac{1}{4}}}{2^{\frac{3}{4}}}=3(ab\cdot bc\cdot ca)^{\frac{1}{4}}\overset{abc=1}{=}3\cdot 1=3$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $abc=1, a,b,c>0$*, we obtain that:* $\sqrt{\frac{a^5+b^5}{a^2+b^2}}+\sqrt{\frac{b^5+c^5}{b^2+c^2}}+\sqrt{\frac{c^5+a^5}{c^2+a^2}}\geq$

$$\geq \sqrt{\frac{(a^3+b^3)(a^2+b^2)}{2(a^2+b^2)}}+\sqrt{\frac{(b^3+c^3)(b^2+c^2)}{2(b^2+c^2)}}+\sqrt{\frac{(c^3+a^3)(c^2+a^2)}{2(c^2+a^2)}}$$

$$=\sqrt{\frac{a^3+b^3}{2}}+\sqrt{\frac{(b^3+c^3)}{2}}+\sqrt{\frac{(c^3+a^3)}{2}}\geq\sqrt{\frac{(a+b)^3}{8}}+\sqrt{\frac{(b+c)^3}{8}}+\sqrt{\frac{(d+a)^3}{8}}$$

$$\geq 3\sqrt[3]{\sqrt{\frac{(a+b)^3(b+c)^3(c+a)^3}{8\times 8\times 8}}}=3\sqrt{\frac{(a+b)(b+c)(c+a)}{8}}\geq 3\sqrt{\frac{8(abc)^2}{8}}=3abc=3\times 1=3$$

Solution 5 by Trinh An-Vietnam

We have: $a^5+b^5=\frac{a^4}{bc}+\frac{b^4}{ca}\geq\frac{(a^2+b^2)^2}{c(a+b)}\Rightarrow\frac{a^5+b^5}{a^2+b^2}\geq\frac{a^2+b^2}{c(a+b)}\geq\frac{(a+b)^2}{2c(a+b)}=\frac{a+b}{2c}\geq\frac{2\sqrt{ab}}{2c}=\frac{1}{c\sqrt{c}}$

$$\Rightarrow\sum\sqrt{\frac{a^5+b^5}{a^2+b^2}}\geq\sum\frac{1}{\sqrt{c\sqrt{c}}}=\frac{3}{\sqrt[3]{\sqrt{abc\sqrt{abc}}}}=3$$

Solution 6 by Marian Ursărescu-Romania

We use breaking method: we show the following inequality:

$$\sqrt{\frac{a^2+b^5}{a^2+b^2}}\geq\sqrt[4]{a^3b^3}\quad (1)$$

Proof: $\frac{a^2+b^5}{a^2+b^2}\geq\sqrt{a^3b^3}\Rightarrow a^5+b^5\geq(a^2+b^2)ab\sqrt{ab}\Rightarrow$

$$a^5-a^3b\sqrt{ab}+b^5-b^3a\sqrt{ab}\geq 0\Rightarrow$$

$$a^3\sqrt{a}(a\sqrt{a}-b\sqrt{b})+b^3\sqrt{4}(b\sqrt{b}-a\sqrt{a})\geq 0\Rightarrow$$

$$(a\sqrt{a}-b\sqrt{b})(a^3\sqrt{a}-b^3\sqrt{b})\geq 0\Leftrightarrow\left(\sqrt{a^3}-\sqrt{b^3}\right)\left(\sqrt{a^5}-\sqrt{b^5}\right)\geq 0$$

obvious, because if $a\geq b\Rightarrow\sqrt{a^3}-\sqrt{b^3}\geq 0$ **and** $\sqrt{a^5}-\sqrt{b^5}\geq 0$

$a\leq b\Rightarrow\sqrt{a^3}-\sqrt{b^3}\leq 0$ **and** $\sqrt{a^5}-\sqrt{b^5}\leq 0$

Using relation (1) $\Rightarrow\sum\sqrt{\frac{a^5+b^5}{a^2+b^2}}\geq\sum\sqrt[4]{a^3b^3}\geq 3\sqrt[3]{\sqrt[4]{a^6b^6c^6}}\Rightarrow\sum\sqrt{\frac{a^5+b^5}{a^2+b^2}}\geq 3$

321. **If** $a,b,c>0, a+b+c=3$ **then:**

$$3+\sum\left(\frac{b}{12a+1}+\frac{c}{6b+1}\right)>\sum\left(\frac{c}{10b+1}+\frac{b}{2a+1}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\textit{If } a,b,c>0, a+b+c=3 \textit{ then } 3+\sum\left(\frac{b}{12a}+\frac{c}{6b+1}\right)>\sum\left(\frac{c}{10b+1}+\frac{b}{2a+1}\right) \quad \textbf{(1)}$$

$$\textit{We have (1)} \Rightarrow a+b+c+\frac{b}{12a+1}+\frac{c}{6b+1}+\frac{c}{12b+1}+\frac{a}{6c+1}+\frac{a}{12c+1}+\frac{b}{6a+1}>$$

$$>\frac{c}{10b+1}+\frac{b}{2a+1}+\frac{a}{10c+1}+\frac{c}{2b+1}+\frac{b}{10a+1}+\frac{a}{2c+1}$$

$$\Rightarrow a\left(1+\frac{1}{12c+1}+\frac{1}{6c+1}-\frac{1}{2c+1}-\frac{1}{10c+1}\right)+$$

$$+b\left(1+\frac{1}{12a+1}+\frac{1}{6a+1}-\frac{1}{2a+1}-\frac{1}{10a+1}\right)+$$

$$+c\left(1+\frac{1}{12b+1}+\frac{1}{6b+1}-\frac{1}{2b+1}-\frac{1}{10b+1}\right)>0$$

$$\Rightarrow a\cdot\frac{1440c^4+720c^3+204c^2+24c+1}{(2c+1)(6c+1)(10c+1)(12c+1)}+b\cdot\frac{1440a^4+720a^3+204a^2+24a+1}{(2a+1)(6a+1)(10a+1)(12a+1)}+$$

$$+c\cdot\frac{1440b^4+720b^3+240b^2+24b+1}{(2b+1)(6b+1)(10b+1)(12b+1)}>0 \quad \textit{(True)} \Rightarrow \textit{ Q.E.D.}$$

322. If $x,y,z>0, n\in\mathbb{N}, n\geq 2$, $x^3+y^3+z^3=3$ then:

$$\sum\frac{x}{y^4+z^4+y^2z^2}\leq\frac{1}{(xyz)^n}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution by Daniel Sitaru-Romania

$$3=x^3+y^3+z^3\overset{AM-GM}{\geq}3xyz\to xyz\leq 1\to (xyz)^2\geq (xyz)^n, n\geq 2$$

$$\sum\frac{x}{y^4+z^4+y^2z^2}\overset{AM-GM}{\leq}\sum\frac{x}{3\sqrt[3]{(yz)^6}}=\frac{1}{3}\sum\frac{x}{y^2z^2}=$$

$$=\frac{1}{3}\sum\frac{x^3}{(xyz)^2}=\frac{1}{3(xyz)^2}\sum x^3\leq\frac{1}{3(xyz)^2}\cdot 3\leq\frac{1}{(xyz)^n}$$

323. If $a,b,c\geq 0$ then:

$$3\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right)\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[4]{ab}+\sqrt[4]{bc}+\sqrt[4]{ca}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Catinca Alexandru-Romania

$$a, b, c \geq 0$$

$$3\sum\sqrt{\frac{a+b}{2}} \geq 2\sum\sqrt{a} + \sum\sqrt[4]{ab}$$

$$\left.\begin{array}{l} 3\sum\sqrt{\frac{a+b}{2}} \overset{MP-MA}{\geq} 3\sum\frac{\sqrt{a}+\sqrt{b}}{2} = 3\sum\sqrt{a} \\ \sum\sqrt{a} = \sum\frac{\sqrt{a}+\sqrt{b}}{2} \overset{MA\geq MG}{\geq} \sum\sqrt[4]{ab} \end{array}\right\} \Rightarrow$$

$$\Rightarrow 3\sum\sqrt{\frac{a+b}{2}} \geq 2\sum\sqrt{a} + \sum\sqrt{a} \geq 2\sum\sqrt{a} + \sum\sqrt[4]{ab}$$

Solution 2 by Ravi Prakash-New Delhi-India

For $x, y \geq 0$*, we first show:* $9(x^4 + y^4) \geq 2(x^2 + xy + y^2)^2$

$$\Leftrightarrow 9(x^4 + y^4) \geq 2(x^4 + y^4 + x^2y^2 + 2x^3y + 2xy^3 + 2x^2y^2)$$

$$\Leftrightarrow 7(x^4 + y^4) - 6x^2y^2 - 4x^3y - 4xy^3 \geq 0$$

$$\Leftrightarrow 3(x^2 - y^2)^2 + 4(x^3 - y^3)(x - y) \geq 0$$

$\Leftrightarrow 3(x^2 - y^2)^2 + 4(x - y)^2(x^2 + xy + y^2) \geq 0$. *Which is true.*

Putting $x = a^{\frac{1}{4}}, y = b^{\frac{1}{4}}$*, we get:* $9(a + b) \geq 2\left(\sqrt{a} + (ab)^{\frac{1}{4}} + \sqrt{b}\right)^2$

$$\Rightarrow \frac{3}{\sqrt{2}}\sqrt{a+b} \geq \sqrt{a} + \sqrt{b} + (ab)^{\frac{1}{4}} \quad (1)$$

Similarly, $3\sqrt{\frac{b+c}{2}} \geq \sqrt{b} + \sqrt{c} + (bc)^{\frac{1}{4}}$ *(2)*

$$3\sqrt{\frac{c+a}{2}} \geq \sqrt{c} + \sqrt{a} + (ca)^{\frac{1}{4}} \quad (3)$$

Adding (1), (2), (3) we get

$$3\left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}}\right) \geq 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) + (ab)^{\frac{1}{4}} + (bc)^{\frac{1}{4}} + (ca)^{\frac{1}{4}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\sqrt[4]{ab}+2\sum_{cyc}\sqrt{a}\le 3\sum_{cyc}\sqrt{a}\left[\because \sum_{cyc}\sqrt{a}\ge\sum_{cyc}\sqrt[4]{ab}\right]$$

$$=3\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\le 3\sum_{cyc}\sqrt{\frac{a+b}{2}}\ [\because \sqrt{x}\ is\ concave]$$

(proved)

324. If $a,b,c>0, a+b+c=3, 0\le x\le 1$ then:

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\le 6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

Because $a+b+c=3\Rightarrow \exists m,n,p>0$ *such that:* $a=\frac{3m}{m+n+p}, b=\frac{3n}{m+n+p}, c=\frac{3p}{m+n+p}$.

Inequality becomes:

$$\frac{m}{m+n+p}\cdot\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\cdot\left(\frac{m}{p}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{m}{n}\right)^x+\frac{p}{m+n+p}\cdot\left(\frac{n}{p}\right)^x+\frac{m}{m+n+p}\cdot\left(\frac{p}{m}\right)^x\le 2 \quad (1)$$

Let $f:(0,+\infty)\to\mathbb{R}, f(\alpha)=\alpha^x; f'(\alpha)=x\alpha^{x-1}, f''(\alpha)=x(x-1)\alpha^{x-2}\Rightarrow f''(x)<0$, *we use Jensen's generalization:* $p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\le f(p_1x_1+p_2x_2+p_3x_3)$ *with* $p_1,p_2,p_3>0\wedge p_1+p_2+p_3=1$. *Let* $p_1=\frac{m}{m+n+p}, p_2=\frac{n}{m+n+p}, p_3=\frac{p}{m+n+p}, x_1=\frac{n}{m}$,

$$x_2=\frac{p}{n}, x_3=\frac{m}{p}\Rightarrow\frac{m}{m+n+p}\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{m}{p}\right)^x\le\left(\frac{n+p+m}{m+n+p}\right)^x=1 \quad (2)$$

Let $p_1=\frac{n}{m+n+p}, p_2=\frac{p}{m+n+p}, p_3=\frac{m}{m+n+p}, x_1=\frac{m}{n}, x_2=\frac{n}{p}, x_3=\frac{p}{m}\Rightarrow$

$$\Rightarrow\frac{n}{m+n+p}\left(\frac{m}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{n}{p}\right)^x+\frac{m}{m+n+p}\cdot\left(\frac{p}{m}\right)^x\le\left(\frac{m+n+p}{m+n+p}\right)^x=1 \quad (3)$$

From (2)+(3) ⇒ (1) its true.

Solution 2 by Lahiru Samarakoon-India

$$\text{Assume, } a+b\overset{?}{\ge} a\left(\frac{b}{a}\right)^n+b\left(\frac{a}{b}\right)^n \quad (1)$$

$$\text{We have to prove: } \frac{a}{a^n}(a^n-b^n)+\frac{b}{b^n}(b^n-a^n)\ge 0$$

$$(a^n-b^n)\left(\frac{a}{a^n}-\frac{b}{b^n}\right)\geq 0;\ (a^n-b^n)\left[(a^x)^{\left(\frac{1}{n}-1\right)}-(b^n)^{\left(\frac{1}{n}-1\right)}\right]\geq 0$$

Consider $p=\frac{1}{n}-1\geq 0\ \ (\because c\leq x\leq 1)$

$$(a^n-b^n)(a^n-b^n)((a^n)^{p-1}+(a^n)^{p-2}(b^n)+\cdots+(b^n)^{p-1})\geq 0$$

$\underbrace{(a^n-b^n)^2}_{(+)}\underbrace{((a^n)^{p-1}+(a^n)^{p-2}\cdot b^n+\cdots+(b^n)^{p-1})}_{(+)}\geq 0$ ***it's true. So, similarly,***

$$b+c\geq b\left(\frac{c}{b}\right)^n+c\left(\frac{b}{c}\right)^n\quad (2)$$

$$a+c\geq a\left(\frac{c}{a}\right)^n+c\left(\frac{a}{c}\right)^n\quad (3)$$

(1)+(2)+(3): $a\left(\frac{b}{a}\right)^n+b\left(\frac{c}{b}\right)^n+c\left(\frac{a}{c}\right)^n+b\left(\frac{a}{b}\right)^n+c\left(\frac{b}{c}\right)^n+a\left(\frac{c}{a}\right)^n\leq 2\sum a$

But $\sum a=3\leq 6$ ***(Proved)***

Solution 3 by Michael Stergiou-Greece

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\leq 6\qquad (1)$$

The function $f(t)=t^x$ ***with*** $f''(t)=(x-1)xt^{(t-2)}<0$ ***for*** $x<1$ ***is concave. Applying the generalized Jensen inequality:***

$$LHS\text{ of (1)}\leq (a+b+c+b+c+a)\left(\frac{a\cdot\frac{b}{a}+b\cdot\frac{c}{b}+c\cdot\frac{a}{c}+b\cdot\frac{a}{b}+c\cdot\frac{b}{c}+a\cdot\frac{c}{a}}{a+b+c+b+c+a}\right)^x=6\cdot 1^x=6.$$

If $x=1$ ***then (1) is directly true. Done.***

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{b}{a}\right)^x=\left(1+\frac{b-a}{a}\right)^x\overset{Bernoulli}{\leq}1+x\left(\frac{b-a}{a}\right)$$

$$\left(\because\frac{b-a}{a}>-1\ as\ 1+\frac{b-a}{a}=\frac{b}{a}>0>-1\ \&0\leq x\leq 1\right)\Rightarrow a\left(\frac{b}{a}\right)^x\leq a+x(b-a)$$

$\left(\because 1+x\left(\frac{b-a}{a}\right)\geq\left(\frac{b}{a}\right)^x>0\right)$***. So,*** $a\left(\frac{b}{a}\right)^x\overset{(1)}{\leq}a+x(b-a)$***. Similarly,***

$$b\left(\frac{c}{b}\right)^x\overset{(2)}{\leq}b+x(c-b),\ c\left(\frac{a}{c}\right)^x\overset{(3)}{\leq}c+x(a-c),\ b\left(\frac{a}{b}\right)^x\overset{(4)}{\leq}b+x(a-b),$$

$$c\left(\frac{b}{c}\right)^x\overset{(5)}{\leq}c+x(b-c),\ a\left(\frac{c}{a}\right)^x\overset{(6)}{\leq}a+x(c-a)$$

(1)+(2)+(3)+(4)+(5)+(6)⇒ LHS

$$\le 2\sum a + x(b-a+c-b+a-c+a-b+b-c+c-a) \overset{\sum a=3}{=} 6 + x\cdot 0 = 6$$

325. If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{(\sin^2 x + \sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\sin x)^{2\sin^2 x}\cdot(\sin y)^{2\sin^2 y}\cdot(\cos x)^{2\cos^2 x}\cdot(\cos y)^{2\cos^2 y}} \le 4$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $\sin^2 x = a, \sin^2 y = b, \cos^2 x = c, \cos^2 y = d$

Then, given inequality $\Leftrightarrow \frac{(a+b)^{a+b}(c+d)^{c+d}}{a^a b^b c^c d^d} \overset{(1)}{\le} 4$

Now, $\sqrt[a+b]{a^a b^b} \overset{weighted\ GM-HM}{\ge} \frac{a+b}{\frac{a}{a}+\frac{b}{b}} = \frac{a+b}{2} \Rightarrow a^a b^b \overset{(a)}{\ge} \frac{(a+b)^{a+b}}{2^{a+b}}$. Similarly, $c^c d^d \overset{(b)}{\ge} \frac{(c+d)^{c+d}}{2^{c+d}}$

(a).(b)$\Rightarrow a^a b^b c^c d^d \ge \frac{(a+b)^{a+b}\cdot(c+d)^{c+d}}{2^{a+b+c+d}} = \frac{(a+b)^{a+b}(c+d)^{c+d}}{2^{(\sin^2 x+\cos^2 x)+(\sin^2 y+\cos^2 y)}} = \frac{(a+b)^{a+b}(c+d)^{c+d}}{4} \Rightarrow$

$\Rightarrow \frac{(a+b)^{a+b}(c+d)^{c+d}}{a^a b^b c^c d^d} \le 4 \Rightarrow$ (1) is true (Proved)

326. Let $a, b, c > 0$ and $a+b+c=3$. Prove that:

$$a\cdot \arcsin\left(\frac{b}{b+1}\right) + b\cdot \arcsin\left(\frac{c}{c+1}\right) + c\cdot \arcsin\left(\frac{a}{a+1}\right) \le \frac{\pi}{2}$$

Proposed by Dimitris Kastriotis-Athens-Greece

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$a+b+c=3$ then $3(ab+bc+ca) \le (a+b+c)^2 \Rightarrow (a+b+c)(ab+bc+ca) \le$

$\le (a+b+c)^2 \Rightarrow ab+bc+ca \le a+b+c$. Let $f(x) = \sin^{-1}\left(\frac{x}{x+1}\right)$ for all $x \in (0,1)$

then $f'(x) = \frac{1}{(1+x)\sqrt{2x+1}} \Rightarrow f''(x) = -\frac{1}{(1+x)^2\sqrt{2x+1}} - \frac{1}{(1+x)(2x+1)^{\frac{3}{2}}} < 0$ for all $x \in (0,1)$

Hence f is concave function, then:

$$\sum_{cyc}\frac{a}{a+b+c}\sin^{-1}\left(\frac{b}{b+1}\right) \le \sin^{-1}\frac{\frac{ab+bc+ca}{a+b+c}}{\frac{ab+bc+ca}{a+b+c}+1} \overset{AM\ge GM}{\le} \sin^{-1}\left(\frac{\sqrt{\frac{ab+bc+ca}{a+b+c}}}{2}\right)$$

$$\leq \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \Rightarrow \sum_{cyc} a\sin^{-1}\left(\frac{b}{b+1}\right) \leq \frac{\pi}{2}$$ ***(proved)***

Solution 2 by Chris Kyriazis-Athens-Greece

I will use that:

1) Function $f(x) = \arcsin\left(\frac{x}{x+1}\right), x > 0$ ***is concave (because*** $f''(x) = -\frac{3x+2}{(x+2)^2(2x+1)^{\frac{3}{2}}} < 0$

2) Function $\arcsin x$ ***is strictly increasing when*** $0 < x < 1$, $\left((\arcsin x)' = \frac{1}{\sqrt{1-x^2}} > 0\right)$

3) $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3} \leq \frac{1}{2}$***, when*** $a, b, c > 0, a + b + c = 3$

Proof:

$$a\frac{b}{b+1} + b\frac{c}{c+1} + c\frac{a}{a+1} \overset{GM-AM}{\leq} a\frac{(b+1)^2}{4(b+1)} + b\frac{(c+1)^2}{4(c+1)} + c\frac{(a+1)^2}{4(a+1)} =$$

$$= \frac{ab + a + bc + b + ca + c}{4} \leq \frac{\left(\frac{a+b+c}{3}\right)^2 + 3}{4} = \frac{6}{4} = \frac{3}{2}$$

So, $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3} \leq \frac{1}{2}$

Now, (using (1)) applying Jensen's inequality with weights a, b, c***, gives then:***

$$LHS \leq (a+b+c)\arcsin\left(\frac{a\frac{b}{b+1} + b\frac{c}{c+1} + c\frac{a}{a+1}}{a+b+c}\right) =$$

$$= 3\arcsin\frac{\left(a\frac{b}{b+1} + b\frac{c}{c+1} + c\frac{a}{a+1}\right)}{3} \underset{(2)}{\overset{(3)}{\leq}} 3\arcsin\left(\frac{1}{2}\right) = 3\cdot\frac{n}{6} = \frac{n}{2}$$

because $\arcsin\left(\frac{1}{2}\right) = \frac{n}{6}$

Solution 3 by Soumava Chakraborty-Kolkata-India

Given inequality can be written as:

$$\left(\frac{a}{\sum a}\right)\sin^{-1}\left(\frac{b}{b+1}\right) + \left(\frac{b}{\sum a}\right)\sin^{-1}\left(\frac{c}{c+1}\right) + \left(\frac{c}{\sum a}\right)\sin^{-1}\left(\frac{a}{a+1}\right) \overset{(1)}{\leq} \frac{\pi}{6}$$

Let $\frac{a}{\sum a} = p_1, \frac{b}{\sum a} = p_2, \frac{c}{\sum a} = p_3$***. Then*** $p_1 + p_2 + p_3 = 1$***. Now,***

$\because f''(x) = -\frac{(3x+2)}{(x+1)^5\left(\frac{2x+1}{(x+1)^2}\right)^{\frac{3}{2}}} < 0, \forall x > 0 \because f(x) = \sin^{-1}\left(\frac{x}{x+1}\right), \forall x > 0$ ***is concave,***

$\therefore$ ***by Jensen, LHS of (1)*** $= p_1 f(b) + p_2 f(c) + p_3 f(a) \overset{(2)}{\leq} f(p_1 b + p_2 c + p_3 a) =$

$$= \sin^{-1}\left(\frac{\frac{ab+bc+bc}{\sum a}}{\frac{\sum ab}{\sum a}+1}\right) = \sin^{-1}\left(\frac{\sum ab}{\sum ab + 3}\right) \because 3\left(\sum ab\right) \leq \left(\sum a\right)^2 = 9 \therefore \sum ab \leq 3$$

$$\therefore 1 - \frac{3}{\sum ab + 3} \leq 1 - \frac{3}{3+3} = \frac{1}{2} \Rightarrow \frac{\sum ab}{\sum ab + 3} \overset{(3)}{\leq} \frac{1}{2}$$

(2),(3) $\Rightarrow$ ***LHS of (1)*** $\leq \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} =$ ***RHS of (1) (proved)***

327. **Let $n \in \mathbb{N}^* \wedge n \geq 2$ and $x_1, x_2, \ldots, x_n \in (0; +\infty)$. Prove:**

$$e^n x_1^{\frac{1}{x_1}} x_2^{\frac{1}{x_2}} \ldots x_n^{\frac{1}{x_n}} \leq e^{x_1+x_2+\cdots+x_n}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Dimitris Kastriotis-Athens-Greece

$$ex^{\frac{1}{x}} \leq e^x, x \in (0,\infty)$$

$$ex^{\frac{1}{x}} \leq e^x \Leftrightarrow 1 + \frac{1}{x}\log x \leq x \Leftrightarrow x + \log x - x^2 \leq 0, x \in (0,\infty)$$

$$f(x) = x + \log x - x^2, x \in (0,\infty);\ f'(x) = 1 + \frac{1}{x} - 2x, x \in (0,\infty)$$

$$f'(x) = 0 \Rightarrow 1 + \frac{1}{x} - 2x = 0 \Rightarrow x = 1$$

$$f''(x) = -\frac{1}{x^2} - 2 < 0, x \in (0,\infty) \Rightarrow \max\{f(x) | 0 < x < \infty\} = f(1) = 0$$

$$\Rightarrow f(x) \leq f(1) = 0 \Rightarrow x \leq x + \log x - x^2 \leq 0, x \in (0,\infty) \Rightarrow ex^{\frac{1}{x}} < e^x, x \in (0,\infty)$$

$$\Rightarrow e^n x_1^{\frac{1}{x_1}} \ldots x_n^{\frac{1}{x_n}} \leq e^{x_1+\cdots+x_n}$$

Solution 2 by Tran Hong-Vietnam

We prove that: $\frac{\ln t}{t} \leq t - 1; \forall t > 0 \Leftrightarrow t^2 - t - \ln t \geq 0$

$$f(t) = t^2 - t - \ln t\,; t > 0;\ f'(t) = 2t - 1 - \frac{1}{t}; f'(t) = 0 \Leftrightarrow \frac{2t^2 - t - 1}{t} = 0 \Leftrightarrow t = 1$$

t	0 $\qquad$ 1 $\qquad$ $+\infty$
$f'(t)$	$- - - - - - - - - - 0 + + + + + + + + +$
$f(t)$	$+\infty \searrow 0 \nearrow +\infty$

$$\Rightarrow f(t) \geq 0 \Rightarrow \frac{\ln t}{t} \leq t - 1; \forall t \geq 0$$

We need to prove that: $\sum_{i=1}^{n} \frac{1}{x_i} \ln x_i \leq \sum_{i=1}^{n} x_i - n$ ***(*)***

We have: $\frac{1}{x_1} \ln x_1 \leq x_1 - 1; \forall x_1 > 0$

$$\frac{1}{x_2} \ln x_2 \leq x_2 - 1; \forall x_2 > 0$$

$$\vdots$$

$$\frac{1}{x_n} \ln x_n \leq x_n - 1; \forall x_n > 0$$

$\Rightarrow$ ***(*) True; equality*** $\Leftrightarrow x_1 = x_2 = \cdots = x_n = 1$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

From these facts: 1. $e^k \geq k + 1, \forall k \in \mathbb{R}$. ***2.*** $x^x \geq x, \forall x \in \mathbb{R}^+$

Hence $x^{x_1 - 1} \geq x_1, e^{x_2 - 1} \geq x_2, e^{x_3 - 1} \geq x_3$

$$e^{x_4 - 1} \geq x_4, e^{x_5 - 1} \geq x^5, \dots, e^{x_n - 1} \geq x^n$$

Hence $x_1^{\frac{1}{x_1}} x_2^{\frac{1}{x_2}} x_3^{\frac{1}{x_3}}, \dots, x_n^{\frac{1}{x_n}} \leq x_1 x_2 x_3 \dots x_n \leq e^{(x_1 - 1)} e^{(x_2 - 1)} e^{(x_3 - 1)} \dots e^{(x_n - 1)} =$

$$= e^{((x_1 - 1) + (x_2 - 1) + (x_3 - 1) + \dots + (x_n - 1))} = e^{((x_1 + x_2 + x_3 + \dots + x_n) - (1 + 1 + 1 + \dots + 1))} =$$

$$= e^{((x_1 + x_2 + x_3 + \dots + x_n) - n)} \Rightarrow e^n \left(x_1^{\frac{1}{x_1}} x_2^{\frac{1}{x_2}} x_3^{\frac{1}{x_3}} \dots x_n^{\frac{1}{x_n}} \right) \leq e^{(x_1 + x_2 + x_3 + \dots + x_n)}$$

Therefore, it is true.

Solution 4 by Nassim Nicholas Taleb-USA

We need to prove:

$$e^n\prod_{i=1}^{n}x_i^{\frac{1}{x_i}}\le e^{\sum_{i=1}^{n}x_i}, so\ n+\sum_{i=1}^{n}\frac{1}{x_i}\log x_i\le\sum_{i=1}^{n}x_i, n+\sum_{i=1}^{n}\frac{1}{x_i}\log x_i-x_i\le 0$$

$\max_{x\in\mathbb{N}}\left(\frac{\log(x)}{x}-x\right)=-1$, ***reached for*** $x=1$, ***so,*** $\max_x\left(\sum_{i=1}^{n}\frac{1}{x_i}\log x_i-x_i\right)=-n$,

which proves the inequality.

328. **If $x,y\ge 0$ then:**

$$(e^x+1)\sqrt{e^y}+(e^y+1)\sqrt{e^x}\le(e^x+1)(e^y+1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $\sqrt{e^x}=a,\sqrt{e^y}=b\because(a^2+1)\frac{(b-1)^2}{2}+(b^2+1)\frac{(a-1)^2}{2}\ge 0\Rightarrow$

$$\Rightarrow(a^2+1)\left[\frac{b^2+1}{2}-b\right]+(b^2+1)\left[\frac{a^2+1}{2}-a\right]\ge 0\Rightarrow$$

$$\Rightarrow(a^2+1)\frac{(b^2+1)}{2}-b(a^2+1)+(b^2+1)\frac{(a^2+1)}{2}-a(b^2+1)\ge 0\Rightarrow$$

$$\Rightarrow\frac{(a^2+1)(b^2+1)}{2}+\frac{(b^2+1)(a^2+1)}{2}\ge a(b^2+1)+b(a^2+1)\Rightarrow$$

$\Rightarrow(a^2+1)(b^2+1)\ge a(b^2+1)+b(a^2+1)$. ***Now, put*** $\sqrt{e^x}=a\Rightarrow e^x=a^2$

$\sqrt{e^x}=b\Rightarrow e^y=b^2$. ***So,*** $(e^x+1)(e^y+1)\ge(e^x+1)\sqrt{e^y}+(e^y+1)\sqrt{e^x}$

Solution 2 by Boris Colakovic-Belgrade-Serbie

$$LHS=e^{x+\frac{y}{2}}+e^{\frac{x}{2}+y}+e^{\frac{x}{2}}+e^{\frac{y}{2}}=e^{\frac{y}{2}}(e^x+1)+e^{\frac{x}{2}}(e^y+1)\le(e^x+1)(e^y+1)\Leftrightarrow$$

$\Leftrightarrow\frac{e^{\frac{y}{2}}}{e^y+1}+\frac{e^{\frac{x}{2}}}{e^x+1}\le 1$ ***true, because of*** $\frac{e^{\frac{x}{2}}}{e^a+1}\le\frac{1}{2}\Leftrightarrow\left(\sqrt{e^x}-1\right)^2\ge 0$. ***Similarly,*** $\frac{e^{\frac{y}{2}}}{e^y+1}\le\frac{1}{2}$

Solution 3 by Ravi Prakash-New Delhi-India

$$e^x+1>2e^{\frac{x}{2}},\forall x>0\Rightarrow e^x+1-e^{\frac{x}{2}}>e^{\frac{x}{2}},\forall x>0\Rightarrow$$

$$\Rightarrow\left(e^x+1-e^{\frac{x}{2}}\right)\left(e^y+1-e^{\frac{y}{2}}\right)>e^{\frac{x}{2}}e^{\frac{y}{2}},\forall x,y,>0\Rightarrow$$

$$\Rightarrow(e^x+1)(e^y+1)>\sqrt{e^y}(e^x+1)+\sqrt{e^x}(e^y+1),\forall x,y>0$$

329. If $a,b,c>0, abc=1$ then:

$$e^{a^{3a^3}}+e^{b^{3b^3}}+e^{c^{3c^3}}\geq 3e$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Antonis Anastasiadis-Greece

From well known inequality: $e^x\geq x+1$

$$\therefore\ x^{3^{x^3}}=e^{3x^3\ln x}\geq 3x^3\ln x+1\quad (1)$$

It is: $3\ln x\cdot(x^3-1)\geq 0,\forall x>0$. *So,* $3x^3\ln x\geq 3\ln x,\forall x>0$

$$(1)\Rightarrow e^{3x^3\ln x}\geq 3\ln x+1=\ln x^3e\Rightarrow x^{3^{x^3}}\geq \ln x^3e\Leftrightarrow e^{x^{3^{x^3}}}\geq x^3e$$

So: $e^{a^{3^{a^3}}}+e^{b^{3^{b^3}}}+e^{c^{3^{c^3}}}\geq a^3e+b^3e+c^3e\Rightarrow LHS\geq e(a^3+b^3+c^3)\overset{AM-GM}{\geq}$

$$\geq e\cdot 3\sqrt[3]{abc}=3e$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$abc=1,a,b,c>0;\ A=\frac{e^{a^{3^{a^3}}}+e^{b^{3^{b^3}}}+e^{c^{3^{c^3}}}}{e}$. *Kindly prove* $A\geq 3$

It's $x^x\geq e^{x-1},\forall x>0$ „=" $x=1$ *(1)*

and $e^x\geq x+1\ \forall x\in\mathbb{R}$ „=" $x=0$ *(2)*

so $A=\frac{e^{a^{3^{a^3}}}+e^{b^{3^{b^3}}}+e^{c^{3^{c^3}}}}{e}=\frac{e^{(a^3)^{a^3}}+e^{(b^3)^{b^3}}+e^{(c^3)^{c^3}}}{e}\overset{(1)}{\geq}\frac{e^{e^{a^3-1}}+e^{e^{b^3-1}}+e^{e^{c^3-1}}}{e}\overset{(2)}{\geq}\frac{e^{a^3}+e^{b^3}+e^{c^3}}{e}$

$=e^{a^3-1}+e^{b^3-1}+e^{c^3-1}\overset{(2)}{\geq}a^3+b^3+c^3\overset{AM-GM}{\geq}3\sqrt[3]{(abc)^3}\overset{abc=1}{=}3$ *(proved)*

Solution 3 by proposer

for $x>0$, *we get* $x^{3x^3}\geq x^{3^{3x^2}}\geq x^{3x}\geq x^3$. *Hence for* $a,b,c>0$ *and* $abc=1$, *we have:*

$$a^{3a^3}b^{3b^3}e^{3c^3}\geq a^3b^3c^3=(abc)^3=1\Rightarrow e^{a^{3a^3}b^{3b^3}c^{3c^3}}\geq e^{(abc)^3}=e^1\Rightarrow$$

$$\Rightarrow e^{\sqrt[3]{a^{3a^3}b^{3b^3}c^{3c^3}}}\geq e^1\Rightarrow e^{3\sqrt{a^{3a^3}b^{3b^3}c^{3c^3}}}\geq e^1\Rightarrow e^{\left(a^{3a^3}+b^{3b^3}+c^{3c^3}\right)}\geq e^3\Rightarrow$$

$$\Rightarrow\sqrt[3]{e^{\left(a^{3a^3}+b^{3b^3}+c^{3c^3}\right)}}\geq e\Rightarrow e^{a^{3a^3}}+e^{b^{3b^3}}+e^{b^{3b^3}}\geq 3e.$$ *Therefore, it is true.*

330. $A = (a_{ij})_{\substack{1\le i\le n,\\ 1\le j\le n}}, a_{ij} = 10i + j, n \ge 2, n \in \mathbb{N}^*$.

Find $X, Y \in M_n(\mathbb{R})$ such that:

$$\det X < 0, \det Y < 0, A + Y = X$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$A = (a_{ij})_{n\times n}$, where $a_{ij} = 10i + j$. Let $x = (x_{ij})_{n\times n}$, where $x_{ij} = a_{ij}$ if $i > j = 0$ if $i < j$; $x_{11} = -1$ and $x_{ii} = a_{ii} + 1, \forall i \ge 2$

Let $Y = (y_{ij})_{n\times n}$, where $y_{ij} = 0$ if $i > j = -a_{ij}$ if $i < j$; $y_{11} = -12 = -(a_{11} + 1)$

$y_{ii} = 1\ \forall i \ge 2$. Note that $A + Y = X$ and $\det(Y) = -12 < 0$ and

$$\det(X) = -(23)(34)\dots(10n + n + 1) < 0$$

331. If $n \in \mathbb{N}, n \ge 2$ then:

$$log(n!) + 1 - n < \sum_{k=2}^{n}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}\right) < log(n!)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

We take the function $f: [k; (k+1)] \to \mathbb{R}$ with $f(x) = \ln(x)$

We know that f is continuous on $[k; (k+1)]$ and derivable on $(k; (k+1))$ so f is a Rolle function $\Rightarrow$ we can apply Lagrange's theorem on $[k; (k+1)]$

$$\frac{f(k+1) - f(k)}{k+1-k} = f'(c_k) \text{ such that } c_k \in [k; k+1]$$

$$\succ \ln(k+1) - \ln(k) = \frac{1}{c_k} \text{ (1) with } k \le c_k \le k+1 \text{ (2)}$$

We obtain from (1) that $\frac{1}{c_1} + \cdots + \frac{1}{c_n} = \ln(n+1)$ and from (2)

We can write that $\frac{1}{2} + \cdots + \frac{1}{n+1} < \ln(n+1) < 1 + \cdots + \frac{1}{n}$

So, $\ln\left(\frac{(n+1)!}{2}\right)+1-n<\sum_{k=2}^{n}\left(\frac{1}{2}+\cdots+\frac{1}{k}\right)<\ln(n!)$

$\ln\left(\frac{(n+1)!}{2}\right)+1-n>\ln(n!)+1-n\Leftrightarrow\frac{n+1}{2}>1\Leftrightarrow n>1$ ***(true because*** $n\ge 2$***)*** $\Rightarrow$

$\ln(n!)+1-n<\sum_{k=2}^{n}\left(\frac{1}{2}+\cdots+\frac{1}{k}\right)<\ln(n!)$ ***(Q.E.D)***

Solution 2 by Khaled Abd Almuti-Damascus-Syria

If $n\in\mathbb{N}, n\ge 2$ ***then:*** $\ln(n!)+1-n<\sum_{k=2}^{k=2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{k}\right)<\ln(n!)$

Note: $\forall t\in[k,k+1]:\frac{1}{k+1}\le\frac{1}{t}\le\frac{1}{k};\ k\le t\le k+1$

$\int_k^{k+1}\frac{1}{k+1}dt\le\int_k^{k+1}\frac{1}{t}dt\le\int_k^{k+1}\frac{1}{k}dt,\ \frac{1}{k+1}[t]_k^{k+1}\le\int_k^{k+1}\frac{dt}{t}\le\frac{1}{k}[t]_k^{k+1}$

$$\frac{1}{k+1}\le\int_k^{k+1}\frac{dt}{t}\le\frac{1}{k}$$

$$\sum_{k=1}^{n-1}\frac{1}{k+1}\le\int_1^{n}\frac{dt}{t}\le\sum_{k=1}^{n-1}\frac{1}{k}$$

$$\sum_{k=1}^{n-1}\frac{1}{k+1}\le\ln(n)\le\sum_{k=1}^{n-1}\frac{1}{k},n\ge 2$$

$$\sum_{k=1}^{k=n-1}\frac{1}{k+1}\le\ln(n)\le\sum_{k=1}^{k=n-1}\frac{1}{k}$$

$$n!=n\cdot(n-1)(n-2)(n-3)\ldots 2\cdot 1$$

$$\ln(n!)=\ln(n)+\ln(n-1)+\ln(n-2)+\ln(n-3)+\cdots+\ln(2)$$

$$\sum_{k=1}^{n-1}\frac{1}{k+1}\le\ln(n)\le\sum_{k=1}^{n-1}\frac{1}{k}$$

$$\sum_{k=1}^{n-2}\frac{1}{k+1}\le\ln(n-1)\le\sum_{k=1}^{n-2}\frac{1}{k}$$

$$\sum_{k=1}^{n-3}\frac{1}{k+1}\le\ln(n-2)\le\sum_{k=1}^{n-3}\frac{1}{k}$$

$$\sum_{k=1}^{n-4}\frac{1}{k+1}\le\ln(n-3)\le\sum_{k=1}^{n-4}\frac{1}{k}$$

$$\sum_{k=1}^{k=2}\frac{1}{k+1}\le \ln(3)\le \sum_{k=1}^{k=2}\frac{1}{k}$$

$$\sum_{k=1}^{k=1}\frac{1}{k+1}\le \ln(2)\le \sum_{k=1}^{k=1}\frac{1}{k}$$

$$\Leftarrow\left\{\begin{array}{c}
\frac{1}{3}\le \ln(2)\le 1, \frac{1}{2}\le \ln(2)\le 1\\
\frac{1}{2}+\frac{1}{3}\le \ln(3)\le 1+\frac{1}{2}\\
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\le \ln(4)\le 1+\frac{1}{2}+\frac{1}{3}\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-2}\le \ln(n-3)\le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-4}\\
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-2}+\frac{1}{n-2}\le \ln(n-2)\le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-4}+\frac{1}{n-3}\\
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}\le \ln(n-1)\le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}\\
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-1}+\frac{1}{n}\le \ln(n)\le 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}
\end{array}\right.$$

$$\sum_{k=1}^{k=n-1}\left(\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{k}\right)\le \ln(n!)$$

$$\ln(n!)\le n-1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\dots+\left(\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-1}\right)$$

$$\ln(n!)-n+1\le \sum_{k=2}^{k=n-1}\left(\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{k}\right)$$

$$\ln(n!)-n+1<\sum_{k=2}^{k=n}\left(\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{k}\right)$$

Solution 3 by Ravi Prakash-New Delhi-India

For $k=2, \ln k-1<0=\ln 2-\ln 2<\frac{1}{2}$ ***(1)***

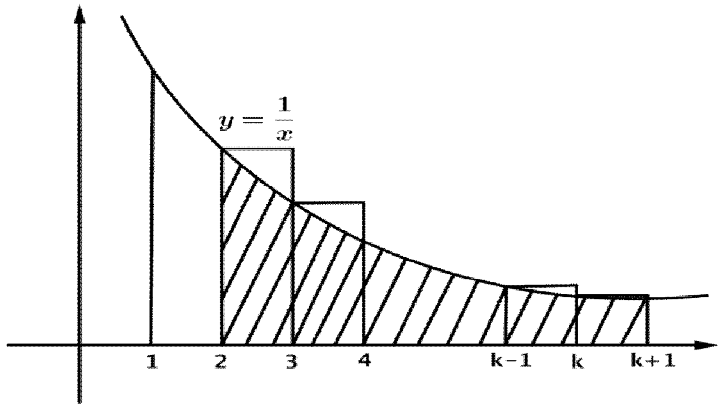

Fig. 1

For $k \geq 3, \ln(k) - 1 < \ln(k) - \ln(2) = \int_2^k \frac{1}{x} dx < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1}$ *[see Fig. 1]*

$$< \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k} \quad (2)$$

$$\Rightarrow \sum_{k=2}^{n} (\ln k - 1) < \sum_{k=2}^{n} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) \quad \text{[using (1), (2)]}$$

$$\Rightarrow \ln(n!) - (n-1) < \sum_{k=2}^{n} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) \quad (3)$$

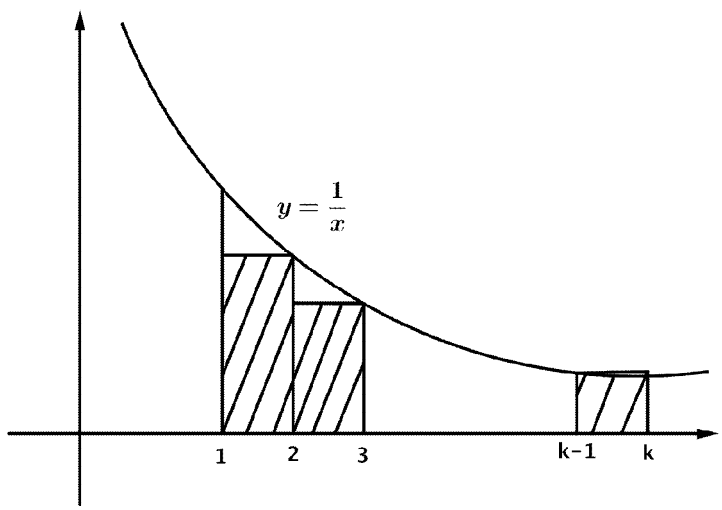

Fig. 2

For $k \geq 2, \ln k = \int_1^k \frac{1}{x} dx > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ *[see Fig. 2]*

$$\Rightarrow \ln(n!) = \sum_{k=2}^{n} \ln k > \sum_{k=2}^{n} \left(\frac{1}{2} + \cdots + \frac{1}{k}\right) \quad (4)$$

From (3), (4) the inequality follows.

332. If $\frac{\sqrt{3}}{3} \le a, b, c \le 1$ then:

$$\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \le \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}\left(\sqrt[3]{abc}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Definition: A function $f: I \to \mathbb{R}$ is said to be a decreasing function on I if $f(y) \ge f(x)$ for all $x \ge y$ where $x, y \in I$

Let $f(x) = \frac{\tan^{-1} x}{x}$ for all $x \in \left[\frac{1}{\sqrt{3}}, 1\right]$, $f'(x) = \frac{1}{x(1+x^2)} - \frac{\tan^{-1} x}{x^2} = \frac{1}{x^2}\left(\frac{x}{1+x^2} - \tan^{-1} x\right)$

Let $\varphi(x) = \frac{x}{1+x^2} - \tan^{-1} x$ for all $x \in \left[\frac{1}{\sqrt{3}}, 1\right]$, $\varphi'(x) = -\frac{2x^2}{(1+x^2)^2} < 0$

For all $x \in \left[\frac{1}{\sqrt{3}}, 1\right]$. Hence φ is decreasing $\therefore \varphi(1) \le \varphi(x) \le \varphi\left(\frac{1}{\sqrt{3}}\right) \Rightarrow$

$$\Rightarrow \frac{\frac{1}{\sqrt{3}}}{1+\frac{1}{3}} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \ge \varphi(x) \Rightarrow \frac{\sqrt{3}}{4} - \frac{\pi}{6} \ge \varphi(x) \Rightarrow 0 > \frac{\sqrt{3}}{4} - \frac{\pi}{6} \ge \varphi(x)$$

$\therefore \tan^{-1} x > \frac{x}{1+x^2}$ hence $f'(x) < 0$. So f is decreasing on $\left[\frac{1}{\sqrt{3}}, 1\right]$

Again, $\sqrt{\frac{ab+bc+ca}{3}} \ge \sqrt[3]{abc}$, so by definition of decreasing function

$$\frac{\tan^{-1}\sqrt[3]{abc}}{\sqrt[3]{abc}} \ge \frac{\tan^{-1}\sqrt{\frac{ab+bc+ca}{3}}}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\therefore \sqrt{\frac{ab+bc+ca}{3}} \tan^{-1}\left(\sqrt[3]{abc}\right) \ge \sqrt[3]{abc} \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)$$

Solution 2 by Michael Sterghiou-Greece

If $\frac{\sqrt{3}}{3} \le a, b, c \le 1$ *then:* $\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \le \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}\left(\sqrt[3]{abc}\right)$ **(*)**

$$\sum_{cyc} ab \geq 3\sqrt[3]{(abc)^2} \to \left(\frac{\sum ab}{3}\right)^{\frac{1}{2}} \geq \sqrt[3]{abc} \quad (1)$$

The function $\frac{\tan^{-1}x}{x} = f(x), \begin{matrix} x > 0 \\ x < 1 \end{matrix}$ *has* $f'(x) = \frac{\frac{x}{x^2+1} - \tan^{-1}x}{x^2}$

The function $g(x) = \frac{x}{x^2+1} - \tan^{-1}x$ *has* $g'(x) = -\frac{2x^2}{(x^2+1)^2} < 0, x > 0$

Hence $g(x) \leq g(0)$ *for* $x > 0$ *or* $g(x) < 0 \to f'(x) < 0$ *and*

$f(x) \downarrow$. *From (1)* $\to \frac{\tan^{-1}\left(\sqrt{\frac{\sum ab}{3}}\right)}{\sqrt{\frac{\sum ab}{3}}} \leq \frac{\tan^{-1}(\sqrt[3]{abc})}{\sqrt[3]{abc}}$ *in the requested interval.*

Solution 3 by Ravi Prakash-New Delhi-India

Let $f(x) = \frac{\tan^{-1}x}{x}, 0 < x \leq 1;\ f'(x) = \left(\frac{x}{1+x^2} - \tan^{-1}x\right)\frac{1}{x^2}, 0 < x < 1$

$$= \frac{x - (1+x^2)\tan^{-1}x}{(1+x^2)x^2}, 0 < x < 1$$

Let $g(x) = x - (1+x^2)\tan^{-1}x, 0 \leq x \leq 1;\ g'(x) = 1 - (1+x^2)\frac{1}{1+x^2} - 2x\tan^{-1}x$

$= -2x\tan^{-1}x < 0$ *for* $0 < x < 1 \Rightarrow g(x)$ *is strictly decreasing on* $[0,1]$.

$\therefore g(x) < g(0)\ \forall x \in (0,1) \Rightarrow x - (1+x^2)\tan^{-1}x < 0; \forall x \in (0,1)$

Thus, $f'(x) < 0$ *for* $0 < x < 1 \Rightarrow f(x)$ *is strictly decreasing on* $(0,1]$

Now, $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1 \Rightarrow \frac{ab+bc+ca}{3} \geq (abc)^{\frac{2}{3}} \Rightarrow \sqrt[3]{abc} \leq \left[\frac{1}{2}(ab+bc+ca)\right]^{\frac{1}{2}} \Rightarrow$

$$\Rightarrow f\left((abc)^{\frac{1}{3}}\right) \geq f\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \Rightarrow \frac{\tan^{-1}(abc)^{\frac{1}{3}}}{(abc)^{\frac{1}{3}}} \geq \frac{\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\Rightarrow \sqrt[3]{abc}\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}}\tan^{-1}(3\sqrt[3]{abc})$$

333. If $a \geq 4, b, c \geq 0, a + c \leq 2b, x, y, z \in \mathbb{R}$ then:

$$(a-3)(c - x^2 - y^2 - z^2) \leq (b - x - y - z)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(a-3)(c-x^2-y^2-z^2)\overset{(1)}{\le}(b-x-y-z)^2$$

$$\textbf{(1)}\Leftrightarrow c(a-3)-(a-3)(\sum x^2)\le b^2+(\sum x)^2-2b(\sum x)$$

$$\Leftrightarrow (a-3)\left(\sum x^2\right)+\left(\sum x\right)^2-2b\left(\sum x\right)+b^2-c(a-3)\overset{(2)}{\ge}0$$

$\because \sum x^2\ge\frac{(\sum x)^2}{3}$ ***&*** $a-3\ge 1>0$, $\therefore$ ***LHS of (2)*** $\ge\left(\frac{a-3}{3}+1\right)(\sum x)^2-2b(\sum x)$

$$+b^2-c(a-3)=\frac{a}{3}\left(\sum x\right)^2-2b\left(\sum x\right)+b^2-c(a-3)$$

$$\overset{(?)}{\ge}0\Leftrightarrow a\left(\sum x\right)^2-6b\left(\sum x\right)+3\{b^2-c(a-3)\}\underset{(3)}{\overset{?}{\ge}}0$$

$\because a\ge 4>0$ ***& LHS of (3) is a quadratic in*** $(\sum x)$ ***&*** $\because \sum x\in\mathbb{R}$ ***(as*** $x,y,z\in\mathbb{R}$***),*** $\therefore$ ***it suffices to prove that the discriminant is*** ≤ 0 ***that is, it suffices to prove:***

$$36b^2-4a\cdot 3\{b^2-c(a-3)\}\le 0\Leftrightarrow 3b^2-a\{b^2-c(a-3)\}\le 0\Leftrightarrow$$

$$\Leftrightarrow ac(a-3)-b^2(a-3)\le 0\Leftrightarrow (a-3)(ac-b^2)\le 0$$

$\because a-3\ge 1>0$, $\therefore$ ***it suffices to prove:*** $ac-b^2\le 0\Leftrightarrow 4b^2\overset{(4)}{\ge}4ac$

But LHS of (4) $\ge(a+c)^2$ $(\because 2b\ge a+c; b\ge 0; a+c\ge 4>0)$

$\overset{?}{\ge}4ac\Leftrightarrow(a-c)^2\ge 0\to$ ***true*** $\Rightarrow$ ***(4) is true (proved)***

334. **Let $x,y\in(0;+\infty)\wedge x+y=1$ and $n\in\mathbb{N}^*$.**

$$\textbf{Prove: }(xy)^n\ge\frac{16^n+1}{4^n}-\frac{1}{x^ny^n}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$\textit{Put } x=\cos^2\theta,\ y=\sin^2\theta,\ 0<\theta<\frac{\pi}{2}$$

$$P=(xy)^n+(xy)^{-n}=(\cos\theta\sin\theta)^{2n}+(\cos\theta\sin\theta)^{-2n}$$

$$\frac{dp}{d\theta}=(2n)(\cos\theta\sin\theta)^{2n-1}(\cos 2\theta)-2n(\cos\theta\sin\theta)^{-2n-1}(\cos 2\theta)$$

$$=2n(\cos 2\theta)(\cos\theta\sin\theta)^{-2n-1}[(\cos\theta\sin\theta)^{4n}-1]$$

As $\cos\theta\sin\theta > 0, 0 < \cos\theta\sin\theta < 1, \frac{dp}{d\theta} < 0$ *if* $0 < \theta < \frac{\pi}{4} = 0$ *if* $\theta = \frac{\pi}{4}$

> 0 *if* $\frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow P$ *is least when* $\theta = \frac{\pi}{4}$*. Thus,* $P \geq P\left(\frac{\pi}{4}\right) = \frac{1}{2^{2n}} + 2^{2n} = \frac{16^n+1}{4^n}$

335. If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{(\tan x + \cot x)(\tan y + \cot y)(\tan z + \cot z)}{(\tan x + \cot y)(\tan y + \cot z)(\tan z + \cot x)} \geq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $\tan x = a, \tan y = b, \tan z = c \because x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow a, b, c > 0$

So, to prove $\frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)} \geq 1$ *or* $\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)$

$$\Rightarrow abc + \frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{1}{abc} \geq abc + a + c + \frac{1}{b} + b + \frac{1}{c} + \frac{1}{a} + \frac{1}{abc} \Rightarrow$$

$$\Rightarrow \frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \geq (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \Rightarrow$$

$$\Rightarrow \left(\frac{a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2}{abc}\right) \geq \left(\frac{a^2bc + b^2ac + c^2ab + ab + bc + ac}{abc}\right)$$

or $(a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2) \geq (a^2bc + b^2ac + c^2ab + ab + bc + ac)$ ***(1)***

$\because$ *we know that* $p^2 + q^2 + r^2 \geq pq + qr + pr$*. Taking* $p = ab, q = bc, r = ac$*, we get*

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + b^2ac + c^2ab \quad (2)$$

Taking $p = a, q = b, r = c$

$$a^2 + b^2 + c^2 \geq ab + bc + ac \quad (3)$$

Adding (2) & (3), we get (1)⇒ (2)+(3)⇒ (1)

So, (1)⇒ $\left(\sum a^2b^2 + \sum a^2\right) \geq \left(\sum a^2bc + \sum ab\right)$*. This is true*

and hence $\frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)} \geq 1$ *or* $\frac{(\tan x+\cot x)(\tan y+\cot y)(\tan z+\cot z)}{(\tan x+\cot y)(\tan y+\cot z)(\tan z+\cot x)} \geq 1$ *(proved)*

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $\tan x = a, \tan y = b, \tan z = c$ *of course,* $a, b, c > 0$ *as* $x, y, z \in \left(0, \frac{\pi}{2}\right)$

using this substitution, given inequality becomes:

$$(a^2+1)(b^2+1)(c^2+1)\overset{(1)}{\geq}(ab+1)(bc+1)(ca+1).$$ ***Now,***

$$ab+1\overset{CBS}{\leq}\sqrt{a^2+1}\sqrt{b^2+1},\ bc+1\overset{CBS}{\leq}\sqrt{b^2+1}\sqrt{c^2+1},\ ca+1\overset{CBS}{\leq}\sqrt{c^2+1}\sqrt{a^2+1}$$

Multiplying last three inequalities, we find (1) is true (Proved)

336.

$$\Omega(x,y)=\sum_{n=1}^{\infty}\frac{2n^2+(2x+2y+5)n+2xy+6x-y}{3^n(n+y)(n+y+1)(n+y+2)},\ x,y>0$$

Prove that:

$$\Omega(x,y)\cdot\Omega(y,x)\leq\frac{1}{9\sqrt[3]{xy}}$$

Proposed by Daniel Sitaru – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\Omega(x,y)=\sum_{n}^{\infty}\frac{2n^2+(2x+2y+5)n+2xy+6x-y}{3^n(n+y)(n+y+1)(n+y+2)}=$$

$$=\sum_{n=1}^{\infty}\left(\frac{n+x}{3^{n-1}(n+y)(n+y+1)}-\frac{n+x+1}{3^n(n+y+1)(n+y+2)}\right)=\frac{x+1}{(y+1)(y+2)}\Rightarrow$$

$$\Rightarrow\Omega(x,y)\cdot\Omega(y,x)=\frac{1}{x+1+1}\cdot\frac{1}{y+1+1}\leq\frac{1}{3\sqrt[3]{x}}\cdot\frac{1}{3\sqrt[3]{y}}$$

$$\Omega(x,y)\cdot\Omega(y,x)\leq\frac{1}{9\sqrt[3]{xy}}$$

337. If $x>0$ then:

$$\left(e^{x^2}+e^{(x+3)^2}\right)\left(\frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\right)>\left(e^{(x+1)^2}+e^{(x+2)^2}\right)\left(\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $f(x)=e^{x^2}-e^{(x+1)^2}\ \forall x>0;\ f'(x)\overset{(1)}{=}-2\left((x+1)e^{(x+1)^2}-xe^{x^2}\right)$

Now, $(x+1)^2(\ln e)>x^2(\ln e)(\because 2x+1>0\ as\ x>0)\Rightarrow e^{(x+1)^2}\overset{(i)}{>}e^{x^2}$

Also, $x+1\overset{(ii)}{>}x\ \&\because x>0\therefore$ *(i).(ii)* $\Rightarrow (x+1)e^{(x+1)^2}-xe^{x^2}>0\Rightarrow$

$\Rightarrow f'(x)<0$ *(by (1))* $\therefore f(x)\downarrow\therefore e^{x^2}-e^{(x+1)^2}<e^{(x+2)^2}-e^{(x+3)^2}\Rightarrow$

$\Rightarrow e^{x^2}+e^{(x+3)^2}\overset{(a)}{>}e^{(x+1)^2}+e^{(x+2)^2}$. *Now, let* $g(x)=\frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}\ \forall x>0$

$$g'(x)=\frac{e^{x+1}(e^x+1)^2-e^x(e^{x+1}+1)^2}{(e^{x+1}+1)^2(e^x+1)^2}=\frac{et(t+1)^2-t(et+1)^2}{(et+1)^2(t+1)^2}\ (t=e^x)$$

$$=\frac{et(t^2+2t+1)-t(e^2t^2+2et+1)}{(et+1)^2(t+1)^2}=\frac{t(1-e)(et^2-1)}{(et+1)^2(t+1)^2}<0$$

$$\left(\because et^2>1\ as\ t=e^x>1\ (\because x>0)\right)\therefore g(x)\downarrow$$

$$\therefore \frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}>\frac{1}{1+e^{x+2}}-\frac{1}{1+e^{x+3}}\Rightarrow$$

$$\Rightarrow \frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\overset{(b)}{>}\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}$$

(a).(b) $\Rightarrow$ ***given inequality is true (proved)***

338.

$$\Omega(x)=-\frac{1}{2}+4\sum_{n=1}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)},x\in\mathbb{R}$$

If $a\in(0,1),b>1$ **then:**

$$\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<1+\Omega(a)\cdot\Omega(b)$$

Proposed by Daniel Sitaru – Romania

Solution by Dimitris Kastriotis-Athens-Greece

$$\Omega(x)=-1+4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)},x\in\mathbb{R}$$

$$\Rightarrow\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<\Omega(a)\Omega(b)+1,0<a<1,b>1$$

$$\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{n+2}\cdot\frac{1}{(n+1)(n+3)}=\frac{1}{n+2}\left(\frac{1}{2(n+1)}-\frac{1}{2(n+3)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\right)$$

$$S_1=\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{2}\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\right)$$

$$=\frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{(N+2)(N+3)}\right)=\frac{1}{4}$$

$$\frac{n}{(n+1)(n+2)(n+3)}=\frac{1}{(n+2)(n+3)}-\frac{1}{(n+1)(n+2)(n+3)}$$

$$S_2=\sum_{n=0}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}=\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+3)}-S_1$$

$$=\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-S_1=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+3}\right)-\frac{1}{4}=\frac{1}{4}$$

$$\Omega(x)=-1+4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)}=-1+4(S_2\cdot x+S_1)=$$

$$=-1+4\left(\frac{1}{4}+\frac{1}{4}x\right)=x$$

$$\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<\Omega(a)\Omega(b)+1$$

$$\Leftrightarrow a^b+b^a-ab-1<0,0<a<1,b>1$$

$$\text{Let } f(b)=a^b+b^a-ab-1,0<a<1,b>1$$

$$f'(b)=a^b\log(a)+ab^{a-1}-a=a^b\log(a)-a(1-b^{a-1})<0\ \forall b>1\Rightarrow f\searrow(1,\infty)$$

$$\text{For } b>1\Leftrightarrow f(b)<f(1)=0\Leftrightarrow a^b+b^a<1+ab,0<a<1,b>1$$

339. Let x,y,z be positive real numbers such that: $x^2+y^2+z^2=3$.

Find the minimum of value:

$$P=\frac{x}{\sqrt{y}+\sqrt{z}}+\frac{y}{\sqrt{z}+\sqrt{x}}+\frac{z}{\sqrt{x}+\sqrt{y}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Do Huu Duc Thinh-Vietnam

Let $x, y, z > 0$ *such that* $x^2 + y^2 + z^2 = 3$. *Find* $Min: P = \sum \frac{x}{\sqrt{y}+\sqrt{z}}$

By Cauchy-Schwarz we have: $P = \sum \frac{x^2}{x\sqrt{y}+x\sqrt{z}} \geq \frac{(x+y+z)^2}{\sum x\sqrt{y}+\sum y\sqrt{x}} \geq \frac{(x+y+z)^2}{2\sqrt{(x+y+z)(xy+yz+zx)}}$

Let $t = x + y + z$ *then* $0 < t \leq 3$ *and* $xy + yz + zx = \frac{t^2-3}{2}$. *We will prove that:*

$$\frac{t^2}{2\sqrt{t \cdot \frac{t^2-3}{2}}} \geq \frac{3}{2} \Leftrightarrow t^4 \geq \frac{9(t^3-3t)}{2} \Leftrightarrow t(2t^3 - 9t^2 + 27) \geq 0 \Leftrightarrow t(t-3)^2(2t+3) \geq 0 \quad \textit{(true)}$$

So, $P \geq \frac{3}{2} \Rightarrow P_{Min} = \frac{3}{2} \Leftrightarrow x = y = z = 1$.

340. If $x, y \in \mathbb{R}$ then:

$$\frac{5\sin^2 x}{1+\cos^2 x} + \frac{5\cos^2 x \cdot \sin^2 y}{1+\sin^2 x + \cos^2 x \cdot \cos^2 y} + \frac{5\cos^2 x \cdot \cos^2 y}{1+\sin^2 x + \cos^2 x \cdot \sin^2 y} \geq 3$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijian

$$\frac{5\sin^2 x}{1+1-\sin^2 x} = \frac{5\sin^2 x}{2-\sin^2 x};$$

$$\frac{5\cos^2 x \cdot \sin^2 y}{1+\sin^2 x + \cos^2 x \cdot (1-\sin^2 y)} = \frac{5\cos^2 x \cdot \sin^2 y}{2-\cos^2 x \cdot \sin^2 y}$$

$$\frac{5\cos^2 x \cdot \cos^2 y}{1+\sin^2 x + \cos^2 x \cdot (1-\cos^2 y)} = \frac{5\cos^2 x \cdot \cos^2 y}{2-\cos^2 x \cdot \cos^2 y}$$

We take the function $f(x) = \frac{5x}{2-x}$, *this function is convex,* $f''(x) = \frac{20}{(2-x)^3} > 0$

then by Jensen's inequality, we have

$$\frac{f(\sin^2 x) + f(\cos^2 x \sin^2 y) + f(\cos^2 x \cos^2 y)}{3} \geq f\left(\frac{\sin^2 x + \cos^2 x \sin^2 y + \cos^2 x \cos^2 y}{3}\right)$$

or $f(\sin^2 x) + f(\cos^2 x \sin^2 y) + f(\cos^2 x \cos^2 y) \geq 3 \cdot f\left(\frac{1}{3}\right)$

(since $\sin^2 x + \cos^2 x \sin^2 y + \cos^2 x \cdot \cos^2 y = 1$*)*

$f\left(\frac{1}{3}\right) = \frac{5 \cdot \frac{1}{3}}{2-\frac{1}{3}} = 1$, *we have:* $f(\sin^2 x) + f(\cos^2 x \cdot \sin^2 y) + f(\cos^2 x \cdot \cos^2 y) \geq 3$

341. If $a,b,c>0$ then:

$$\frac{9+4a+4a^2}{1+a}+\frac{9+4b+4b^2}{1+b}+\frac{9+4c+4c^2}{1+c}\ge 24$$

Proposed by Eliezer Okeke-Nigeria

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to\mathbb{R},f(a)=\frac{9+4a+4a^2}{1+a},f'(a)=\frac{(2a+5)(2a-1)}{(1+a)^2}$$

$$min(f(a))=f\left(\frac{1}{2}\right)=8\to f(a)\ge 8;\ f(a)+f(b)+f(c)\ge 8+8+8=24$$

342. If $a,b,c,d\in\mathbb{N}-\{0\},a>b>c>d$ then:

$$bd(2^a-1)(2^c-1)>ac(2^b-1)(2^d-1)$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan

$$bd(2^a-1)(2^c-1)>ac(2^b-1)(2^d-1)\quad (1)$$

$$(1)\Rightarrow\frac{2^a-1}{a}\cdot\frac{2^c-1}{c}>\frac{2^b-1}{b}\cdot\frac{2^d-1}{d}$$

denote $f(x)=\frac{2^x-1}{x}$, *we prove that* f *increasing function*

$$f'(x)=\frac{2^x\cdot\ln 2\cdot x-2^x+1}{x}=\frac{2^x(\ln 2^x-1)+1}{x^2}>0\Rightarrow f\uparrow$$

then we have $\otimes\begin{cases}\frac{2^a-1}{a}>\frac{2^b-1}{b} & (2)\\ \frac{2^c-1}{c}>\frac{2^d-1}{d} & (3)\end{cases}\Rightarrow f(a)\cdot f(c)>f(b)\cdot f(d)$

343. If $x,y>0,x+2y\le 5,3x+y\ge 7,(x+2y)(3x+y)\ge 20$ then:

$$4x+3y\ge 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Van Nho-Nghe An-Vietnam

Put: $a=x+2y,b=3x+y\to a\le 5,b\ge 7$ *(*) and* $ab\ge 20$

a, b are two solutions of the quadratic triangles: $f(t) = t^2 - (a+b)t + ab$

Because (*) then $1.\ f(5) \le 0$ and $1.\ f(7) \le 0 \to 25 - 5(a+b) + ab \le 0 \to$

$\to 5(a+b) \ge 25 + ab \ge 25 + 20 = 45 \to a + b \ge 9 \to 4x + 3y \ge 9$ (done)

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $a = x + 2y$ & $b = 3x + y$. Then, $a \le 5, b \ge 7, ab \ge 20$. We have $b - 2 \ge 5 \ge a \Rightarrow$

$$\Rightarrow b - a \ge 2 \Rightarrow (b-a)^2 \ge 4 \Rightarrow (a+b)^2 - 4ab \ge 4 \Rightarrow (a+b)^2 \ge 4 + 4ab \overset{ab\ge 20}{\ge} 84 \Rightarrow$$

$$\Rightarrow a + b \ge \sqrt{84} > \sqrt{81} = 9 \therefore a + b > 9 \Rightarrow 4x + 3y > 9 \text{ or, } 4x + 3y \ge 9 \text{ (proved)}$$

Solution 3 by Tran Hong-Vietnam

Let $y = tx; t > 0$. We have:

$$\left.\begin{array}{l} x + 2y \le 5 \Leftrightarrow x \le \dfrac{5}{1+2t} \\ 3x + y \ge 7 \Leftrightarrow x \ge \dfrac{7}{3+t} \end{array}\right\} \Rightarrow \frac{7}{3+t} \le x \le \frac{5}{1+27} \Rightarrow 9t \le 8 \Rightarrow t \le \frac{8}{9}$$

$$(x+2y)(3x+y) \ge 20 \Leftrightarrow x^2 \ge \frac{20}{(1+2t)(3+2t)} \Rightarrow x^2 \ge \frac{324}{175}; \forall t \le \frac{8}{9} \Rightarrow x \ge \frac{18\sqrt{7}}{35} \text{ (1)}$$

We need to prove: $4x + 3y \ge 9 \Leftrightarrow x \ge \frac{9}{4+3t}$. In fact: $\frac{7}{3+t} \ge \frac{9}{4+3t} \Leftrightarrow \frac{12t+1}{(3+t)(4+3t)} > 0$ (true)

$$\Rightarrow x \ge \frac{7}{3+t} \ge \frac{9}{4+3t}; \forall t \in \left(0, \frac{8}{9}\right] \text{ and } t \le \frac{8}{9} \Rightarrow \frac{7}{3+t} \ge \frac{9}{5} > \frac{18\sqrt{7}}{35} \text{ (true)} \Rightarrow \text{(proved)}$$

344. If $0 < x < \frac{\pi}{2}$ then:

$$\pi \cdot e^{\sum_{k=1}^{n} \log\left(\cos\left(\frac{x}{2^k}\right)\right)} > 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $0 < x < \frac{\pi}{2}; 0 < \cos\left(\frac{x}{2^k}\right) < 1, \forall k \in \mathbb{N}$. Let $a_n = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right) \dots \cos\left(\frac{x}{2^n}\right)$

Note $a_{n+1} < a_n \Rightarrow < a_n >$ is a strictly decreasing sequence. Also

$$2^n \sin\left(\frac{x}{2^n}\right) a_n = 2^{n-1}\left[2\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^n}\right)\right]\cos\left(\frac{x}{2^{n-1}}\right) \dots \cos\left(\frac{x}{2}\right) =$$

$$= 2^{n-2}\left[2\sin\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^{n-1}}\right)\right] \dots \cos\left(\frac{x}{2}\right) = \dots = \sin x \Rightarrow a_n = \frac{\sin(x)}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin x}{x} \cdot \frac{\frac{n}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}(1) = \frac{\sin x}{x}$$

As $< a_n >$ ***is strictly increasing and*** $\lim_{n\to\infty} a_n = \frac{\sin x}{x}$

$$a_n > \frac{\sin x}{x}; \forall n \in \mathbb{N} \quad \textbf{(1)}$$

$$\left[\frac{\sin x}{x} = g/b(a_n)\right]$$

Also, for $0 < x < \frac{\pi}{2}$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x) < 0 \Rightarrow \frac{\sin x}{x} \textit{ is strictly decreasing on } \left(0, \frac{\pi}{2}\right] \Rightarrow$$

$$\Rightarrow \frac{\sin x}{x} > \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \textit{ for } 0 < x < \frac{\pi}{2} \quad \textbf{(2)}$$

From (1), (2): $a_n > \frac{2}{\pi}, \forall n \in \mathbb{N}$***. Now,***

$$\sum_{k=1}^{n} \log\left(\cos\frac{x}{2^k}\right) = \log a_n > \log\left(\frac{2}{\pi}\right) \Rightarrow \prod e^{\sum_{k=1}^{n} \log\cos\left(\frac{x}{2^k}\right)} > \prod e^{\log\left(\frac{2}{\pi}\right)} = 2$$

Solution 2 by Michel Rebeiz-Lebanon

$$0 < x < \frac{\pi}{2}. \textit{ Let } P_n = e^{\sum_{k=1}^{n} \ln\cos\frac{x}{2^k}} = \cos\frac{x}{2} \dots \cos\frac{x}{2^n} \overset{or}{\equiv} \sin\frac{x}{2^n} \cdot P_n = \frac{1}{2^n}\sin x$$

$$\therefore \left[\frac{1}{2}\sin 2x = \sin x \cdot \cos x\right]. \textit{ Let } f(x) = \frac{\sin x}{x}$$

x	0	$2^{\frac{x}{n}}$	x	$\frac{\pi}{2}$
$f(x)$	1	↘		$\frac{2}{\pi}$

$$1 > \frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}} > \frac{\sin x}{x} > \frac{2}{\pi} \Rightarrow \frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} > 1 \Rightarrow \frac{\sin x}{x} \cdot \frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} > \frac{\sin x}{x} > \frac{2}{\pi}$$

$$\pi \cdot P_n = \pi \cdot \frac{1}{2^n} \cdot \frac{\sin x}{\sin\frac{x}{2^n}} = \pi \cdot \frac{\sin x}{x} \cdot \frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} > \pi \cdot \frac{2}{\pi} = 2$$

Solution 3 by Shafiqur Rahman-Bangladesh

$$\sum_{k=1}^{n}\log\left(\cos\left(\frac{x}{2^k}\right)\right)=\log\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)=\log\prod_{k=1}^{n}\frac{\sin\left(\frac{x}{2^{k-1}}\right)}{2\sin\left(\frac{x}{2^k}\right)}=\log\frac{\sin(x)}{2^n\sin\left(\frac{x}{2^n}\right)}\ge$$

$$\ge\lim_{n\to\infty}\log\frac{\sin(x)}{2^n\sin\left(\frac{x}{2^n}\right)}=\log\left(\frac{\sin x}{x}\right)>\log\left(\frac{2}{\pi}\right)\left[For\ 0<x<\frac{\pi}{2},\sin x>\frac{2x}{\pi}\right]$$

$$\therefore\pi-e^{\sum_{k=1}^{n}\log\left(\cos\left(\frac{x}{2^k}\right)\right)}>2$$

Solution 4 by Sagar Kumar-Patna Bihar-India

$$\pi\cdot e^{\sum_{k=1}^{n}\log\left(\cos\left(\frac{x}{2^n}\right)\right)}>2\Rightarrow \textit{To prove}\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)>\frac{2}{\pi}\Rightarrow S=\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

$$S=\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cdot\ldots\cdot\cos\left(\frac{x}{2^n}\right);\ x=2^n\theta$$

$$S=\cos(\theta)\cos(2\theta)\cdot\ldots\cdot\cos(2^{n-1}\theta);\ S=\frac{\sin(2^n\theta)}{2^n\sin\theta}=\frac{\sin\left(2^n\cdot\frac{x}{2^n}\right)}{2^n\sin\left(\frac{x}{2^n}\right)};\ S=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$\lim_{n\to\infty}S=\lim_{n\to\infty}\frac{x\left(\frac{\sin x}{x}\right)}{x\frac{\sin\left(\frac{x}{2^n}\right)}{\left(\frac{x}{2^n}\right)}}=1>\frac{2}{\pi}.\ \textit{Hence proved.}$$

345. Let x,y,z be positive real numbers such that $x+y+z=3$. Find the minimum of value:

$$P=\frac{x}{\sqrt{\frac{y^4+z^4}{2}}+2yz}+\frac{y}{\sqrt{\frac{z^4+x^4}{2}}+2zx}+\frac{z}{\sqrt{\frac{x^4+y^4}{2}}+2xy}+\frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Amit Dutta-Jamshedpur-India

Applying Cauchy's Schwarz inequality:

$$\left(\sqrt{2(y^4+z^4)}+2yz\right)^2\le(1^2+1^2)(2(y^4+z^4)+4y^2z^2)\le2\left(2(y^4+z^4+2y^2z^2)\right)$$

$$\le4(y^2+z^2)^2\Rightarrow\sqrt{2(y^4+z^4)}+2yz\le2(y^2+z^2)$$

$$\sqrt{2(y^4+z^4)}\le2(y^2-yz+z^2)\Rightarrow\sqrt{\frac{y^4+z^4}{2}}\le(y^2-yz+z^2)\Rightarrow$$

$\Rightarrow \sqrt{\frac{y^4+z^4}{2}} + 2yz \le (y^2 + yz + z^2)$ ***Similarly,*** $\sqrt{\frac{x^4+z^4}{2}} + 2xz \le (x^2 + xz + z^2)$

$$\sqrt{\frac{x^4 + y^4}{2}} + 2xy \le (x^2 + xy + y^2)$$

$$P \ge \frac{x}{y^2 + yz + z^2} + \frac{y}{x^2 + xz + z^2} + \frac{z}{x^2 + xy + y^2} + \frac{\sqrt[5]{x} + \sqrt[5]{y} + \sqrt[5]{z}}{18}$$

$$P \ge \sum \frac{x^2}{xy^2 + xyz + xz^2} + \frac{\sqrt[5]{x} + \sqrt[5]{y} + \sqrt[5]{z}}{18}$$

$$P \overset{Bergstrom}{\ge} \frac{(x + y + z)^2}{(xy^2 + x^2y + xyz) + (y^2z + z^2y + xyz) + (x^2z + z^2x + xy^2)} +$$

$$+ \frac{\sqrt[5]{x} + \sqrt[5]{y} + \sqrt[5]{z}}{18}$$

$$P \ge \frac{(x + y + z)^2}{(x + y + z)(xy + yz + xz)} + \frac{\sqrt[5]{x} + \sqrt[5]{y} + \sqrt[5]{z}}{18}$$

$$P \ge \frac{(x+y+z)}{\sum xy} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18},\ P \ge \frac{3}{\sum xy} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

Using AM-GM

$$\sqrt[5]{x} + \sqrt[5]{x} + \sqrt[5]{x} + \sqrt[5]{x} + \sqrt[5]{x} + x^3 + x^3 + x^3 + 1 + 1 \ge 10x \Rightarrow 5(x)^{\frac{1}{5}} + 3 \cdot x^3 + 2 \ge 10x$$

$$5(y)^{\frac{1}{5}} + 3y^3 + 2 \ge 10y,\ 5(z)^{\frac{1}{5}} + 3z^3 + 2 \ge 10z$$

Adding these: $5\left(x^{\frac{2}{5}} + y^{\frac{2}{5}} + z^{\frac{2}{5}}\right) + 3(x^3 + y^3 + z^3) + 6 \ge 10(x + y + z) \Rightarrow$

$$\Rightarrow 5\left(x^{\frac{1}{5}} + y^{\frac{1}{5}} + z^{\frac{1}{5}}\right) \ge 10 \cdot (3) - 6 - 3(x^3 + y^3 + z^3) \ge 30 - 6 - 3(x^3 + y^3 + z^3)$$

$$5\left(x^{\frac{1}{5}} + y^{\frac{1}{5}} + z^{\frac{1}{5}}\right) \ge 24 - 3(x^3 + y^3 + z^3) \quad (1)$$

Now, since $x + y + z = 3 \Rightarrow (x - 3) = -y - z \Rightarrow (x - 3) < 0 \because y, z > 0$

Similarly, $(y - 3) < 0, (z - 3) < 0$

Clearly, $(x - 3)(x - 1)^2 + (y - 3)(y - 1)^2 + (z - 3)(z - 1)^2 \le 0 \Rightarrow$

$$\Rightarrow \sum x^3 - 5\sum x^2 + 7\sum x - 9 \le 0$$

$$\sum x^3 \le 5\left(\sum x^2\right) + 9 - 7\sum x \le 5\left[(x + y + z)^2 - 2\sum xy\right] + 9 - 7 \times (3)$$

$$\le 5\left(3^2-2\sum xy\right)+9-21\le 45-10\sum xy-12;\ \sum x^3\le 33-10\sum xy$$

$$\because P\ge \frac{3}{\sum xy}+\left\{\frac{(x)^{\frac{1}{5}}+(y)^{\frac{1}{5}}+(z)^{\frac{1}{5}}}{18}\right\},\ P\ge \frac{3}{\sum xy}+\left[\frac{24-3(\sum x^3)}{90}\right]\ \textit{\{From (1)\}}$$

$$P\ge \frac{3}{\sum xy}+\frac{24-3(33-10\sum xy)}{90},\ P\ge \frac{3}{\sum xy}+\left(\frac{30\sum xy-75}{90}\right)$$

$$P\ge \frac{3}{\sum xy}+\frac{\sum xy}{3}-\frac{75}{90},\ P\ge \frac{3}{\sum xy}+\frac{\sum xy}{3}-\frac{5}{6},\ P\overset{AM-GM}{\ge} 2-\frac{5}{6};\ P\ge \frac{7}{6}$$

∴ minimum value of P is $\left(\frac{7}{6}\right)$. Equality occurs when $(x=y=z=1)$.

346. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$, $\sin x+\sin y+\sin z=1$ then:

$$\cos^2 x\cdot\cos^2 y\cdot\cos^2 z\ge 512\sin^2 x\cdot\sin^2 y\cdot\sin^2 z$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $\sin x=a,\sin y=b,\sin z=c\because x,y,z\in\left(0,\frac{\pi}{2}\right)\therefore a,b,c\in(0,1)$ *&* $\sum a=1$. *Now,*

$$\cos^2 x\cos^2 y\cos^2 z=(1-a^2)(1-b^2)(1-c^2)=$$

$$=\{(a+b+c)^2-a^2\}\{(a+b+c)^2-b^2\}\{(a+b+c)^2-c^2\}=$$

$$=(2a+b+c)(2b+c+a)(2c+a+b)(a+b)(b+c)(c+a)\overset{Cesaro}{\ge}$$

$$\ge\{(a+b)+(c+a)\}\{(b+c)+(a+b)\}\{(b+c)+(c+a)\}8abc$$

$$\overset{A-G}{\ge}\left\{2\sqrt{(a+b)(c+a)}\right\}\left\{2\sqrt{(b+c)(a+b)}\right\}\left\{2\sqrt{(b+c)(c+a)}\right\}8abc=$$

$$=64abc(a+b)(b+c)(c+a)\overset{Cesaro}{\ge}64abc\cdot 8abc=512a^2b^2c^2=$$

$$=512\sin^2 x\sin^2 y\sin^2 z\ \ \textit{(proved)}$$

347. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$, $x+y+z=\pi$ then:

$$\frac{xy(\tan x+\sin x)}{x^2+\sin x\cdot\tan x}+\frac{yz(\tan y+\sin y)}{y^2+\sin y\cdot\tan y}+\frac{zx(\tan z+\sin z)}{z^2+\sin z\cdot\cos z}>\pi$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$0<x<\frac{\pi}{2},\tan x>x \text{ and } x>\sin x\Rightarrow(\tan x-x)(x-\sin x)>0$$

$$\Rightarrow x\tan x-x^2-\sin x\tan x+x\sin x>0\Rightarrow x(\tan x+\sin x)>x^2+\sin x\tan x\Rightarrow$$

$$\Rightarrow\frac{xy(\tan x+\sin x)}{x^2+\sin x\tan x}\overset{(1)}{>}y.\text{ Similarly, }\frac{yz(\tan y+\sin y)}{y^2+\sin y\tan y}\overset{(2)}{>}z\ \&\ \frac{zx(\tan z+\sin z)}{z^2+\sin z\tan z}\overset{(3)}{>}x$$

$$(1)+(2)+(3)\Rightarrow LHS>x+y+z=\pi\ \text{(Proved)}$$

348. If $x_1,x_2,\dots x_n>0,n\in\mathbb{N},n\geq 2,x_1x_2\cdot\ldots\cdot x_n=1$ then:

$$\frac{x_1e^{x_1}+x_2e^{x_2}+\cdots+x_ne^{x_n}}{x_1+x_2+\cdots+x_n}\geq e$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to\mathbb{R},f(x)=xe^x,f'(x)=(x+1)e^x>0,f-increasing,$$

$$f''(x)=(x+2)e^x>0,f-convexe$$

$$\sum_{i=1}^{n}f(x_i)=\sum x_ie^{x_i}\overset{JENSEN}{\geq}nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\overset{AM-GM}{\geq}n\cdot\frac{1}{n}\sum_{i=1}^{n}x_i\cdot e^{\frac{1}{n}\sum_{i=1}^{n}x_i}$$

$$\frac{x_1e^{x_1}+x_2e^{x_2}+\cdots+x_ne^{x_n}}{x_1+x_2+\cdots+x_n}\geq e^{\frac{1}{n}\sum_{i=1}^{n}x_i}\overset{AM-GM}{\geq}e^{\sqrt[n]{\prod_1^n x_i}}=e^1=e$$

349. If $x,y,z,t\in\left(0,\frac{\pi}{2}\right)$ then:

$$64\cdot\cos x\cdot\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\cdot\sin(z-t)\leq 1$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

We must show this:

$$\cos x\cos z\cdot\sin y\sin t\,(\sin x\cos y-\cos x\sin y)(\sin z\cos t-\cos z\sin t)\leq\frac{1}{64}\quad (1)$$

We show this: $\cos x\cdot\sin y\,(\sin x\cos y-\cos x\sin y)\leq\frac{1}{8}$ *(2)*

$$\left.\begin{array}{c}\cos x = a, \sin y = b \ \ (2) \Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)} - ab\right) \le \frac{1}{8} \\ \textit{But } \sqrt{(1-a^2)(1-b^2)} \le \frac{2-a^2-b^2}{2}\end{array}\right\} \Rightarrow$$

$$\Rightarrow ab\left(\frac{2-a^2-b^2}{2} - ab\right) \le \frac{1}{8} \Leftrightarrow ab(2-a^2-b^2-2ab) \le \frac{1}{4} \Leftrightarrow$$

$$4ab(2-(a+b)^2) \le 1 \quad \textit{(3)}$$

$$\textit{But } (a+b)^2 \ge 4ab \Rightarrow -(a+b)^2 \le -4ab \quad \textit{(4)}$$

From (3) + (4) $\Rightarrow 4ab(2-4ab) \le 1 \Leftrightarrow 8ab - 16a^2b^2 \le 1 \Leftrightarrow 16a^2b^2 - 8ab + 1 \ge 0 \Leftrightarrow$

$(4ab-1)^2 \ge 0$ *true (equality for* $a = b = \frac{1}{2}$*).*

Similarly: $\cos z \sin t \sin(z-t) \le \frac{1}{8}$ *(5)*

From (2)+(5) $\Rightarrow \cos x \cos z \cdot \sin y \sin t \cdot \sin(x-y)\sin(z-t) \le 1,$

with equality for $x = z = \frac{\pi}{3}$ *and* $y = t = \frac{\pi}{6}$*.*

350. If $x, y, z \in (0,1)$ then:

$$\sum_{cyc(x,y,z)} \frac{y(\sin^{-1} x + \tan^{-1} x)}{x^2 + \tan^{-1} x \cdot \sin^{-1} x} > 3$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

$$\because \sin^{-1} x > x \Rightarrow (\sin^{-1} x - x) > 0 \quad \textit{(1)}$$

$$\tan^{-1} x < x \Rightarrow (x - \tan^{-1} x) > 0 \quad \textit{(2)}$$

Multiplying (1) & (2): $(\sin^{-1} x - x)(x - \tan^{-1} x) > 0 \Rightarrow$

$$\Rightarrow x\sin^{-1} x - \sin^{-1} x \tan^{-1} x - x^2 + x\tan^{-1} x > 0 \Rightarrow$$

$$\Rightarrow x(\sin^{-1} x + \tan^{-1} x) > x^2 + \tan^{-1} x \sin^{-1} x$$

$$\Rightarrow \frac{y(\sin^{-1} x + \tan^{-1} x)}{x^2 + \tan^{-1} x \sin^{-1} x} > \frac{y}{x} \quad \textit{(3)}$$

Similarly, $\frac{z(\sin^{-1} y + \tan^{-1} y)}{y^2 + \tan^{-1} y \sin^{-1} y} > \frac{z}{y}$ *(4)*

$$\frac{x(\sin^{-1} z + \tan^{-1} z)}{z^2 + \tan^{-1} z \sin^{-1} z} > \frac{x}{z} \quad \textit{(5)}$$

Adding (1), (2), (3):

$$\sum_{cyc(x,y,z)} \frac{y(\sin^{-1}x + \tan^{-1}x)}{x^2 + \tan^{-1}x\sin^{-1}x} > \left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) \overset{AM-GM}{>} 3\sqrt[3]{\frac{xyz}{xyz}} > 3$$

(proved)

351. **If $x \geq 0$ then:**

$$\sin x\,(16\sin^4 x + 5) \leq 5x(4x^2 + 1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $x = 0$, the inequality clearly holds. For $0 < x \leq 1; 0 < \sin x < x \leq 1$

$$\sin^5 x < \sin^3 x < x^3 \Rightarrow$$

$$\Rightarrow 16\sin^5 x < 16x^3 < 20x^3 \quad (1)$$

Also, for $0 < x \leq 1$, $\sin x \leq x \Rightarrow$

$$\Rightarrow 5\sin x \leq 5x \quad (2)$$

Adding (1) and (2), we get, for $0 < x \leq 1$

$$16\sin^5 x + 5\sin x \leq 20x^3 + 5x \Rightarrow \sin x\,(16\sin^4 x + 5) \leq 5x(4x^2 + 1)$$

If $x > 1$, then $LHS \leq 21$ and $RHS \geq 25$. Thus, for all $x \geq 0$, the inequality holds

Solution 2 by Nguyen Van Nho-Nghe An-Vietnam

$\oplus$ ***case 1. If $x \geq 1$ then $LHS \leq 1.(16 + 5) = 21(|\sin x| \leq 1)$ and $RHS \geq 25 \rightarrow LHS < RHS$***

$\oplus$ ***case 2. If $0 \leq x < 1$ then $\sin x \leq x \rightarrow LHS \leq x(16x^4 + 5) \overset{??}{\leq} 5x(4x^2 + 1) \rightarrow$ (*)***

(*) $\leftrightarrow x(16x^4 - 20x^2) \leq 0 \leftrightarrow 4x^3(4x^2 - 5) \leq 0$ true because $0 \leq x < 1$

From case 1 and case 2 then the inequality is true .

352. **If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, $\cos x \cdot \cos y \cdot \cos z = \frac{\sqrt{2}}{2}$ then:**

$$15(\cos 2x + \cos 2y + \cos 2z) + 6(\cos 4x + \cos 4y + \cos 4z) +$$
$$+ \cos 6x + \cos 6y + \cos 6z \geq 18$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos 6x = 4\cos^3 2x - 3\cos 2x \overset{(i)}{=} 4(2\cos^2 x - 1)^3 - 3\cos 2x$$

$$6\cos 4x = 6(2\cos^2 2x - 1) \overset{(ii)}{=} 12(2\cos^2 x - 1)^2 - 6$$

(i)+(ii)$\Rightarrow 15\cos 2x + 6\cos 4x + \cos 6x$

$$= 12\cos 2x + 4\left(2\cos^2 x - 1\right)^3 + 12\left(2\cos^2 x - 1\right)^2 - 6$$

$$= 24\cos^2 x + 4(2\cos^2 x - 1)^3 + 12(2\cos^2 x - 1)^2 - 18 \overset{(a)}{=} 32\cos^2 x - 10$$

Similarly, $15\cos 2y + 6\cos 4y + \cos 6y \overset{(b)}{=} 32\cos^6 y - 10$

& $15\cos 2z + 6\cos 4z + \cos 6z \overset{(c)}{=} 32\cos^6 z - 10$

(a)+(b)+(c)$\Rightarrow$ ***LHS*** $= 32\sum\cos^6 x - 30 \overset{?}{\geq} 18 \Leftrightarrow \sum\cos^6 x \underset{(1)}{\overset{?}{\geq}} \frac{3}{2}$

$$\because x, y, z \in \left(0, \frac{\pi}{2}\right), \therefore \cos^6 x, \cos^6 y, \cos^6 z > 0$$

$$\therefore \sum\cos^6 x \overset{A-G}{\geq} 3\sqrt[3]{\cos^6 x \cdot \cos^6 y \cdot \cos^6 z} = 3(\cos x\cos y\cos z)^2 = 3\left(\frac{2}{4}\right) = \frac{3}{2} \Rightarrow$$

$\Rightarrow$ ***(1) is true (Proved)***

Solution 2 by Tran Hong-Vietnam

$$\cos 6x = 32t^6 - 48t^4 + 18t^2 - 1;\ \cos 4x = 8t^4 - 8t^2 + 1;\ \cos 2x = 2t^2 - 1$$

$$(t = \cos x > 0) \Rightarrow \cos 6x + 6\cos 4x + 15\cos 2x = 32t^6 - 10.$$

Same: $\cos 6y + 6\cos 4x + 15\cos 2y = 32u^6 - 10.\ (u = \cos y > 0)$

and: $\cos 6z + 6\cos 4z + 15\cos 2z = 32v^6 - 10.$

$$(v = \cos z > 0) \Rightarrow LHS = 32(t^6 + u^6 - v^6) - 30$$

$$= 2\{(16t^6 - 5) + (16u^6 - 5) + (16v^6 - 5)\}$$

$$\overset{(Jensen)}{\geq} 2 \cdot 3\left\{16\left(\frac{u+t+v}{3}\right)^6 - 5\right\} \geq 2 \cdot 3 \cdot 3 = 18.$$

$\left(u + t + v \geq 3\sqrt[3]{utv} = 3\sqrt[3]{\frac{\sqrt{2}}{2}}\right)$ ***Proved.***

Solution 3 by Remus Florin Stanca-Romania

$$\cos 4x = 2\cos^2 2x - 1; \cos 4y = 2\cos^2 2y - 1; \cos 4z = 2\cos^2 2z - 1$$

$\cos(6x) = 4\cos^3 2x - 3\cos 2x$***. The inequality can be written as:***

$$15(\cos 2x + \cos 2y + \cos 2z) + 6(2\cos^2 2x + 2\cos^2 2y + 2\cos^2 2z - 3) +$$

$$+4(\cos^3 2x+\cos^3 2y+\cos^3 2z)-3(\cos 2x+\cos 2y+\cos 2z)\ge 18$$

Let $\cos 2x=a, \cos 2y=b, \cos 2z=c$

$$\Leftrightarrow 15(a+b+c)+12(a^2+b^2+c^2)+4(a^3+b^3+c^3)-3(a+b+c)\ge 36$$

$$\Leftrightarrow 12(a+b+c)+4(a^3+b^3+c^3)+12(a^2+b^2+c^2)\ge 36$$

$$\Leftrightarrow 3(a+b+c)+a^3+b^3+c^3+3(a^2+b^2+c^2)\ge 9$$

$$\Leftrightarrow a^3+3a^2+3a+1+b^3+3b^2+3b+1+c^3+3c^2+3c+1\ge 12$$

$$\Leftrightarrow (a+1)^3+(b+1)^3+(c+1)^3\ge 12$$

$$\cos 2x=2\cos^2 x-1\Rightarrow 2\cos^2 x=\cos 2x+1, x\in\left(0;\frac{\pi}{2}\right)\Rightarrow$$

$$\Rightarrow \cos x>0\Rightarrow \cos x=\sqrt{\frac{\cos 2x+1}{2}}=\sqrt{\frac{a+1}{2}}$$

$$\cos x\cos y\cos z=\frac{\sqrt{2}}{2}\Leftrightarrow \frac{\sqrt{(a+1)(b+1)(c+1)}}{2\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow \sqrt{(a+1)(b+1)(c+1)}=2$$

$$\Leftrightarrow (a+1)(b+1)(c+1)=4$$

$$(a+1)^3+(b+1)^3+(c+1)^3\ge 3\sqrt[3]{(a+1)^3(b+1)^3(c+1)^3}=3(a+1)(b+1)(c+1)=3\cdot 4=12$$

$$\Leftrightarrow 15(\cos 2x+\cos 2y+\cos 2z)+6(\cos 4x+\cos 4y+\cos 4z)+\cos 6x+\cos 6y+\cos 6z\ge 18$$

(Q.E.D.)

353. Find $x,y,z\in\left(0,\frac{\pi}{2}\right]$ such that:

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}\le 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Given inequality $\Rightarrow$

$$1-\frac{\sin^2 x(1+\sin^2 y)(1+\sin^2 z)+\sin^2 y(1+\sin^2 z)+\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}\ge\frac{1}{8\sin x\sin y\sin z}$$

$$\Rightarrow \frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}\ge\frac{1}{8\sin x\sin y\sin z}$$

$$\Rightarrow (1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)\overset{(1)}{\le} 8\sin x\sin y\sin z$$

But $(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z) \overset{A-G}{\underset{(2)}{\geq}} 8\sin x\sin y\sin z$

(1), (2) $\Rightarrow (1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z) = 8\sin x\sin y\sin z$

$\Rightarrow$ *Equality case of (2) occurs* $\Rightarrow 1+\sin^2 x = 2\sin x$, *etc* $\Rightarrow (\sin x - 1)^2 = 0$, *etc*

$\Rightarrow \sin x = 1$, *etc* $\Rightarrow x = y = z = \frac{\pi}{2}$

Solution 2 by Tran Hong-Vietnam

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}$$

$$=1-\frac{1}{1+\sin^2 x}+\frac{1}{1+\sin^2 x}-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}-$$

$$-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}=$$

$$=1-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}$$

$$\overset{(Cauchy)}{\geq} 1-\frac{1}{2\sin x\cdot 2\sin y\cdot 2\sin z}+\frac{1}{8\sin x\sin y\sin z}=1$$

$$\Rightarrow LHS = RHS \Leftrightarrow \sin x = \sin y = \sin z \Leftrightarrow x = y = z = \frac{\pi}{2}$$

354. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, $\sin x + \sin y + \sin z = 1$ then:

$$\cos^2 x\cdot\cos^2 y\cdot\cos^2 z \geq 512\sin^2 x\cdot\sin^2 y\cdot\sin^2 z$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $(1-\sin^2 x)(1-\sin^2 y)(1-\sin^2 z) \geq 512\sin^2 x\cdot\sin^2 y\cdot\sin^2 z$ *(1)*

Let $\sin x = m, \sin y = n, \sin z = p, m, n, p > 0$ *(2)*

From (1)+(2) we must show:

$(1-m^2)(1-n^2)(1-p^2) \geq 512m^2n^2p^2$, *with* $m, n, p > 0 \wedge m+n+p = 1$ *(3)*

Let $m = \frac{a}{a+b+c}, n = \frac{b}{a+b+c}, n = \frac{c}{a+b+c}, a, b, c > 0$ *(4)*

Form (3)+(4) we must show:

$$[(a+b+c)^2-a^2][(a+b+c)^2-b^2][(a+b+c)^2-c^2] \geq 512a^2b^2c^2 \Leftrightarrow$$

$$\left[\left(\frac{a+b+c}{a}\right)^2-1\right]\left[\left(\frac{a+b+c}{b}\right)^2-1\right]\left[\left(\frac{a+b+c}{c}\right)^2-1\right]\geq 512\Leftrightarrow$$

$$\left[\left(\frac{b+c}{a}+1\right)^2-1\right]\left[\left(\frac{a+c}{b}+1\right)^2-1\right]\left[\left(\frac{a+b}{c}\right)^2-1\right]\geq 512\Leftrightarrow$$

$$\left[\left(\frac{b+c}{a}\right)^2+2\left(\frac{b+c}{a}\right)\right]\left[\left(\frac{a+c}{b}\right)^2+2\left(\frac{b+c}{a}\right)\right]\left[\left(\frac{a+b}{c}\right)^2+2\left(\frac{a+b}{c}\right)\right]\geq 512\Leftrightarrow$$

$$\left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right)\left(\frac{a+b}{c}\right)\left(\frac{b+c+2a}{a}\right)\left(\frac{a+c+2b}{b}\right)\left(\frac{a+b+2c}{c}\right)\geq 512 \quad \textit{(5)}$$

$$\left.\begin{array}{r}\textit{But } \frac{b+c}{a}\geq 2\sqrt{bc}\\ \frac{a+c}{b}\geq 2\sqrt{ac}\\ \frac{a+b}{c}\geq 2\sqrt{ab}\end{array}\right\}\Rightarrow \frac{b+c}{a}\cdot\frac{a+c}{b}\cdot\frac{a+b}{c}\geq 2^3 \quad \textit{(6)}$$

$$\frac{b+c+a+a}{a}\geq 4\sqrt[4]{a^2bc} \textit{ and similarly}\Rightarrow$$

$$\frac{b+c+2a}{a}\cdot\frac{a+c+2b}{b}\cdot\frac{a+b+2c}{c}\geq 2^6 \quad \textit{(7)}$$

From (6)+(7)⇒ its true.

Solution 2 by Soumava Chakraborty-Kolkata-India

If $x,y,z\in\left(0,\frac{\pi}{2}\right)\mid \sin x+\sin y+\sin z=1$***, then:***

$\cos^2 x\cos^2 y\cos^2 z\geq 512\sin^2 x\sin^2 y\sin^2 z$. ***Let*** $\sin x=a, \sin y=b, \sin z=c$

$\because x,y,z\in\left(0,\frac{\pi}{2}\right)\therefore a,b,c\in(0,1)$ ***&*** $\sum a=1$

Now, $\cos^2 x\cos^2 y\cos^2 z=(1-a^2)(1-b^2)(1-c^2)$

$$=\{(a+b+c)^2-a^2\}\{(a+b+c)^2-b^2\}\{(a+b+c)^2-c^2\}$$

$$=(2a+b+c)(2b+c+a)(2c+a+b)(a+b)(b+c)(c+a)$$

$$\overset{Cesaro}{\geq}\{(a+b)+(c+a)\}\{(b+c)+(a+b)\}\{(b+c)+(c+a)\}8abc$$

$$\overset{Cesaro}{\geq}\left\{2\sqrt{(a+b)(c+a)}\right\}\left\{2\sqrt{(b+c)(a+b)}\right\}\left\{2\sqrt{(b+c)(c+a)}\right\}8abc$$

$$=64abc(a+b)(b+c)(c+a)\overset{Cesaro}{\geq}64abc\cdot 8abc$$

$$=512a^2b^2c^2=512\sin^2 x\sin^2 y\sin^2 z \quad \textit{(Proved)}$$

355. **If $n\in\mathbb{N}, n\geq 1$ then:**

$$2+\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\cdot\ldots\cdot(1+H_k)}{H_1H_2\cdot\ldots\cdot H_{k+1}}<\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

Let $U_k=\frac{(1+H_1)(1+H_2)\ldots(1+H_k)}{H_1H_2\ldots H_k}\Rightarrow LHS=2+\sum_{k=1}^{n}(U_{k+1}-U_k)=2+U_{k+1}-U_1=U_{k+1}$

$$U_{k+1}=\prod_{k=1}^{n}\left(\frac{H_k}{H_{k+1}}+\frac{1}{H_{k+1}}\right)\le\prod_{k=1}^{n}\left(1+\frac{1}{H_{k+1}}\right)$$

$$\le\left[\frac{1}{n+1}\left\{1+\sum_{k=1}^{n}\left(1+\frac{1}{H_{k+1}}\right)\right\}\right]^{n+1}=\left[1+\frac{1}{n+1}\left(\sum_{k=1}^{n}\frac{1}{H_{k+1}}\right)\right]^{n+1}<\left[1+\frac{1}{n+1}\left(\sum_{k=1}^{n+1}\frac{1}{H_k}\right)\right]^{n+1}$$

Proved.

Solution 2 by Remus Florin Stanca-Romania

The inequality can be written as: $\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}-\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\cdot\ldots\cdot(1+H_k)}{H_1\cdot H_2\cdot\ldots H_{k+1}}>2$

$$\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}+1=\frac{\sum_{k=1}^{n+1}\frac{1}{H_k}}{n+1}+\frac{n+1}{n+1}=\frac{\sum_{k=1}^{n+1}\left(\frac{1}{H_k}+1\right)}{n+1}>\sqrt[n+1]{\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)}\succ$$

$$\succ\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}>\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)$$

We need to prove that $\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\left(1+\frac{1}{H_2}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}\ge 2$

$$\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}=\Omega(n+1)=$$

$$=\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_{n+1}}\right)-\left(\frac{1}{H_1}+1\right)\cdot\ldots\cdot\left(1+\frac{1}{H_n}\right)\cdot\frac{1}{H_{n+1s}}-\sum_{k=1}^{n-1}\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}=$$

$$=\prod_{k=1}^{n}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n-1}(1+H_1)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}=\Omega(n)$$

$$\Rightarrow\Omega(n+1)=\Omega(n)=\cdots=\Omega(2)=\left(1+\frac{1}{H_1}\right)\left(1+\frac{1}{H_2}\right)-\left(1+\frac{1}{H_1}\right)\frac{1}{H_2}=$$

$$=1+\frac{1}{H_1}=2\Rightarrow\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\ldots\left(1+\frac{1}{H_k}\right)\frac{1}{H_{k+1}}\ge 2\Leftrightarrow$$

$$\Leftrightarrow\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}-\sum_{k=1}^{n}\frac{(1+H_1)\cdot\ldots\cdot(1+H_k)}{H_1\cdot\ldots\cdot H_{k+1}}>\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\frac{(1+H_1)\cdot\ldots\cdot(1+H_k)}{H_1\cdot\ldots\cdot H_k\cdot H_{k+1}}\geq 2$$

$$\Leftrightarrow 2+\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\ldots(1+H_k)}{H_1H_2\ldots H_{k+1}}<\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}$$

(Q.E.D)

356. If $0 < x, y, z, t < \frac{\pi}{2}$ then:

$$\sum_{cyc(x,y,z,t)}(\sin^2 x+\csc^2 x)^3+\sum_{cyc(x,y,z,t)}(\cos^2 x+\sec^2 x)^3\geq 125$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\textit{For } 0<\theta<\frac{\pi}{2};\ f(\theta)=\left(\sin^2\theta+\frac{1}{\sin^2\theta}\right)^3+\left(\cos^2\theta+\frac{1}{\cos^2\theta}\right)^3=$$

$$=\sin^6\theta+\cos^6\theta+3(\sin^2\theta+\cos^2\theta)+3\left(\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right)+\left(\frac{1}{\sin^6\theta}+\frac{1}{\cos^6\theta}\right)$$

$$\geq\sin^6\theta+\cos^6\theta+3+\frac{6}{\sin\theta\cos\theta}+\frac{2}{\sin^3\theta\cos^3\theta}=$$

$$=\sin^6\theta+\cos^6\theta+3+\frac{12}{\sin 2\theta}+\frac{16}{(\sin^3 2\theta)}$$

$$\textit{But } \sin^6\theta+\cos^6\theta\geq 2\left(\frac{1}{\sqrt{2}}\right)^6\ \textit{[Using derivatives]}$$

$$\therefore f(\theta)\geq\frac{1}{4}+3+12+16=31\frac{1}{4}=\frac{125}{4}\ \therefore \textit{For } 0<x,y,z,t<\frac{\pi}{2}$$

$$\sum\left(\sin x+\frac{1}{\sin x}\right)^3+\sum\left(\cos x+\frac{1}{\cos x}\right)^3\geq 4\left(\frac{125}{4}\right)=125$$

Solution 2 by Tran Hong-Vietnam

$$\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^3+\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^3\geq$$

$$\frac{\left(1+\frac{1}{\sin^2 x\cos^2 x}\right)^3}{4}=\frac{\left(1+\frac{4}{\sin^2 2x}\right)^3}{4}\geq\frac{5^3}{4}\ \textit{(1)}$$

Same:

$$(\sin^2 y +)^3 + \left(\cos^2 y + \frac{1}{\cos^2 y}\right)^3 \ge \frac{5^3}{4} \quad (2)$$

$$\left(\sin^2 z + \frac{1}{\sin^2 z}\right)^3 + \left(\cos^2 z + \frac{1}{\cos^2 z}\right)^3 \ge \frac{5^3}{4} \quad (3)$$

$$\left(\sin^2 t + \frac{1}{\sin^2 t}\right)^3 + \left(\cos^2 t + \frac{1}{\cos^2 t}\right)^3 \ge \frac{5^3}{4} \quad (4)$$

From (1)+(2)+(3)+(4)⇒ LHS $\ge 4 \cdot \frac{5^3}{4} = 125.$

357. If $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$2 \cdot (\sin x)^{1-\sin x} \cdot (1 - \sin x)^{\sin x} \le 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rade Krenkov-Strumica-Macedonia

Let $\sin x = a < 1$. ***We have:*** $2a^{1-a}(1-a)^a \le 1; 2a\left(\frac{1-a}{a}\right)^a \le 1; 2a\left(1 + \frac{1-a}{a} - 1\right)^a \le 1$

$2a\left(1 + \frac{1-2a}{a}\right)^a \le 1$. ***Using Bernouly's inequality we have:*** $LHS \le 2a(1 + 1 - 2a)$

$LHS \le 2a(2 - 2a)$. ***We have to prove:*** $2a(2 - 2a) \le 1$.

Now, we have $4a^2 - 4a + 1 \ge 0; (2a - 1)^2 \ge 0$ ***(true)***

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Let $a = \sin x, b = 1 - \sin x$, ***then we must prove:*** $2a^b \cdot b^a \le 1$ ***(*)***

Proof. We take the logarithms:

$$a^b \cdot b^a \le \frac{1}{2} \Rightarrow \ln(a^b \cdot b^a) \le \ln\frac{1}{2} \Rightarrow b \ln a + a \ln b \le \ln\frac{1}{2} \quad (**)$$

By Jensen's inequality, we have: $b \ln a + a \ln b \le \ln 2ab$ ***(***)***

because $\frac{2ab}{a+b} \le \frac{a+b}{2}$ ***(where*** $a + b = 1$***). We have:***

$$b \ln a + a \ln b \le \ln 2ab \le \ln\frac{a+b}{2}, \text{ since } \ln\frac{a+b}{2} = \ln\frac{1}{2} \Rightarrow (**)$$

Solution 3 by Nguyen Van Nho-Nghe An-Vietnam

$$x \in \left(0; \frac{\pi}{2}\right) \to \sin x, 1 - \sin x > 0$$

$$\to LHS \overset{AM-GM}{\le} 2(\sin x(1 - \sin x) + 1(1 - \sin x)\sin x) = 4 \sin x (1 - \sin x)$$

$$\overset{AM-GM}{\leq} 4\left(\frac{\sin x+1-\sin x}{2}\right)^2 = 1 = RHS \ \textit{(done)}$$

$$\textit{Equality} \leftrightarrow x = \frac{\pi}{6}$$

Solution 4 by Boris Colakovic-Belgrade-Serbia

$$a = \sin x > 0\ \forall x \in \left(0, \frac{\pi}{2}\right);\ b = 1 - \sin x > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$$

$a, b > 0$ ***from weighted GM-AM inequality*** $\Rightarrow$

$$2a^b \cdot b^a \leq 2\left(\frac{ab+bc}{a+b}\right)^{a+b} = 2\left(\frac{2ab}{a+b}\right)^{a+b} \leq 2\left(\frac{a+b}{2}\right)^{a+b} \textit{ or}$$

$$2(\sin x)^{1-\sin x}(1-\sin x)^{\sin x} \leq 2\left(\frac{\sin x + 1 - \sin x}{2}\right)^{\sin x+1-\sin x} = 1$$

358. If $x, y, z > 0$ then:

$$e^{x^2+y^2+z^2} \geq 2exyz\sqrt{2e}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Michael Sterghiou-Greece

$$e^{\sum_{cyc} x^2} \geq 2exyz\sqrt{2e} \ \textit{(1)}$$

(1) $\overset{AM-GM}{\rightarrow} LHS \geq e^{3r^{\frac{2}{3}}} \geq 2r(8e^3)^{\frac{1}{2}}$ ***or it is enough that*** $3r^{\frac{2}{3}} \geq \ln r + \frac{3}{2}(\ln 2 + 1)$ ***where***

$r = abc > 0$***. The function*** $f(r) = 3r^{\frac{2}{3}} - \ln r - \frac{3}{2}(\ln 2 + 1)$ ***has***

$$f\left(\frac{1}{2\sqrt{2}}\right) = 0, f'(r) = \frac{2r^{\frac{2}{3}}-1}{r},$$

$f'(r) = 0$ ***for*** $r = \frac{1}{2\sqrt{2}}$; $f'(r) \leqslant 0$ ***for*** $r \leqslant \frac{1}{2\sqrt{2}}$ ***and*** $f''(r) = \frac{3-2r^{\frac{2}{3}}}{3r^2} > 0$ ***for*** $r = \frac{1}{2\sqrt{2}}$ ***hence***

$r = \frac{1}{2\sqrt{2}}$ ***is a global min for*** $f(r)$ ***and therefore*** $f(r) > f\left(\frac{1}{2\sqrt{2}}\right) = 0$ ***. Done!***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{(1)} \Leftrightarrow x^2 + y^2 + z^2 \geq \ln(xyz) + \frac{3}{2}\ln(2e) \Leftrightarrow$$

$$x^2 + y^2 + z^2 \overset{(2)}{\geq} \ln x + \ln y + \ln z + \frac{3}{2}\ln(2e)$$

Let $f(x)=x^2-\ln x-\frac{1}{2}\ln(2e)\ \forall x>0$*. Then* $f'(x)=2x-\frac{1}{x}=0\Rightarrow x=\frac{1}{\sqrt{2}}$

Also $f''\left(\frac{1}{\sqrt{2}}\right)=\left(\frac{1}{x^2}+2\right)\Big|_{x=\frac{1}{\sqrt{2}}}>0$

$\therefore f(x)$ *attains a minima at* $x=\frac{1}{\sqrt{2}}$ *&* $\because f(x)$ *never attains a maxima* $\forall x>0$*,*

$$\therefore f(x)_{\min}=f\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}-\ln\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{2}(\ln 2e)=\frac{1}{2}+\frac{1}{2}\ln 2-\frac{1}{2}\ln 2-\frac{1}{2}=0$$

$$\therefore \forall x>0,\ x^2-\ln x-\frac{1}{2}(\ln 2e)\ge 0\Rightarrow x^2\overset{(i)}{\ge}\ln x+\frac{1}{2}(\ln 2e)$$

Similarly, $y^2\overset{(ii)}{\ge}\ln y+\frac{1}{2}(\ln 2e)$ *&* $z^2\overset{(iii)}{\ge}\ln z+\frac{1}{2}(\ln 2e)$

(i)+(ii)+(iii)⇒ (2) is true (proved)

359. **If** $x\in\left(0,\frac{\pi}{2}\right)$ **then:**

$$\pi\left(\frac{\sin x}{x}+\frac{\cos x}{\frac{\pi}{2}-x}\right)>4+(\pi-2)(\sin x+\cos x)$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

$$\pi\left(\frac{\sin x}{x}+\frac{\cos x}{\frac{\pi}{2}-x}\right)>4+(\pi-2)(\sin x+\cos x)\quad (1)$$

Lemma 1. $x\in\left(0,\frac{\pi}{4}\right):\sin x>x-\frac{x^3}{6}$

Lemma 2. $x\in\left(0,\frac{\pi}{4}\right):\cos x>1-\frac{x^2}{2}+\frac{x^4}{48}$

Solution: (1) can be written as: $\overbrace{\left(\frac{\pi}{x}-\pi+2\right)}^{>0}\sin x+\overbrace{\left(\frac{\pi}{\frac{\pi}{2}-x}-\pi+2\right)}^{>0}\cos x>4$ ***(2)***

$f(x)=LHS$ ***of (2). We observe that*** $f(x)$ ***has*** $x=\frac{\pi}{4}$ ***as symmetry axis as*** $f(x)=f\left(\frac{\pi}{2}-x\right)$***. We have*** $\lim_{x\to 0_+}f(x)=\lim_{x\to\frac{\pi}{2}}f(x)=4,\ f\left(\frac{\pi}{4}\right)=\sqrt{2}(6-\pi)>4(\sim 4.042)$.

It is easy to show also that $f'\left(\frac{\pi}{4}\right)=0$***. We need to prove that*** $f(x)$ ***lies on and over the***

line $y = 4$ *in the interval* $\left(0,\frac{\pi}{4}\right)$ *as symmetry will take care of the interval* $\left[\frac{\pi}{4},\frac{\pi}{2}\right)$.

Consider the function $g(x)$ *in* $\left(0,\frac{\pi}{4}\right]$: $g(x) = \left(\frac{\pi}{x} - \pi + 2\right)\left(x - \frac{x^3}{6}\right) + \left(\frac{\pi}{\frac{\pi}{2}-x} - \pi + 2\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{48}\right)$. *We will show that* $g(x) > 4$ *in* $\left(0,\frac{\pi}{4}\right]$. *Indeed* $g(x) \to 4$ *when* $x \to 0_+$ *and*

$g\left(\frac{\pi}{4}\right) > 4$.

$$g''(x) = \underbrace{-\frac{1}{4}(\pi - 2)x^2}_{T_1} + \underbrace{\frac{1}{8}(7\pi - 16)x}_{T_2} + \underbrace{\frac{-768\pi + 96\pi^3 - \pi^5}{48(2x - \pi)^3}}_{T_3}$$

$$+ \underbrace{\frac{1}{48}(-96 + 32\pi - \pi^2)}_{T_4}$$

$T_1 < 0, T_3 < 0$ *and* $T_4 < 0$. *The max of* T_2 *is* $\frac{1}{8}(7\pi - 16)\cdot\frac{\pi}{4} < |T_4|$ *therefore* $g''(x) < 0$

This means $g'(x) \downarrow$ *with only one root* x_0 *in* $\left(0,\frac{\pi}{4}\right)$ *in which* $g''(x_0) < 0$ *therefore* x_0 *is a max. Using Lemmas now therefore* ≥ 4. *The same applies by symmetry in* $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$.

The proof is complete.

Lemma 2: Consider $h(x) = \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{48}$ *over* $\left(0,\frac{\pi}{4}\right]$. $h(0) = 0$,

$$h'(x) = -\sin x + x - \frac{4x^3}{48}, h'(0) = 0, h''(x) = \cos x + 1 - \frac{12x^2}{48} > 0$$

As for $x \leq \frac{\pi}{4}\cos x > 0, 1 - \frac{12x^2}{48} > 0$. *So* $h'(x) \uparrow$ *and* $> h'(0) = 0 \to h(x) \uparrow$ *and*

$h(x) > h(0) = 0$

Lemma 1: Easy in a similar manner.

360. If $2\sin^2 x + 2\sin^2 y = 1, x, y \in \left(0,\frac{\pi}{2}\right)$ then:

$$2\tan x\tan y + 2\tan x + 2\tan y < 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

$$\sin^2 x + \sin^2 y = \frac{1}{2} \Leftrightarrow \frac{1}{2} = \frac{\tan^2 x}{1 + \tan^2 x} + \frac{\tan^2 y}{1 + \tan^2 y} \overset{(Schwarz)}{\geq}$$

$$\geq \frac{(\tan x + \tan y)^2}{2 + (\tan x + \tan y)^2 - 2\tan x \tan y} \Leftrightarrow 2(\tan x + \tan y)^2 \leq$$

$$\leq 2 + (\tan x + \tan y)^2 - 2\tan x \tan y \Leftrightarrow (\tan x + \tan y)^2 + 2\tan x \tan y \leq 2$$

$$\Rightarrow \tan x + \tan y + \tan x \tan y \leq \tan x + \tan y + \frac{2 - (\tan x + \tan y)^2}{2} = \frac{-u^2 + 2u + 2}{2} \leq \frac{3}{2}$$

$$\forall u \in (0;2) \ \left(u = \tan x + \tan y, 0 < x, y < \frac{\pi}{4} \Rightarrow 0 < u < 2\right)$$

Solution 2 by Khanh Hung Vu-Vietnam

If $2\sin^2 x + 2\sin^2 y = 1, x, y \in \left(0; \frac{\pi}{2}\right)$ **then** $2\tan x \tan y + 2\tan x + 2\tan y < 3$ ***(1)***

We have $2\sin^2 x + 2\sin^2 y = 1 \Rightarrow \sin^2 x + \sin^2 y = \frac{1}{2} \Rightarrow 1 - \cos^2 x + 1 - \cos^2 y = \frac{1}{2}$

$$\Rightarrow \cos^2 x + \cos^2 y = \frac{3}{2} \Rightarrow \frac{1}{1+\tan^2 x} + \frac{1}{1+\tan^2 y} = \frac{3}{2} \quad (2)$$

Put $\tan x = a, \tan y = b \Rightarrow a, b \in (0; +\infty)$

We have the equation (2) equivalent to: $\frac{1}{1+a^2} + \frac{1}{1+b^2} = \frac{3}{2} \Rightarrow \frac{a^2+b^2+2}{a^2b^2+a^2+b^2+1} = \frac{3}{2} \Rightarrow$

$$\Rightarrow 2(a^2 + b^2 + 2) = 3(a^2b^2 + a^2 + b^2 + 1) \Rightarrow$$

$$\Rightarrow 3a^2b^2 + a^2 + b^2 = 1 \Rightarrow 3a^2b^2 + (a+b)^2 - 2ab = 1 \quad (3)$$

On the other hand, we have

$$(a+b)^2 \geq 4ab \Rightarrow -3a^2b^2 + 2ab + 1 \geq 4ab \Rightarrow -3a^2b^2 - 2ab + 1 \geq 0 \Rightarrow$$

$\Rightarrow 0 < ab \leq \frac{1}{3}$. ***That means the equation (3) is equivalent to***

$a + b = \sqrt{-3a^2b^2 + 2ab + 1}$. ***We have the inequality (1) equivalent to***

$$2ab + 2a + 2b < 3 \Rightarrow 2ab + 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 \Rightarrow$$

$$\Rightarrow 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 - 2ab \Rightarrow 4(-3a^2b^2 + 2ab + 1) < 4a^2b^2 - 12ab + 9 \Rightarrow$$

$$\Rightarrow 16a^2b^2 - 20ab + 5 > 0 \Rightarrow 16\left(ab - \frac{5+\sqrt{5}}{8}\right)\left(ab - \frac{5-\sqrt{5}}{8}\right) > 0$$

(True since $ab - \frac{5+\sqrt{5}}{8} < 0$ ***and*** $ab - \frac{5-\sqrt{5}}{8} < 0$ ***by*** $0 < ab \leq \frac{1}{3}$***)***

So, (1) is true $\Rightarrow 2\tan x \tan y + 2\tan x + 2\tan y < 3$

361. If $x,y>0, xy\ge\frac{1}{8}$ then:

$$\frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}}>\frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Vietnam

By BCS inequality and AM-GM inequality, we have:

$$\frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}}\ge\frac{(x+y)^2}{\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}}\ge\frac{4xy}{\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}}\ge\frac{1}{2\left(\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}\right)}\quad \text{(Since } xy\ge\tfrac{1}{8}\text{) (1)}$$

We need to prove that $\frac{1}{2\left(\sin\frac{3\pi}{11}+\sin\frac{4\pi}{11}\right)}>\frac{1}{\left(\cos\frac{2\pi}{11}+\cos\frac{5\pi}{11}\right)^2}$ *(2)*

Put $t=\frac{\pi}{11}\Rightarrow 11t=\pi\Rightarrow 4t=\pi-7t\Rightarrow \sin 4t=\sin(\pi-7t)=\sin 7t$

We have inequality (2) equivalent to $\frac{1}{2(\sin 3t+\sin 4t)}>\frac{1}{(\cos 2t+\sin 5t)^2}$ *(3)*

We have $(\cos 2t+\sin 5t)^2=\left(\sin\left(\frac{\pi}{2}-2t\right)+\sin 5t\right)^2=\left(2\sin\left(\frac{\pi}{4}+\frac{3t}{2}\right)\cos\left(\frac{\pi}{4}-\frac{7t}{2}\right)\right)^2$

$$\Rightarrow(\cos 2t+\sin 5t)^2=4\sin^2\left(\frac{\pi}{4}+\frac{3t}{2}\right)\cos^2\left(\frac{\pi}{5}-\frac{7t}{2}\right)=\left[1-\cos\left(\frac{\pi}{2}+3t\right)\right]\left[1+\cos\left(\frac{\pi}{2}-7t\right)\right]$$

$$\Rightarrow(\cos 2t+\sin 5t)^2=(1+\sin 3t)(1+\sin 7t)\quad (4)$$

We have $(\sin\sin 3t-1)(\sin 7t-1)>0\Rightarrow \sin 3t\cdot\sin 7t-\sin 3t-\sin 7t+1>0$

$$\Rightarrow \sin 3t\cdot\sin 7t+\sin 3t+\sin 7t+1>2(\sin 3t+\sin 7t)$$

$$\Rightarrow(1+\sin 3t)(1+\sin 7t)>2(\sin 3t+\sin 7t)\Rightarrow(1+\sin 3t)(1+\sin 7t)>2(\sin 3t+\sin 4t)$$

$$\overset{(4)}{\Rightarrow}(\cos 2t+\sin 5t)^2>2(\sin 3t+\sin 4t)\Rightarrow \text{(3) true}\Rightarrow\text{(2) true}$$

From (1) and (2) $\Rightarrow\frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}}>\frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}$ *(Q.E.D.)*

362. If $0<a\le b<\frac{\pi}{2}$ then:

$$(a+b)\left(\sin(\sqrt{ab})-\cos(\sqrt{ab})\right)\le 2\sqrt{ab}\left(\sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$(a+b)\left(\sin(\sqrt{ab})-\cos(\sqrt{ab})\right)\le 2\sqrt{ab}\left(\sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)\right)\quad (1)$$

$$(1)\to \frac{\sin\sqrt{ab}}{\sqrt{ab}}-\frac{\cos(\sqrt{ab})}{\sqrt{ab}}\le \frac{\sin\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}-\frac{\cos\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}\quad (2)$$

Consider the function $f(x)=\frac{\sin x}{x}-\frac{\cos x}{x}$*. As* $f'(x)>0$ *in* $\left(0,\frac{\pi}{2}\right)$

$f(x)$ *is increasing in* $\left(0,\frac{\pi}{2}\right)$ *(*)*

Therefore: $\sqrt{ab}\le\frac{a+b}{2}\to f(\sqrt{ab})\le f\left(\frac{a+b}{2}\right)$ *so (1) true*

$$f'(x)=\frac{1}{x^2}\left[(x-1)\sin x+(x+1)\cos x\right].$$

Consider $g(x)=(x-1)\sin x+(x+1)\cos x$

$$g'(x)=x(\cos x-\sin x)\begin{cases}\ge 0\ x\le\frac{\pi}{4}\to g(x)\uparrow\to g(x)>g(0)>0\\ \le 0\ x\ge\frac{\pi}{4}\to g(x)\downarrow\to g(x)>g\left(\frac{\pi}{2}\right)>0\end{cases}$$

In any case $g(x)>0\to f'(x)>0\to f(x)\uparrow$

Solution 2 by Remus Florin Stanca-Romania

$$\sin\alpha-\cos\alpha=\left(\frac{\sqrt2}{2}\sin\alpha-\frac{\sqrt2}{2}\cos\alpha\right)\cdot\sqrt2=\sin\left(\alpha-\frac{\pi}{4}\right)\sqrt2\Rightarrow$$

$$\Rightarrow\sin\sqrt{ab}-\cos\sqrt{ab}=\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\sqrt2 \text{ and } \sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)=\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\sqrt2$$

The inequality becomes $\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\sqrt2(a+b)\le 2\sqrt{ab}\sqrt2\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\Leftrightarrow$

$$\Leftrightarrow\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)(a+b)\le 2\sqrt{ab}\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\Leftrightarrow$$

$$\Leftrightarrow\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\cdot\frac{1}{\sqrt{ab}}\le\frac{2}{a+b}\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)$$

Let be the function $f:\left(0;\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\sin\left(x-\frac{\pi}{4}\right)\cdot\frac{1}{x}$

$$f'(x)=\frac{\cos\left(x-\frac{\pi}{4}\right)x-\sin\left(x-\frac{\pi}{4}\right)}{x^2}=\cos\left(x-\frac{\pi}{4}\right)\cdot\frac{x-\tan\left(x-\frac{\pi}{4}\right)}{x^2}$$

$$x\in\left(0;\frac{\pi}{2}\right)\Rightarrow x-\frac{\pi}{4}\in\left(-\frac{\pi}{4};\frac{\pi}{4}\right)\Rightarrow\cos\left(x-\frac{\pi}{4}\right)\ge 0$$

$$g:\left(0;\frac{\pi}{2}\right)\to\mathbb{R}\ g(x)=x-\tan\left(x-\frac{\pi}{4}\right)\Rightarrow g'(x)=1-\frac{1}{\cos^2\left(x-\frac{\pi}{4}\right)}<0$$

$\succ$ ***g is a decreasing function and because*** $g\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-1>0\succ$

$$\succ g(x)>0\ \forall x\in\left(0;\frac{\pi}{2}\right)\succ f'(x)=\cos\left(x-\frac{\pi}{4}\right)\cdot\frac{x-\tan\left(x-\frac{\pi}{4}\right)}{x^2}>0\succ$$

$\Rightarrow$ ***f is an increasing function (1)***

$\sqrt{ab}$ ***and*** $\frac{a+b}{2}\in\left(0;\frac{\pi}{2}\right)$ ***and*** $\sqrt{ab}\le\frac{a+b}{2}\overset{(1)}{\succ}\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\frac{1}{\sqrt{ab}}\le\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\frac{2}{a+b}\succ$ ***Q.E.D.***

Solution 3 by Tran Hong-Vietnam

Inequality $\Leftrightarrow\frac{\sin\sqrt{ab}-\cos\sqrt{ab}}{\sqrt{ab}}\le\frac{\sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}$ ***(*)***

Let $f(x)=\frac{\sin x-\cos x}{x}\forall x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f'(x)=\frac{(x-1)\sin x+(x+1)\cos x}{x^2}$

We prove: $g(x)=(x-1)\sin x+(x+1)\cos x>0,\forall x\in\left(0,\frac{\pi}{2}\right)$

$$\Rightarrow g'(x)=x(\cos x-\sin x)=0\Leftrightarrow\cos x=\sin x\Leftrightarrow x=\frac{\pi}{4}$$

$$\Rightarrow g'(x)>0\Leftrightarrow x\in\left(0,\frac{\pi}{4}\right),g'(x)<0\Leftrightarrow x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

$\Rightarrow g(x)>\min\left\{g\left(\frac{\pi}{2}\right),g(0)\right\}=\frac{\pi}{2}-1>0\Rightarrow f'(x)>0\forall x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f(x)\nearrow$ ***on*** $\left(0,\frac{\pi}{2}\right)$

Because: $0<\sqrt{ab}\le\frac{a+b}{2}\le b<\frac{\pi}{2}\Rightarrow f(\sqrt{ab})\le f\left(\frac{a+b}{2}\right)\Rightarrow$ ***(*) true. Equality*** $\Leftrightarrow a=b$.

363. If $x\in\left[0,\frac{\pi}{14}\right)$ then:

$$(\cos 3x)^{21}\cdot(\cos 5x)^7\cdot(\cos 7x)\le(\cos x)^{413}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

If $x\in\left[0,\frac{\pi}{14}\right]$, ***then:*** $(\cos 3x)^{21}(\cos 5x)^7(\cos 7x)\le(\cos x)^{413}$

we have: $\cos 3x=4\cos^3 x-3\cos x\le\cos^9 x\leftrightarrow\cos x\left[\cos^8 x-4\cos^2 x+3\right]\ge 0$

$\leftrightarrow\cos x\left[(\cos^4 x-1)^2+2(\cos^2 x-1)^2\right]\ge 0$ ***(true)***

$$\to \cos 3x \le \cos^9 x \to (\cos 3x)^{21} \le (\cos x)^{189}$$

Similar: $\cos 5x \le \cos^{25} x \to (\cos 5x)^7 \le (\cos x)^{175}\ \ \forall x \in \left[0, \frac{\pi}{14}\right)$

$$\cos 7x \le \cos^{49} x\ \ \forall x \in \left[0; \frac{\pi}{14}\right)$$

$$\Rightarrow (\cos 3x)^{21} \cdot (\cos 7x)^7 \le (\cos x)^{189} \cdot (\cos x)^{175} (\cos x)^{49} = (\cos x)^{413}$$

$\to$ ***Q.E.D; Equality occurs if*** $x = 2k\pi; (k \in \mathbb{Z})$

Solution 2 by Tran Hong-Vietnam

For $x \in \left[0, \frac{\pi}{14}\right)$ ***we have:*** $1 \ge t = \cos x > \cos\frac{\pi}{14} \approx 0,975$

$$\because \{(t^4 - 1)^2 + 4(t-1)^2\} \ge 0$$

$$\Leftrightarrow t\{t^8 - 4t^2 + 3\} \ge 0 \Leftrightarrow t^9 \ge 4t^3 - 3t = \cos 3x \quad \textit{(1)}$$

$$t^{25} \ge 16t^5 - 20t^3 + 5t \Leftrightarrow t\{t^{24} - 16t^4 + 20t^2 - 5\} \ge 0$$

$$\Leftrightarrow t(t-1)^2(t+1)^2(t^{20} + 2t^{18} + 3t^{16} + 4t^{14} + 5t^{12} + 6t^{10} + 7t^8 + 8t^6 + 9t^4 + 10t^2 - 5) \ge 0 \ \ \textit{(true)}$$

$$\Leftrightarrow t^{25} \ge \cos 5x \quad \textit{(2)}$$

$$t^{49} \ge 64t^7 - 112t^5 + 56t^3 - 7t$$

$$\Leftrightarrow t(t-1)^2(t+1)^2(t^{44} + 2t^{42} + 3t^{40} + 4t^{38} + \cdots + 20t^6 + 21t^4 - 42t^2 + 7) \ge 0$$

$$\textit{(true)} \Leftrightarrow t^{49} \ge \cos 7x \quad \textit{(3)}$$

$\Rightarrow LHS \le (t^9)^{21} \cdot \left(t^{25}\right)^7 \cdot t^{49} = t^{413} = (\cos x)^{413}$; ***Equality*** $\Leftrightarrow \cos x = 1 \overset{\left[0,\frac{\pi}{14}\right)}{\Leftrightarrow} x = 0.$

364. If $m, n \in \mathbb{N}, m, n \ge 1$ then:

$$3(m+n) + \log(m! \cdot n!)^{10} \ge 6\sqrt{m \cdot n \cdot H_m \cdot H_n}$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

$$3(a+b) + \log(a! \cdot b!)^{10} \ge 6\sqrt{abH_a \cdot H_b} \quad \textit{(1)}$$

$$H_a = \sum_k^a \frac{1}{k} \le 1 + \log a \text{ and } H_b \le 1 + \log b$$

$$\textit{(1)} \to 3(a+b) + 10 \cdot \log(a! \cdot b!) \ge 6\sqrt{ab} \cdot \sqrt{(1+\log a)(1+\log b)} \quad \textit{(2)}$$

But $3(a+b) \ge 6\sqrt{ab}$ ***and*** $\sqrt{(1+\log a)(1+\log b)} \le \frac{1+\log a+1+\log b}{2} = 1 + \frac{1}{2}(\log a + \log b)$

From (2) we have stronger inequality

$$10\log(a!\cdot b!)\ge 6\sqrt{ab}\left[\frac{1}{2}(\log a+\log b)+1-1\right]=3\sqrt{ab}(\log a+\log b)$$

and as $\sqrt{ab}\le\frac{a+b}{2}$ *the even stronger* $20[\log a!+\log b!]\ge 3(a+b)(\log a+\log b)$ ***(3)***

Equality throughout for $a=b=1$. *We observe that if* $a+b\le 6$ *then* ***(3)*** *holds as it can be written as* $20\left[\sum_{k=1}^{a-1}\log k+\sum_{k=1}^{b-1}\log i\right]+20(\log a+\log b)\ge 18(\log a+\log b)$

So, ***(3)*** *must be shown for* $a+b\ge 7$. *Using the Stirling formula*

$\log a=a\log a-a+\theta\ (\theta>0)$ *we obtain the stronger inequality*

$$f(a,b)=(17a-3b)\ln a+(17b-3a)\ln b-20(a+b)\quad (4)$$

With $a+b\ge 7$. *Assume WLOG* $a\ge b, a=b+x, x\ge 0$

(4)$\rightarrow f(x,b)=14\log(b+x)-3x\log b+17x\log(b+x)-40b+14b\log b-20x$

Assume b *fixed and* $b,x\in\mathbb{R}^+$: $f''(x)=\frac{20b+17x}{b+x^2}>0$ *so*

$$f'(x)\uparrow\rightarrow f'(x)=-\frac{3(x+2b)}{x+b}+17\log(x+b)-3\log b\ge f'(0)=14\log b-6>0$$

for $b\ge 2$. *Thus for* $b\ge 2$ $f'(x)>0\rightarrow f(x)\uparrow\rightarrow f(x)>f(0)\rightarrow$

$\rightarrow f(x)>4b(7\log b-10)>0$ *for* $b\ge 5$. *Therefore* $\forall b\ge 5$ $f(x,b)>0$ *or* $f(a,b)>0$ *for* $a\ge b\ge 5$. *Now we have only the following cases:*

$b=1\rightarrow f(a,b)=(17a-3)\ln a-20(a+1)>0$ *for* $a\ge 6$ *as can easily be shown* $f'(a)>0$ *for* $a\ge 6$ *and* $f(a)>f(6)>0$. *In a similar way we meet the cases* $b=2, a\ge 5$ $b=3, a\ge 4$; $b=4, a\ge 4$. *All cases have been exhausted and the proof is complete.*

365. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\ge\sqrt{2}(1+\tan x)(1+\tan y)(1+\tan z)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

We shall prove: $\frac{2}{\sqrt{\cos x\cos y\cos(x-y)}}\ge\sqrt{2}\sqrt{(1+\tan x)(1+\tan y)}$

$$\Leftrightarrow\frac{4}{\cos x\cos y\cos(x-y)}\ge\frac{(\sin x+\cos x)(\sin y+\cos y)}{\cos x\cos y}$$

$$\Leftrightarrow\frac{2}{\cos(x-y)}\ge(\sin x+\cos x)(\sin y+\cos y)$$

$$\Leftrightarrow 2 \overset{(1)}{\geq} \cos(x-y)\,(\sin x+\cos x)(\sin y+\cos y)$$

$$\left(\because 0<\cos(x-y)\leq 1 \;as\; -\frac{\pi}{2}<x-y<\frac{\pi}{2}\right)$$

But $\sin x+\cos x \underset{(a)}{\overset{CBS}{\leq}} \sqrt{2}\sqrt{\sin^2 x+\cos^2 x}=\sqrt{2}$

& $\sin y+\cos y \underset{(b)}{\overset{CBS}{\leq}} \sqrt{2}\sqrt{\sin^2 y+\cos^2 y}=\sqrt{2}$

Hence, $0<(\sin x+\cos x)(\sin y+\cos y) \underset{(i)}{\overset{by\,(a).(b)}{\leq}} 2$ *& also,* $0<\cos(x-y) \underset{(ii)}{\leq} 1$

(i).(ii) ⇒ *(1) is true* ⇒

$$\frac{2}{\sqrt{\cos x\cos y\cos(x-y)}} \overset{(m)}{\geq} \sqrt{2}\sqrt{(1+\tan x)(1+\tan y)}$$

Similarly, $\frac{2}{\sqrt{\cos y\cos z\cos(y-z)}} \overset{(n)}{\geq} \sqrt{2}\sqrt{(1+\tan y)(1+\tan z)}$ *&*

$$\frac{2}{\sqrt{\cos z\cos x\cos(z-x)}} \overset{(p)}{\geq} \sqrt{2}\sqrt{(1+\tan z)(1+\tan x)}$$

(m).(n).(p) ⇒ *given inequality is true (Proved)*

Solution 2 by Tran Hong-Vietnam

Inequality ⇔ $\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}$

$$\geq \sqrt{2}\cdot\frac{(\sin x+\cos x)(\sin y+\cos y)(\sin z+\cos z)}{\cos x\cos y\cos z};$$

$$\Leftrightarrow 8 \geq \{\cos(x-y)\,(\sin x+\cos x)^2\} \times \{\cos(y-z)\,(\sin y+\cos y)^2\} \times$$

$$\times \{\cos(x-z)\,(\sin z+\cos z)^2\} \text{ (*)}$$

$$\therefore 0<x,y,z<\frac{\pi}{2} \Rightarrow 0<\cos(x-y),\cos(y-z),\cos(z-x)\leq 1 \quad (1)$$

$$\therefore (\sin x+\cos x)^2=1+\sin 2x\leq 2 \quad (2)$$

$$\therefore (\sin y+\cos y)^2=1+\sin 2y\leq 2 \quad (3)$$

$$\therefore (\sin z+\cos z)^2=1+\sin 2z\leq 2 \quad (4)$$

From (1), (2), (3), (4) we have

$RHS_{(*)} \leq 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2=8$ *(proved)*

Solution 3 by Remus Florin Stanca-Romania

$$\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\sqrt{2}(1+\tan x)(1+\tan y)(1+\tan z)$$

$$\Leftrightarrow\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\sqrt{2}\cdot\frac{(\sin x+\cos x)(\sin y+\cos y)(\sin z+\cos z)}{\cos x\cos y\cos z}$$

$$\Leftrightarrow\frac{1}{\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\left(\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x\right)\left(\frac{\sqrt{2}}{2}\sin y+\frac{\sqrt{2}}{2}\cos y\right)\left(\frac{\sqrt{2}}{2}\sin z+\frac{\sqrt{2}}{2}\cos z\right)$$

$$\Leftrightarrow\frac{1}{\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\cos\left(x-\frac{\pi}{4}\right)\cos\left(y-\frac{\pi}{4}\right)\cos\left(z-\frac{\pi}{4}\right)$$

$$0\leq\cos(x-y)\leq 1$$

$$0\leq\cos(y-z)\leq 1$$

$$0\leq\cos(z-x)\leq 1$$

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$$\Rightarrow\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}\leq 1\Rightarrow\frac{1}{\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq 1$$

$$0\leq\cos\left(x-\frac{\pi}{4}\right)\leq 1$$

$$0\leq\cos\left(y-\frac{\pi}{4}\right)\leq 1$$

$$0\leq\cos\left(z-\frac{\pi}{4}\right)\leq 1$$

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$$\Rightarrow\cos\left(x-\frac{\pi}{4}\right)\cos\left(y-\frac{\pi}{4}\right)\cos\left(z-\frac{\pi}{4}\right)\leq 1$$

$$\frac{1}{\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq 1\geq\cos\left(x-\frac{\pi}{4}\right)\cos\left(y-\frac{\pi}{4}\right)\cos\left(z-\frac{\pi}{4}\right)$$

$$\Leftrightarrow\frac{1}{\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\cos\left(x-\frac{\pi}{4}\right)\cos\left(y-\frac{\pi}{4}\right)\cos\left(z-\frac{\pi}{4}\right)$$

$$\Leftrightarrow\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\sqrt{2}(1+\tan x)(1+\tan y)(1+\tan z)$$ ***(Q.E.D.)***

366. If $a,b,c\geq 1$ then:

$$\frac{a}{c\cdot\log(eb-\log b)}+\frac{b}{a\cdot\log(ec-\log c)}+\frac{c}{b\cdot\log(ea-\log a)}\geq\frac{9}{a+b+c}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{a}{c\ln(eb-\ln b)}\geq\frac{9}{\sum_{cyc}a}\quad (1)$$

$$LHS\ (1)\overset{BCS}{\geq}\frac{\left(\sum_{cyc}\frac{\sqrt{a}}{\sqrt{c}}\right)^2}{\sum_{cyc}\ln(eb-\ln b)}\overset{AM-GM}{\geq}\frac{9}{\sum\ln(eb-\ln b)}\text{ which must be}\geq\frac{9}{a+b+c}\text{ or}$$

$\sum_{cyc}a\geq\sum_{cyc}\ln(ea-\ln a)$ **(2). Consider** $f(x)=x-\ln x-1;x\geq 1$

$f'(x)=1-\frac{1}{x}\geq 0\to f(x)\uparrow\to f(x)\geq f(1)=0$. **Therefore**

$a>\ln a+1=\ln(ea)\geq\ln(ea-\ln a)$. **Cyclic application gives (2). We are done.**

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c\geq 1$ **we get that:** $\frac{a}{c\log(eb-\log b)}+\frac{b}{a\log(ec-\log c)}+\frac{c}{b\log(ea-\log a)}$

$$\geq\frac{\left(\sqrt{\frac{a}{e}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}\right)^2}{\log(eb-\log b)+\log(ec-\log c)+\log(ea-\log a)}$$

$$\geq\frac{9}{(\log eb+\log ec+\log ea)}:\log(ex-\log x)\leq\log(ex),x\geq 1$$

$\geq\frac{9}{b+c+a}:\log ex<x,x\geq 1$. **Therefore, it is to be true.**

Solution 3 by Tran Hong-Vietnam

$$LHS=\frac{\left(\sqrt{\frac{a}{c}}\right)^2}{\log(eb-\log b)}+\frac{\left(\sqrt{\frac{b}{a}}\right)^2}{\log(ec-\log c)}+\frac{\left(\sqrt{\frac{c}{b}}\right)^2}{\log(ea-\log a)}\overset{(Schwarz)}{\geq}$$

$$\frac{\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}\right)^2}{\log(eb-\log b)+\log(ec-\log c)+\log(ea-\log a)}\overset{(Cauchy)}{\geq}$$

$$\frac{9}{\log(eb-\log b)+\log(ec-\log c)+\log(ea-\log a)}\quad (*)$$

Let $f(x)=x-\log(ex-\log x)$ **with** $x\geq 1$

$$\Rightarrow f'(x)=1-\left(\frac{e-\frac{1}{x}}{ex-\log x}\right)=\frac{ex-\log x+\frac{1}{x}-e}{ex-\log x};$$

$$g(x)=ex-\log x+\frac{1}{x}-e\ \ (\forall x\geq 1)$$

$$^*\, g'(x) = e - \frac{1}{x} - \frac{1}{x^2};\ g''(x) = \frac{1}{x^2} + \frac{2}{x^3} > 0$$

$$\Rightarrow g'(x) \nearrow \text{ on } [1;+\infty) \Rightarrow g'(x) \ge g'(1) = e - 2 > 0$$

$$\Rightarrow g(x) \nearrow \text{ on } [1;+\infty) \Rightarrow g(x) \ge g(1) = 0$$

$$\Rightarrow f'(x) \ge 0\ \forall x \ge 1 \Rightarrow f(x) \nearrow \text{ on } [1;+\infty)$$

$$\Rightarrow f(x) \ge f(1) = 0 \Rightarrow f(x) = x - \log(ex - \log x) \ge 0; \forall x \ge 1 \quad (^{**})$$

Using inequality (**) with $a, b, c \ge 1$ ***we have*** $f(a) + f(b) + f(c) \ge 0$

$$\Leftrightarrow \sum a \ge \sum \log(ea - \log a) \Rightarrow (^*) \ge \frac{9}{\sum a} = \frac{9}{a+b+c}.$$ ***Equality*** $\Leftrightarrow a = b = c = 1.$

367. **If** $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 6, 0 \le x \le 1$ **then:**

$$a\left(\frac{b}{a}\right)^x + b\left(\frac{c}{b}\right)^x + c\left(\frac{a}{c}\right)^x + b\left(\frac{a}{b}\right)^x + c\left(\frac{b}{c}\right)^x + a\left(\frac{c}{a}\right)^x \ge 12$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\sum\left[a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x\right] \ge 12$$

$$12 = 2 \cdot 6 = 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac} \Rightarrow \sum\left[a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x - 2\sqrt{ab}\right] \ge 0 \Rightarrow$$

$$\Rightarrow \sum\left[\sqrt{a}\sqrt{\frac{b}{a}}^x - \sqrt{b}\sqrt{\frac{a}{b}}^x\right]^2 \ge 0.\ \textit{true}$$

Solution 2 by Michael Sterghiou-Greece

$$\sum_{cyc} a\left(\frac{b}{a}\right)^x + \sum_{cyc} b\left(\frac{a}{b}\right)^x = \sum_{cyc}\left[a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x\right] \ge \sum_{cyc} 2\sqrt{ab\left(\frac{b}{a}\cdot\frac{a}{b}\right)^x} = 2\sum_{cyc}\sqrt{ab} = 12$$

Equality for $a = b = c = 2$ ***or*** $x = \frac{1}{2}$

368. **If** $a, b, c, x, y, z > 0, a + b + c = x + y + z = 1$ **then:**

$$\frac{(a+x)^{a+x} \cdot (b+y)^{b+y} \cdot (c+z)^{c+z}}{a^a \cdot b^b \cdot c^c \cdot x^x \cdot y^y \cdot z^z} \le 4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{(a+x)^{(a+x)}(b+y)^{(b+y)}(c+z)^{(c+z)}}{a^a b^b c^c x^x y^y z^z}=$$

$$=\left(\frac{a+x}{a}\right)^a\left(\frac{a+x}{x}\right)^x\left(\frac{b+y}{b}\right)^b\left(\frac{b+y}{y}\right)^y\left(\frac{c+z}{c}\right)^c\left(\frac{c+z}{z}\right)^z=$$

$$=\left(1+\frac{x}{a}\right)^a\left(\frac{a}{x}+1\right)^x\left(1+\frac{y}{b}\right)^b\left(\frac{b}{y}+1\right)^y\left(1+\frac{z}{c}\right)^c\left(\frac{c}{z}+1\right)^z$$

$$\leq\left(\frac{a+x+a+x+y+b+b+y+c+z+c+z}{a+b+c+x+y+z}\right)^{a+b+c+x+y+z}=\left(\frac{4}{2}\right)^2=4$$ ***. Therefore, it's true.***

Solution 2 by Sudhir Jha-Kolkata-India

Considering $\frac{a+x}{a},\frac{b+y}{b},\frac{c+z}{c},\frac{a+x}{x},\frac{b+y}{y}$ ***&*** $\frac{c+z}{z}$ ***with associated weights***

a,b,c,x,y,z ***respectively. Then applying weighted GM*** $\leq$ ***weighted AM***

We get, $\left[\left(\frac{a+x}{x}\right)^a\left(\frac{b+y}{b}\right)^b\left(\frac{c+z}{c}\right)^c\left(\frac{a+x}{x}\right)^x\left(\frac{b+y}{y}\right)^y\left(\frac{c+z}{z}\right)^z\right]^{\frac{1}{a+b+c+x+y+z}}\leq$

$$\leq\frac{a+x+b+y+c+z+a+x+b+y+c+z}{a+b+c+x+y+z}$$

$$\Rightarrow\left[\frac{(a+x)^{a+x}(b+y)^{b+y}(c+z)^{c+z}}{a^a b^b c^c x^x y^y z^z}\right]^{\frac{1}{2}}\leq 2\Rightarrow\frac{(a+x)^{a+x}(b+y)^{b+y}(c+z)^{c+z}}{a^a b^b c^c x^x y^y z^z}\leq 4$$ ***(Proved)***

Solution 3 by Tran Hong-Vietnam

We have:Inequality $\Leftrightarrow$

$$\frac{1}{2}\left[a\ln\left(1+\frac{x}{a}\right)+x\ln\left(1+\frac{a}{x}\right)+b\ln\left(1+\frac{y}{b}\right)+y\ln\left(1+\frac{b}{y}\right)+c\ln\left(1+\frac{z}{c}\right)+z\ln\left(1+\frac{c}{z}\right)\right]\leq\ln 2\quad (*)$$

Using Jensen's inequality with $f(u)=\ln u$***:***

$$LHS_{(*)}=\frac{1}{2}af\left(1+\frac{x}{a}\right)+\frac{1}{2}xf\left(1+\frac{a}{x}\right)+\frac{1}{2}bf\left(1+\frac{y}{b}\right)+\frac{1}{2}yf\left(1+\frac{b}{y}\right)+$$

$$+\frac{1}{2}cf\left(1+\frac{z}{c}\right)+\frac{1}{2}zf\left(1+\frac{c}{z}\right)\leq$$

$$\leq\ln\left\{\frac{1}{2}a\left(1+\frac{x}{a}\right)+\frac{1}{2}x\left(1+\frac{a}{x}\right)+\frac{1}{2}y\left(1+\frac{b}{y}\right)+\frac{1}{2}c\left(1+\frac{z}{c}\right)+\frac{1}{2}z\left(1+\frac{c}{z}\right)\right\}$$

$=\ln\{(a+x+b+y+c+z)\}=\ln 2$***. Proved. Equality*** $\Leftrightarrow a=b=c=x=y=z=\frac{1}{3}$***.***

369. If $x, y, z > 0$ then:

$$(x+y+z)\left(\frac{\sqrt{3}}{3}+\tan 20^\circ\right) > 4\sum_{cyc}\left(\frac{xy}{x\cot 50^\circ + y\cot 10^\circ}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We first show $\tan 10^\circ \tan 50^\circ = \tan 30^\circ \tan 20^\circ \Leftrightarrow \sin 50^\circ \sin 10^\circ \cos 30^\circ \cos 20^\circ$

$$= \sin 30^\circ \cos 10^\circ \sin 20^\circ \cos 50^\circ$$

$$LHS = \frac{\sqrt{3}}{4}[2\sin 50^\circ \cos 20^\circ]\sin 10^\circ = \frac{\sqrt{3}}{4}[\sin 70^\circ + \sin 30^\circ]\sin 10^\circ$$

$$= \frac{\sqrt{3}}{8}[2\sin 70^\circ \sin 10^\circ + \sin 10^\circ] = \frac{\sqrt{3}}{8}[\cos 60^\circ - \cos 80^\circ + \sin 10^\circ] = \frac{\sqrt{3}}{8}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{16}$$

$$RHS = \frac{1}{4}[2\cos 50^\circ \sin 20^\circ]\cos 10^\circ = \frac{1}{4}[\sin 70^\circ - \sin 30^\circ]\cos 10^\circ$$

$$= \frac{1}{8}[2\sin 70^\circ \cos 10^\circ - \cos 10^\circ] = \frac{1}{8}[\sin 80^\circ + \sin 60^\circ - \cos 10^\circ] = \frac{\sqrt{3}}{16}$$

For $a, b > 0$

$$\frac{4ab}{a\cot 50^\circ + b\cot 10^\circ} \le \frac{2ab}{\sqrt{ab\cot 50^\circ \cot 10^\circ}} = 2\sqrt{ab}\sqrt{\tan 50^\circ \tan 10^\circ}$$

$$= 2\sqrt{ab\tan 30^\circ \tan 20^\circ} \le a\tan 30^\circ + b\tan 20^\circ$$

$$\therefore \sum_{cyc}\frac{4xy}{x\cot 50^\circ + y\cot 10^\circ} \le \sum_{cyc}(x\tan 30^\circ + y\tan 20^\circ)$$

$$= (x+y+z)(\tan 30^\circ + \tan 20^\circ) = (x+y+z)\left(\frac{\sqrt{3}}{3}+\tan 20^\circ\right)$$

370. If $0 < a < b$ then:

$$4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right) < 5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right) < 6\left(\frac{5^a}{6^a}-\frac{5^b}{6^b}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right) < 5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right) < 6\left(\frac{5^a}{6^a}-\frac{5^b}{6^b}\right) \quad \textit{(1)}$$

Consider the function $[4,\infty)\to\mathbb{R}: f(x)=\left(\frac{x}{x-1}\right)^{b-a}-\frac{b-a}{a+x-1}-1=$

$$=\left(1+\frac{1}{x-1}\right)^{b-a}-\frac{b-a}{a+x-1}-1\overset{Bernoulli}{\geq}1+\frac{b-a}{x-1}-\frac{b-a}{a+x-1}-1=$$

$$=(b-a)\left(\frac{1}{x-1}-\frac{1}{a+x-1}\right)>0$$

Consider now the function $[4,\infty)\to\mathbb{R}: g(x)=x\left[\left(\frac{x-1}{x}\right)^{a}-\left(\frac{x-1}{x}\right)^{b}\right]$

$g'(x)=\frac{1}{x-1}\left[\left(\frac{x-1}{x}\right)^{a}(a+x-1)-\left(\frac{x-1}{x}\right)^{b}(b+x-1)\right]$***. Assuming*** $(x-1)g'(x)>0$

$\left(\frac{x-1}{x}\right)^{a}(a+x-1)>\left(\frac{x-1}{x}\right)^{b}(b+x-1)\leftrightarrow\left(\frac{x}{x-1}\right)^{b-a}>\frac{b-a}{a+x-1}+1\leftrightarrow f(x)>0$ ***which is valid. Therefore*** $g'(x)>0$ ***and*** $g(x)\uparrow$

As $4<5<6\to f(4)<f(5)<f(6)\to$ ***(1) is true.***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $0<a<b$ ***we have:*** $4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right)<5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)$

If $4\times 5\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right)\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)<5\times 5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)^2$

If $20\left(\sqrt{\frac{3^a}{5^a}}-\sqrt{\frac{3^b}{5^b}}\right)^2<25\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)^2$

If $20\left(\frac{3^a}{5^a}+\frac{3^b}{5^b}-2\sqrt{\frac{3^a3^b}{5^a5^b}}\right)<25\left(\frac{4^{2a}}{5^{2a}}+\frac{4^{2b}}{5^{2b}}-\frac{24^a4^b}{5^a5^b}\right)$ ***and it's true since***

$20\frac{3^a}{5^a}<25\frac{4^{20}}{5^{2a}}$; $20\frac{3^b}{5^b}<25\cdot\frac{4^{2b}}{5^{2b}}$***. Hence*** $4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right)<5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)$

Similarly, we have $5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)<6\left(\frac{5^a}{6^a}-\frac{5^b}{6^b}\right)$***.***

That is $4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right)<5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)<6\left(\frac{5^a}{6^a}-\frac{5^b}{6^b}\right)$***. Therefore, it's true.***

371. If $0\leq x\leq\frac{\pi}{4}$ **then:**

$$\sin x+\cos x+\sin x\cdot\tan x+x^2\geq 1+x\cdot\sin x+x\cdot\tan x$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } f(x)=\sin x(\cos x)^{-\frac{1}{2}}-x\ \forall x\in\left[0,\frac{\pi}{2}\right)$$

$$f'(x)=\frac{\sin^2 x}{2(\cos x)^{\frac{3}{2}}}+\sqrt{\cos x}-1\ \&f''(x)=\frac{3\sin^3 x+2\cos^2 x\sin x}{4(\cos x)^{\frac{5}{2}}}\geq 0,\forall x\in\left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow f'(x)\geq f'(0)\forall x\in\left[0,\frac{\pi}{2}\right)\Rightarrow f'(x)\geq 0\forall x\in\left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow f(x)\geq f(0)\forall x\in\left[0,\frac{\pi}{2}\right)\Rightarrow f(x)\geq 0\ \ \forall x\in\left[0,\frac{\pi}{2}\right)$$

$$\therefore x\in\left[0,\frac{\pi}{4}\right],\sin x(\cos x)^{-\frac{1}{2}}-x\geq 0$$

$$\Rightarrow \sin x\geq x\sqrt{\cos x}\Rightarrow \sin^2 x\geq x^2\cos x\Rightarrow \sin x\tan x\geq x^2\Rightarrow \sin x\tan x+x^2\overset{(1)}{\geq}2x^2$$

$$\textit{Case (1) } x\in\left[\frac{1}{\sqrt{2}},\frac{\pi}{4}\right]\therefore x\geq\frac{1}{\sqrt{2}}\Rightarrow 2x^2-1\overset{(2)}{\geq}0$$

$$\textit{(1), (2)}\Rightarrow \sin x\tan x+x^2\overset{(3)}{\geq}1$$

$$\textit{(3)}\Rightarrow \textit{it suffices to prove: } \sin x+\cos x\overset{(4)}{\geq}x\sin x+x\tan x$$

$$\because \frac{1}{\sqrt{2}}\leq x\leq\frac{\pi}{4},\therefore \cos x\geq \sin x,\therefore \textit{LHS of (4)}\geq 2\sin x\overset{?}{\geq}x\sin x+\frac{x\sin x}{\cos x}$$

$$\Leftrightarrow 2\cos x\overset{?}{\geq}x\cos x+x\Leftrightarrow (2-x)\cos x\overset{?}{\geq}x$$

$$\Leftrightarrow \cos x\underset{(5)}{\overset{?}{\geq}}\frac{x}{2-x}\left(\because 2-x>0\ as\ \frac{1}{\sqrt{2}}\leq x\leq\frac{\pi}{4}\right)$$

$$\because \frac{1}{\sqrt{2}}\leq x\leq\frac{\pi}{4},\therefore \cos x\geq\frac{1}{\sqrt{2}}\overset{?}{\geq}\frac{x}{2-x}\Leftrightarrow \frac{1}{2}\overset{?}{\geq}\frac{x^2}{(2-x)^2}\Leftrightarrow 4+x^2-4x\overset{?}{\geq}2x^2$$

$$\Leftrightarrow x^2+4x-4\underset{(6)}{\overset{?}{\leq}}0\ \because x\leq\frac{\pi}{4},\therefore \textit{LHS of (6)}\leq\frac{\pi^2}{16}+\frac{4\pi}{4}-4$$

$$=\frac{\pi^2+16\pi-64}{16}<0\Rightarrow \textit{(6)}\Rightarrow\textit{(5)}\Rightarrow\textit{(4) is true}\Rightarrow\textit{given inequality is true}$$

$$\textit{Case 2) } x\in\left[0,\frac{1}{\sqrt{2}}\right)$$

$$\because x\geq\sin x\therefore x^2\geq x\sin x\Rightarrow x^2\cos x\overset{(i)}{\geq}x\sin x\cos x$$

$$\textit{Again, } \cos x>\frac{1}{\sqrt{2}}\left(\because x<\frac{1}{\sqrt{2}}<\frac{\pi}{4}\right)>x\Rightarrow \cos x>0\Rightarrow \sin x\cos x\overset{(ii)}{\geq}x\sin x$$

Lastly, $1 \overset{(iii)}{\geq} \cos x$***. Now, given inequality*** $\Leftrightarrow$

$$\sin x\cos x + \cos^2 x + \sin^2 x + x^2\cos x \geq \cos x + x\sin x\cos x + x\sin x$$

$$\Leftrightarrow x^2\cos x + \sin x\cos x + 1 \overset{(7)}{\geq} x\sin x\cos x + x\sin x + \cos x$$

(i)+(ii)+(iii)⇒ (7) is true ⇒ given inequality is true.

Combining both cases, we conclude that: given inequality is true $\forall x \in \left[0, \frac{\pi}{4}\right]$ ***(proved)***

372. If $a, b, c \in \mathbb{N}^*$

$$\Omega(a,b) = \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots + \frac{b(b-1)\cdot \ldots \cdot 2\cdot 1}{(a+b-1)(a+b-2)\cdot \ldots \cdot a}$$

then:

$$b\cdot\Omega(a,b) + c\cdot\Omega(b,c) + a\cdot\Omega(c,a) \geq a+b+c$$

Proposed by Daniel Sitaru – Romania

Solution by Lahiru Samarakoon-Sri Lanka

$$\Omega(a,b) = \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots + \frac{b(b-1)\ldots 2\cdot 1}{(a+b-1)(a+b-2)\ldots a}$$

Then, $b\cdot\Omega(a,b) + c\cdot\Omega(b,c) + a\cdot\Omega(c,a) \geq a+b+c$***. By adding last three parts,***

$$\Omega(a,b) = \frac{b}{a+b-1} + \cdots + \underbrace{\frac{b(b-1)\ldots 2}{(a+b-1)\ldots(a+1)} + \frac{b(b-1)\ldots 2\cdot 1}{(a+b-1)\ldots a}}_{\Downarrow}$$

$$\frac{b}{(a+b-1)} + \cdots + \frac{b(b-1)\ldots 2(a+1)}{(a+b-1)(a+b-2)\ldots(a+1)a}$$

$$\vdots$$

$$\Omega(a,b) = \frac{b}{(a+b-1)} + \frac{b(b-1)}{(a+b-1)a} = \frac{b(a+b-1)}{(a+b-1)a} = \frac{b}{a}$$

So, similarly, $\Omega(b,c) = \frac{c}{b}$ ***and*** $\Omega(c,a) = \frac{a}{c}$ $\therefore$ ***LHS*** $= b\Omega(a,b) + c\Omega(b,c) + a\Omega(c,a)$

$$= \frac{b^2}{a} + \frac{c^2}{b} + \frac{a^2}{c} \geq \frac{(b+c+a)^2}{(a+b+c)} = (b+c+a) \textit{ (proved)}$$

373. If $a, b, c > 0, a+b+c = 64$ **then:**

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$$\frac{\csc^4\left(\frac{\pi}{7}\right)}{\sqrt{ab}}+\frac{\csc^4\left(\frac{2\pi}{7}\right)}{\sqrt{bc}}+\frac{\csc^4\left(\frac{3\pi}{7}\right)}{\sqrt{ca}}>1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

By Bergström's inequality: $LHS \geq \frac{\left(\csc^2\left(\frac{\pi}{4}\right)+\csc^2\left(\frac{2\pi}{7}\right)+\csc^2\left(\frac{2\pi}{7}\right)\right)^2}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}$ ***(1)***

On the other side: $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$ ***(2)***

From (2) to (1) we get: $LHS \geq \frac{\left(\csc^2\left(\frac{\pi}{7}\right)+\csc^2\left(\frac{2\pi}{7}\right)+\csc^2\left(\frac{3\pi}{7}\right)\right)^2}{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}} = \frac{\left(\csc^2\left(\frac{\pi}{7}\right)+\csc^2\left(\frac{2\pi}{7}\right)+\csc^2\left(\frac{3\pi}{7}\right)\right)^2}{64}$ ***(3)***

Let $\left.\begin{array}{l} z = \cos\left(\frac{\pi}{7}\right)+i\sin\left(\frac{\pi}{7}\right) \\ \frac{1}{z} = \cos\left(\frac{\pi}{7}\right)-i\sin\left(\frac{\pi}{7}\right)\end{array}\right\} \Rightarrow \sin\frac{\pi}{7} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$ ***(4)***

$$\sin\left(\frac{2\pi}{7}\right) = \frac{1}{2i}\left(z^2-\frac{1}{z^2}\right) \quad (5)$$

$$\sin\left(\frac{3\pi}{7}\right) = \frac{1}{2i}\left(z^3-\frac{1}{z^3}\right) \quad (6)$$

$$z^7 = -1 \quad (7)$$

From (4), (5), (6) and (7) we get:

$$\csc^2\left(\frac{\pi}{7}\right)+\csc^2\left(\frac{2\pi}{7}\right)+\csc^2\left(\frac{3\pi}{7}\right) = -\frac{4}{\left(z-\frac{1}{z}\right)^2}-\frac{4}{\left(z^2-\frac{1}{z^2}\right)^2}-\frac{4}{\left(z^3-\frac{1}{z^3}\right)^2} =$$

$$= -4\left(\frac{z^2}{(z^2-1)^2}+\frac{z^4}{(z^4-1)^2}+\frac{z^6}{(z^6-1)^2}\right) =$$

$$= -4\left(\frac{z^2(z^4-1)^2(z^6-1)^2+z^4(z^2-1)^2(z^6-1)^2+z^6(z^2-1)(z^4-1)^2}{(z^2-1)^2(z^4-1)^2(z^6-1)^2}\right)$$

$$= -4\left(\frac{z^{22}+z^{20}-3z^{18}-3z^{16}-2z^{14}+12z^{12}-2z^{10}-3z^8-3z^6+z^4+z^2}{z^{24}-2z^{22}-z^{20}+2z^{18}+3z^{16}-6z^{12}+3z^8+2z^6-z^4-2z^2+1}\right) =$$

$$= -4\left(\frac{-z+z^6-3z^4-3z^2-2\cdot 127^5+2z^3+3z-3z^6+z^4+z^2}{-z^3\cdot 2z-z^6+2z^4+3z^2+6z^5-3z+2z^6-z^4\cdot z^3+1}\right)$$

$$= -4\left(\frac{-2z^6-12z^5-2z^2+2z^3-2z^2+2z-2}{z^6+6z^5+z^4-z^3+z^2-z+1}\right) =$$

$$= -4 \cdot (-2) = 8 \quad (8)$$

By substituting (8) to (3) we get: $LHS \geq \frac{8^2}{64} = \frac{64}{64} = 1$. ***Q.E.D.***

Solution 2 by Ruangkhaw Chaoka-Chiangrai-Thailand

$$a, b, c > 0; a + b + c = 64. \textbf{ Prove that: } K = \frac{\csc^4\left(\frac{\pi}{7}\right)}{\sqrt{ab}} + \frac{\csc^4\left(\frac{2\pi}{7}\right)}{\sqrt{bc}} + \frac{\csc^4\left(\frac{3\pi}{7}\right)}{\sqrt{ca}} > 1$$

Part I: $\csc^2\left(\frac{\pi}{7}\right) + \csc^2\left(\frac{2\pi}{7}\right) + \csc^2\left(\frac{3\pi}{7}\right) = ??$

$$\because \sin(nx) = 2^{n-1} \cdot \prod_{k=0}^{n-1} \sin\left(x + \frac{k\pi}{n}\right) \Rightarrow \ln\sin(nx) = \ln 2^{n-1} + \sum_{k=0}^{n-1} \ln\sin\left(x + \frac{k\pi}{n}\right)$$

$$\textit{Diff; } n\cot(nx) = 0 + \sum_{k=0}^{n-1} \cot\left(x + \frac{k\pi}{n}\right) \Rightarrow \textit{Diff; } -n^2\csc^2(nx) = -\sum_{k=0}^{n-1} \csc^2\left(x + \frac{k\pi}{n}\right)$$

$$\sum_{k=1}^{n-1} \csc^2\left(x + \frac{k\pi}{7}\right) = n^2\csc^2(nx) - \csc^2 x$$

$$\lim_{x\to 0}\left(\sum_{k=1}^{n-1} \csc^2\left(x + \frac{k\pi}{7}\right)\right) = \lim_{x\to 0}\left(n^2\csc^2(nx) - \csc^2 x\right) = \lim_{x\to 0} \frac{1}{x^2}\left(\left(\frac{nx}{\sin(nx)}\right)^2 - \left(\frac{x}{\sin x}\right)^2\right)$$

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{7}\right) = \lim_{x\to 0}\left(\frac{nx}{\sin(nx)} + \frac{x}{\sin x}\right) \cdot \lim_{x\to 0} \frac{1}{x^2}\left(\frac{nx}{\sin(nx)} - \frac{x}{\sin x}\right)$$

$$= (1+1) \cdot \lim_{x\to 0} \frac{n\sin x - \sin(nx)}{x\sin x\sin(nx)}; \lim_{x\to 0} \frac{\sin t}{t} = 1$$

$$= 2 \cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right) \cdot \lim_{x\to 0} \frac{1}{n}\left(\frac{nx}{\sin(nx)}\right) \cdot \lim_{x\to 0}\left(\frac{n\sin x - \sin(nx)}{x^3}\right)$$

$$= 2 \cdot 1 \cdot \frac{1}{n} \cdot 1 \cdot \lim_{x\to 0} \frac{1}{x^3}\left(\left(nx - \frac{nx^3}{3!} + \frac{nx^5}{5!} - \cdots\right) - \left(nx - \frac{(nx)^3}{3!} + \frac{(nx)^5}{5!} - \cdots\right)\right)$$

$$= \frac{2}{n} \cdot \lim_{x\to 0}\left(\frac{n^3 - n}{3!} + \frac{(n - n^5)x^2}{5!} + \cdots\right) = \frac{n^2 - 1}{3}; \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

$$n \to 7; \csc^2\left(\frac{\pi}{7}\right) + \csc^2\left(\frac{2\pi}{7}\right) + \csc^2\left(\frac{3\pi}{7}\right) = \frac{7^2 - 1}{2 \cdot 3} = 8$$

Part II: $\because \left(\sqrt{a} - \sqrt{b}\right)^2 + \left(\sqrt{b} - \sqrt{c}\right)^2 + \left(\sqrt{c} - \sqrt{a}\right)^2 \geq 0$

$$64 = a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \rightarrow \textit{(a) holds at } a = b = c = \frac{64}{3} \rightarrow \textit{(1)}$$

$$\because K\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\overset{CS}{\geq}\left(\csc^2\left(\frac{\pi}{7}\right)+\csc^2\left(\frac{2\pi}{7}\right)+\csc^2\left(\frac{3\pi}{7}\right)\right)^2=8^2\to(b)$$

Holds at $\frac{\csc^4\left(\frac{\pi}{7}\right)}{ab}=\frac{\csc^4\left(\frac{2\pi}{7}\right)}{bc}=\frac{\csc^4\left(\frac{3\pi}{7}\right)}{ca}\to$ *(2)*

(a)· (b); $K\geq 1$ *holds at (1)* $\wedge$ *(2)* $\Leftrightarrow \csc\left(\frac{\pi}{7}\right)=\csc\left(\frac{2\pi}{7}\right)=\csc\left(\frac{3\pi}{7}\right)$ *not true!*

No hold point! $\Rightarrow\therefore K>1$*, now, the proof is complete!*

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{\csc^4\left(\frac{\pi}{7}\right)}{\sqrt{ab}}+\frac{\csc^4\left(\frac{2\pi}{7}\right)}{\sqrt{bc}}+\frac{\csc^4\left(\frac{3\pi}{7}\right)}{\sqrt{ca}}\overset{(1)}{>}1$$

LHS of (1) $\underset{(a)}{\overset{Bergstrom}{>}}\frac{\left(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta\right)^2}{\sum\sqrt{ab}};\left(\theta=\frac{\pi}{7}\right)\overset{CBS}{\geq}\frac{\left(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta\right)^2}{\sum a}$

$$=\frac{\left(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta\right)^2}{64}\quad\left(\because\sum a=64\right)$$

Now, $\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta=\left(\csc\theta+\csc 2\theta+\csc 3\theta\right)^2-$

$$-2\left(\csc\theta\csc 2\theta+\csc 2\theta\csc 3\theta+\csc 3\theta\csc\theta\right)\overset{(b)}{=}P^2-2Q\ \textit{(say)}$$

$$P=\frac{1}{\sin\theta}+\frac{1}{\sin 2\theta}+\frac{1}{\sin 3\theta}$$

$$\overset{(i)}{=}\frac{\sin 2\theta-\sin 3\theta+\sin 3\theta\sin\theta+\sin\theta\sin 2\theta}{\sin\theta\sin 2\theta\sin 3\theta}$$

Numerator of above $=\sin 3\theta\left(\sin 2\theta+\sin\theta\right)+\sin\theta\sin 2\theta=$

$$=\sin 3\theta\left(\sin 2\theta+\sin 6\theta\right)+\sin\theta\sin 2\theta\quad\left(\because\theta=\pi-6\theta\right)$$

$$=2\sin 3\theta\sin 4\theta\cos 2\theta+\sin\theta\sin 2\theta=\left(\sin 5\theta+\sin\theta\right)\sin 4\theta+\sin\theta\sin 2\theta$$

$$=\sin 2\theta\sin 4\theta+\sin\theta\sin 4\theta+\sin\theta\sin 2\theta\quad\left(\because 5\theta=\pi-2\theta\right)$$

$$=\sin 2\theta\left(\sin 3\theta+\sin\theta\right)+\sin\theta\sin 4\theta\quad\left(\because 4\theta=\pi-3\theta\right)$$

$$=2\sin^2 2\theta\cos\theta+2\sin\theta\sin 2\theta\cos 2\theta=2\sin 2\theta\left(\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\right)$$

$$\overset{(ii)}{=}2\sin 2\theta\sin 3\theta$$

(i), (ii) $\Rightarrow P^2\overset{(m)}{=}\frac{4}{\sin^2\theta}$*. Now,* $Q=\frac{2\cos 2\theta}{\sin^2\theta}\Leftrightarrow\frac{2\cos 2\theta}{\sin^2\theta}=\frac{\sin 3\theta+\sin\theta+\sin 2\theta}{\sin\theta\sin 2\theta\sin 3\theta}$

$$\Leftrightarrow\left(2^2\right)\cos 2\theta\sin 2\theta\sin 3\theta=2\sin 3\theta\sin\theta+2\sin^2\theta+2\sin\theta\sin 2\theta$$

$$\Leftrightarrow 2\sin 4\theta\sin 3\theta=\cos 2\theta-\cos 4\theta+1-\cos 2\theta+\cos\theta-\cos 3\theta$$

$$\Leftrightarrow\cos\theta-\cos\pi=1+\cos\theta-\left(\cos 3\theta+\cos 4\theta\right)$$

$$\Leftrightarrow 1+\cos\theta = 1+\cos\theta - \{\cos(\pi-4\theta)+\cos 4\theta\}$$

$$\Leftrightarrow 0 = -(-\cos 4\theta + \cos 4\theta) \Leftrightarrow 0 = 0 \to \textit{true}$$

$$\Rightarrow Q = \frac{2\cos 2\theta}{\sin^2\theta} = \frac{2\,(1-2\sin^2\theta)}{\sin^2\theta} = \frac{2}{\sin^2\theta} - 4 \Rightarrow 2Q\theta \overset{(n)}{=} \frac{4}{\sin^2\theta} - 8$$

(m), (n), (b) $\Rightarrow \csc^2\theta + \csc^2 2\theta + \csc^2 3\theta \overset{(c)}{=} 8$

(a), (c) $\Rightarrow$ *LHS of (1)* $> \frac{8^2}{64} = 1$ *(proved)*

374. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\tan x + \tan y + \tan z > \tan x \cdot \tan y \cdot \tan z - \frac{1}{\cos x \cdot \cos y \cdot \cos z}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 < x < \frac{\pi}{2};\ 0 < y < \frac{\pi}{2};\ 0 < z < \frac{\pi}{2}$$

$$\Rightarrow x+y+z \in \left(0, \frac{3\pi}{2}\right) \Rightarrow -1 < \sin(x+y+z) < 1 \Rightarrow \sin(x+y+z) > -1$$

$$\Rightarrow \sin x\cos y\cos z + \sin y\cos x\cos z + \sin z\cos y\cos x - \sin x\sin y\sin z > -1$$

Dividing throughout by $\cos x\cos y\cos z$

$$\Rightarrow \tan x + \tan y + \tan z - \tan x\tan y\tan z > -\frac{1}{\cos x\cos y\cos z}$$

$$\Rightarrow \tan x + \tan y + \tan z > \tan x\tan y\tan z - \frac{1}{\cos x\cos y\cos z}\ \textit{(proved)}$$

Solution 2 by Lahiru Samarakoon-Sri Lanka

We have to prove:

$$\sin x\cos y + \cos x\sin y\cos z + \cos x\cos y\sin z > \sin x\cos y\cos z - 1$$

$$\sin x\,(\cos y\cos z - \sin y\cos z) + \cos x\,(\sin y\cos z + \cos y\cos z) > -1$$

$$\sin x\cos(y+z) + \cos x\sin(y+z) + 1 > 0$$

Here, $x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin(x+y+z) + 1 > 0$. *It's true* $0 < x+y+z < \frac{3\pi}{2}$

Solution 3 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$\tan x+\tan y+\tan z>\sin x\sin y\sin z(\sec x\sec y\sec z)-\sec x\cdot\sec y\sec z$$

$$\Leftrightarrow \tan x+\tan y+\tan z+\sec x\sec y\sec z(1-\sin x\sin y\sin z)\overset{(1)}{>}0$$

Now, $1>\sin x, 1>\sin y, 1>\sin z$ **&** $\because 0<x,y,z<\frac{\pi}{2}, \therefore \sin x,\sin y,\sin z>0$

$$\therefore 1>\sin x\sin y\sin z\Rightarrow 1-\sin x\sin y\sin z\overset{(a)}{>}0$$

$$\&\because \sec x\sec y\sec z>1\overset{(b)}{>}0\ \left(\text{as } 0<x,y,z<\frac{\pi}{2}\right)$$

$$\therefore (a),(b)\Rightarrow \sec x\sec y\sec z(1-\sin x\sin y\sin z)\overset{(c)}{>}0$$

$$\&\because \sum\tan x\overset{(d)}{>}0\ \left(\text{as } 0<x,y,z<\frac{\pi}{2}\right)$$

$\therefore$ ***(c)+(d)***$\Rightarrow \sum\tan x+\sec x\sec y\sec z(1-\sin x\sin y\sin z)>0\Rightarrow$ ***(1) is true***

(Proved)

375. If $a,b,c>0$ then:

$$\tan^{-1}\left(\frac{(2a+b)(b+2a)}{9ab}\right)+\tan^{-1}\left(\frac{(2b+c)(c+2b)}{9bc}\right)+\tan^{-1}\left(\frac{(2c+a)(2a+c)}{9ca}\right)\ge\frac{3\pi}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$y=\arctan x\leftrightarrow x=\tan y$, *where* $x\in\mathbb{R}, y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *for* $a,b,c>0$, *we have*

$$\arctan\left(\frac{(2a+b)(2b+a)}{9ab}\right)+\arctan\left(\left(\frac{(2b+c)(2c+b)}{9bc}\right)\right)+\arctan\left(\frac{(2c+a)(2a+c)}{9ca}\right)=$$

$$=\arctan\left(\frac{2a^2+2b^2+5ab}{9ab}\right)+\arctan\left(\frac{2b^2+2c^2+5bc}{9bc}\right)+\arctan\left(\frac{2c^2+2a^2+5ca}{9ca}\right)$$

$$\ge\arctan\left(\frac{9ab}{9ab}\right)+\arctan\left(\frac{abc}{9bc}\right)+\arctan\left(\frac{9ca}{9ca}\right):\arctan\ \textit{is increasing function}$$

$$=\arctan(1)+\arctan(1)+\arctan(1)=$$

$=3\arctan(1):\tan\frac{\pi}{4}\Rightarrow\arctan(1)=\frac{\pi}{4},\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)=3\left(\frac{\pi}{4}\right)=\frac{3\pi}{4}$. *Therefore, it's true.*

Solution 2 by Ravi Prakash-New Delhi-India

For $x, y > 0, \frac{2x+y}{3} \geq (x^2y)^{\frac{1}{3}} \Rightarrow \frac{2x+y}{3} \cdot \frac{x+2y}{3} \geq [(x^2y)(xy^2)]^{\frac{1}{3}}$

$$\Rightarrow \frac{2x+y}{3} \cdot \frac{x+2y}{3} \geq xy \Rightarrow \frac{(2x+y)(x+2y)}{9xy} \geq 1 \Rightarrow \tan^{-1}\left(\frac{(2x+y)(x+2y)}{9xy}\right) \geq \frac{\pi}{4}$$

Thus, $\tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right) + \tan^{-1}\left(\frac{(2b+c)(b+2c)}{9bc}\right) + \tan^{-1}\left(\frac{(2a+c)(a+2c)}{9ac}\right) \geq \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} = \frac{3}{4}\pi$

Solution 3 by Tran Hong-Vietnam

Let $f(x) = \tan^{-1}(x)$ **with** $x \geq 1 \Rightarrow f'(x) = \frac{1}{1+x^2} > 0\ \forall x \geq 1 \Rightarrow f \nearrow$ **on** $[1; +\infty)$

$\Rightarrow f(x) \geq f(1) = \tan^{-1}(1) = \frac{\pi}{4}\ (\forall x \geq 1)$**. We have:**

$$(2a+b)(2b+a) = (a+a+b)(b+b+a) \overset{Cauchy}{\geq} 3\sqrt[3]{a^2b} \cdot 3\sqrt[3]{b^2a} = 9ab$$

$$\Rightarrow X = \frac{(2a+b)(2b+a)}{9ab} \geq 1$$

Same: $Y = \frac{(2b+c)(2c+b)}{9bc} \geq 1$ **and** $Z = \frac{(2c+a)(2a+b)}{9ac} \geq 1$

$\Rightarrow f(X) + f(Y) + f(Z) \geq 3f(1) = 3 \cdot \frac{\pi}{4}$**. Proved. Equality** $\Leftrightarrow a = b = c.$

376. If $a, b, c, d, e, f > 0, a + d = b + e = c + f = 5$ **then:**

$$(a+b+c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lahiru Samarakoon-Sri Lanka

$$(a+b+c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$$

We can simplify, $\frac{(b+c)}{d} + \frac{(a+c)}{e} + \frac{(a+b)}{f} \leq 2\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$

$$\frac{(5-e+5-f)}{d} + \frac{(5-d+5-f)}{e} + \frac{(5-d+5-e)}{f} \leq 2\left(\frac{5}{d} - 1 + \frac{5}{e} - 1 + \frac{5}{f} - 1\right)$$

$6 \leq \left(\frac{e}{d} + \frac{d}{e}\right) + \left(\frac{f}{d} + \frac{d}{f}\right) + \left(\frac{e}{f} + \frac{f}{e}\right)$**. By AM-GM:** $\left(\frac{e}{d} + \frac{d}{e}\right) \geq 2$

Similarly, $\left(\frac{f}{d} + \frac{d}{f}\right) \geq 2$ **and** $\left(\frac{e}{f} + \frac{f}{e}\right) \geq 2$**. So,** $\sum\left(\frac{e}{d} + \frac{d}{e}\right) \geq 6$ ***(proved)***

Solution 2 by Michael Sterghiou-Greece

$a + d = b + e = c + f = 5$ **then:** $(a+b+c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$ ***(1)***

$\frac{a}{d} = 1 + \frac{5}{d}$ ***and cyclic application gives:*** $\frac{a}{d} + \frac{b}{e} + \frac{c}{f} = 5\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) - 3$

(1) becomes $(15 - a - b - c)\left(\frac{1}{a} + \frac{1}{e} + \frac{1}{f}\right) \geq 9$ ***(2). But***

$15 - a - b - c = a + e + f \rightarrow$ ***(2) is true.***

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Since $a + d = b + e = c + f = 5, a, b, c, d, e, f > 0$

We give $x = \frac{5x}{x+y}, d = \frac{5y}{x+y}, b = \frac{5z}{z+t}, e = \frac{5t}{z+t}, c = \frac{5m}{m+n}, f = \frac{5n}{m+n}$

Hence $\frac{a}{d} + \frac{b}{e} + \frac{c}{f} = \frac{x}{y} = \frac{z}{f} + \frac{m}{n}$

$$\frac{a}{e} + \frac{b}{f} + \frac{c}{d} = \frac{x}{x+y} \cdot \frac{(z+f)}{f} + \frac{z}{(z+f)} \cdot \frac{(m+n)}{n} + \frac{m}{(m+n)} \cdot \frac{(x+y)}{y}$$

$$\frac{a}{f} + \frac{b}{d} + \frac{c}{e} = \frac{x}{(x+y)} \cdot \frac{(m+n)}{x} + \frac{z}{(z+t)} \cdot \frac{(x+y)}{y} + \frac{y}{(m+n)} \cdot \frac{z+t}{f}$$

and from expanding and reducing, we have:

$$x^2n^2tz + y^2z^2mn + m^2t^2xy + n^2x^2t^2 + x^2y^2z^2 + m^2y^2t^2 \geq$$

$$\geq 3xyzmtn + n^2xyzt + y^2mnzt + t^2mnxy$$

Hence $\frac{x}{y} + \frac{z}{t} + \frac{m}{n} \geq \frac{x}{(x+y)} \cdot \frac{(z+t)}{t} + \frac{z}{(z+t)} \cdot \frac{(m+n)}{n} + \frac{m}{(m+n)} \cdot \frac{(x+y)}{y}$***. That is*** $\frac{a}{d} + \frac{b}{e} + \frac{c}{f} \geq \frac{a}{e} + \frac{b}{f} + \frac{c}{d}$

Similarly, we get $\frac{a}{d} + \frac{b}{d} + \frac{c}{f} \geq \frac{a}{f} + \frac{b}{d} + \frac{c}{e}$

Hence $3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right) \geq \frac{a}{d} + \frac{b}{e} + \frac{c}{f} + \left(\frac{a}{e} + \frac{b}{f} + \frac{e}{d}\right) + \left(\frac{a}{e} + \frac{b}{f} + \frac{c}{d}\right)$

Therefore, $(a + b + c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$ ***is true.***

Solution 4 by Shreeyes Biswal-India

$$(a + b + c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$$

$$\Leftrightarrow \left(15 - (d + e + f)\right)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{5-d}{d} + \frac{5-c}{c} + \frac{5-f}{f}\right)$$

$$\Leftrightarrow 15\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) - (d + c + f)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{5}{d} + \frac{5}{e} + \frac{5}{f} - 3\right)$$

$$\Leftrightarrow 15\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) - (d + e + f)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 15\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) - 9$$

$$\Leftrightarrow -(d+e+f)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right) \le -9 \Leftrightarrow (d+e+f)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right) \ge 9$$

$\Leftrightarrow \frac{d+e+f}{3} \ge \frac{3}{\frac{1}{d}+\frac{1}{c}+\frac{1}{f}}$. ***Which is always true as AM*** $\ge$ ***HM*** $(d,e,f>0)$

377. **If $a,b,c>1$ then:**

$$\frac{\sin\left(\frac{2}{a+b}\right)\sin\left(\frac{2}{b+c}\right)\sin\left(\frac{2}{c+a}\right)}{\sin\left(\frac{1}{\sqrt{ab}}\right)\sin\left(\frac{1}{\sqrt{bc}}\right)\sin\left(\frac{1}{\sqrt{ca}}\right)} \ge \left(\frac{8abc}{(a+b)(b+c)(c+a)}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Vietnam

$\left(\frac{a+b}{2}\right)^2 \sin\frac{2}{a+b} \ge ab\sin\frac{1}{\sqrt{ab}}$ ***(*);*** $(a,b>1)$. ***Let*** $f(t)=t^2\sin\frac{1}{t}\ (t>1)$

$$\Rightarrow f'(t)=2t\sin\frac{1}{t}-\cos\frac{1}{t}=\cos\frac{1}{t}\left(2t\tan\frac{1}{t}-1\right) > \cos\frac{1}{t} > 0$$

$\left(\because \tan\frac{1}{t} > \frac{1}{t}; \cos\frac{1}{t} > 0\ \forall t>1\right) \Rightarrow f(t)\nearrow$ ***on*** $(1;+\infty)$

Hence, $\sqrt{ab} \le \frac{a+b}{2} \Rightarrow f(\sqrt{ab}) \le f\left(\frac{a+b}{2}\right) \Rightarrow$ ***(*) true.*** $\Rightarrow \prod\left(\frac{a+b}{2}\right)^2\sin\frac{2}{a+b} \ge \prod ab\sin\frac{1}{\sqrt{ab}}$

$\Leftrightarrow \frac{\prod\sin\frac{2}{a+b}}{\prod\sin\frac{1}{\sqrt{ab}}} \ge 4^3\frac{a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2} \Leftrightarrow \frac{\prod\sin\frac{2}{a+b}}{\prod\sin\frac{1}{\sqrt{ab}}} \ge \frac{(8abc)^2}{[(a+b)(b+c)(c+a)]^2}$. ***Proved***

378. **If $a,b,c\in\mathbb{N}, a,b,c\ge 4$ then:**

$$\sqrt[a+1]{b}+\sqrt[a+1]{c}+\sqrt[b+1]{a}+\sqrt[b+1]{c}+\sqrt[c+1]{a}+\sqrt[c+1]{b} \le 6\sqrt[4]{4}$$

Proposed by Daniel Sitaru – Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c\in\mathbb{N}$ ***and*** $a,b,c\ge 4$. ***We have these facts:***

1. $a^{\frac{1}{a+1}} \ge b^{\frac{1}{b+1}} \Leftrightarrow b\ge a\ge 4 \because a^{b+1}\ge b^{a+1}, 4\le a\le b$

2. $a^{\frac{1}{a+1}}+b^{\frac{1}{b+1}}+c^{\frac{1}{c+1}} \ge a^{\frac{1}{b+1}}+b^{\frac{1}{c+1}}+c^{\frac{1}{a+1}}$

$$3.\ a^{\frac{1}{a+1}}+b^{\frac{1}{b+1}}+c^{\frac{1}{c+1}}\ge a^{\frac{1}{c+1}}+c^{\frac{1}{b+1}}+b^{\frac{1}{a+1}}$$

Consider, $\sqrt[b+1]{a}\sqrt[c+1]{a}\sqrt[a+1]{b}\sqrt[c+1]{b}\sqrt[a+1]{c}\sqrt[b+1]{c}\le 6\sqrt[5]{4}$

If $a^{\frac{1}{b+1}}a^{\frac{1}{c+1}}b^{\frac{1}{a+1}}b^{\frac{1}{c+1}}c^{\frac{1}{a+1}}c^{\frac{1}{b+1}}\le 6\cdot 4^{\frac{1}{5}}$. *If* $\frac{\left(a^{\frac{1}{b+1}}+a^{\frac{1}{c+1}}+b^{\frac{1}{a+1}}+b^{\frac{1}{c+1}}+c^{\frac{1}{a+1}}+c^{\frac{1}{b+1}}\right)^6}{6}\le 6\cdot 4^{\frac{1}{5}}$

If $\frac{a^{\frac{6}{b+1}}+a^{\frac{6}{c+1}}+b^{\frac{6}{a+1}}+b^{\frac{6}{c+1}}+c^{\frac{6}{a+1}}+c^{\frac{6}{b+1}}}{6}\le 6\cdot 4^{\frac{1}{5}}$

If $\left(a^{\frac{6}{b+1}}+b^{\frac{6}{c+1}}+c^{\frac{6}{a+1}}\right)+\left(a^{\frac{6}{c+1}}+c^{\frac{b}{b+1}}+b^{\frac{6}{a+1}}\right)\le 36\cdot 4^{\frac{1}{5}}$

If $\left(a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\right)+\left(a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\right)\le 36\cdot 4^{\frac{1}{5}}$

If $a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\le 18\cdot 4^{\frac{1}{5}}$. *If* $3a^{\frac{b}{a+1}}\le 18\cdot 4^{\frac{1}{5}}, 4\le a\le b\le c$. *If* $a^{\frac{6}{a+1}}\le 6\times 4^{\frac{1}{5}}$

If $a^{30}\le 6^{5(a+1)}4^{(a+1)}$ *and it's true because*

$$4^{30}\le 6^{25}\cdot 4^5$$

$$5^{30}\le 6^{30}\cdot 4^6$$

$$6^{30}\le 6^{35}\cdot 4^7$$

$$\vdots$$

Therefore, it's true.

379. If $a,b,c,d>0, a+b+c+d=1$ then:

$$\frac{ab}{1+c+d}+\frac{ac}{1+b+d}+\frac{ad}{1+b+c}+\frac{bc}{1+a+d}+\frac{bd}{1+a+c}+\frac{cd}{1+a+b}\le\frac{1}{4}$$

Proposed by Vasile Mircea Popa – Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c,d>0$ *and* $a+b+c+d=1$, *we have:*

$$\frac{ab}{1+c+d}+\frac{ac}{1+b+d}+\frac{ad}{1+b+c}+\frac{bc}{1+a+d}+\frac{bd}{1+a+c}+\frac{cd}{1+a+b}$$

$$=\frac{ab}{a+b+c+d+c+d}+\frac{ac}{a+b+c+d+b+d}+\frac{ad}{a+b+c+d+b+c}+$$

$$+\frac{bc}{a+b+c+d+a+d}+\frac{bd}{a+b+c+d+a+c}+\frac{cd}{a+b+c+d+a+b}\le$$

$$\leq \frac{1}{4}\left[\frac{ab}{a+c+d}+\frac{ab}{b+c+d}+\frac{ac}{a+b+d}+\frac{ac}{c+b+d}+\frac{ad}{a+b+c}+\frac{ad}{d+b+c}+ \\ +\frac{bc}{b+a+d}+\frac{bc}{c+a+d}+\frac{bd}{b+a+c}+\frac{bd}{d+a+c}+\frac{cd}{c+a+b}+\frac{cd}{d+a+b}\right]$$

$$\frac{1}{4}\left[\frac{cd+ad+bd}{a+b+c}+\frac{ab+bc+bd}{a+c+d}+\frac{ab+ac+ad}{b+c+d}+\frac{bc+cd+ac}{a+b+d}\right]$$

$$=\frac{1}{4}\left[\frac{d(c+a+b)}{(a+b+c)}+\frac{b(a+c+d)}{(a+c+d)}+\frac{a(b+c+d)}{(b+c+d)}+\frac{c(b+d+a)}{(b+d+a)}\right]$$

$=\frac{1}{4}(a+b+c+d)=\frac{1}{4}$. ***Therefore, it's true.***

380. If $x,y,z,t>0$ then:

$$4\left(\left(x-\sqrt{xy}+y\right)\left(z-\sqrt{zt}+t\right)\right)^2\geq (x^2+y^2)(z^2+t^2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$x^2+y^2=(x+y)^2-2xy=(x^2+y^2)^2-\left(\sqrt{2xy}\right)^2$$

$$x^2+y^2=\left(x^2+y^2+\sqrt{2xy}\right)\left(x^2+y^2-\sqrt{2xy}\right)$$

$$\because GM\geq AM\Rightarrow\left[2\left(x^2+y^2+\sqrt{2xy}\right)\left(x^2+y^2-\sqrt{2xy}\right)\right]^{\frac{1}{2}}\leq$$

$$\leq\frac{\left(2+\sqrt{2}\right)\left(x+y-\sqrt{2xy}\right)+\left(2-\sqrt{2}\right)\left(x+y+\sqrt{2xy}\right)}{2}$$

$$\leq\frac{4(x+y)-4\sqrt{xy}}{2}\leq 2\left(x+y-\sqrt{xy}\right)$$

$$\Rightarrow 2\left(x^2+y^2+\sqrt{2xy}\right)\left(x^2+y^2-\sqrt{2xy}\right)\leq 4\left(x+y-\sqrt{xy}\right)^2$$

But $x^2+y^2=\left(x^2+y^2+\sqrt{2xy}\right)\left(x^2+y^2-\sqrt{2xy}\right)$

$$\Rightarrow 2(x^2+y^2)\leq 4\left(x+y-\sqrt{xy}\right)^2$$

$$\Rightarrow (x^2+y^2)\leq 2\left(x+y-\sqrt{xy}\right)^2\quad \textit{(1)}$$

In this same way, $(z^2+t^2)\leq 2\left(z+t-\sqrt{zt}\right)^2$ ***(2)***

Multiplying (1) & (2): $(x^2+y^2)(z^2+t^2)\leq 4\left[\left(x+y-\sqrt{xy}\right)\left(z+t-\sqrt{zt}\right)\right]^2$

or $4\left[\left(x-\sqrt{xy}+y\right)\left(z-\sqrt{zt}+t\right)\right]^2 \ge (x^2+y^2)(z^2+^2)$ ***(proved)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$x+y \ge 2\sqrt{xy} \text{ (True); } \left(x+y-2\sqrt{xy}\right)^2 \ge 0;\ (x+y)^2+4xy-4\sqrt{xy}(x+y) \ge 0$$

$$x^2+y^2+6xy-4\sqrt{xy}\cdot(x+y) \ge 0;\ 2(x^2+y^2)+6xy-4\sqrt{xy}(x+y) \ge x^2+y^2$$

$$2\left(x^2+y^2+3xy-2\sqrt{xy}(x+y)\right) \ge x^2+y^2$$

$$\left.\begin{matrix} 2\left(x-\sqrt{xy}+y\right)^2 \ge x^2+y^2 \\ similarly: 2\left(z-\sqrt{zt}+t\right)^2 \ge z^2+t^2 \end{matrix}\right\} \Rightarrow$$

$$4\left(x-\sqrt{xy}+y\right)^2\left(z-\sqrt{zt}+t\right)^2 \ge (x^2+y^2)(z^2+t^2)$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $a,b>0$***, we have:*** $a^2+b^2+6ab=(a+b)^2+4ab \ge 4(a+b)\sqrt{ab}$

$$=4a\sqrt{ab}+4b\sqrt{ab}$$

$$\Rightarrow a^2+b^2+6ab-4a\sqrt{ab}-4b\sqrt{ab} \ge 0$$

$$\Rightarrow 2(a^2+b^2)+6ab-4a\sqrt{ab}-4b\sqrt{ab} \ge a^2+b^2$$

$$\Rightarrow 2\left(a^2+b^2+ab+2ab-2a\sqrt{ab}-2b\sqrt{ab}\right) \ge a^2+b^2$$

$$\Rightarrow 2\left(a-\sqrt{ab}+b\right)^2 \ge a^2+b^2$$

Hence: $2\times 2\left(\left(x-\sqrt{xy}+y\right)\left(z-\sqrt{zt}+t\right)\right)^2 \ge (x^2+y^2)(z^2+t^2)$

$4\left(\left(x\sqrt{xy}+y\right)\left(z-\sqrt{zt}+t\right)\right)^2 \ge (x^2+y^2)(z^2+t^2)$***. Therefore, it's true.***

Solution 4 by Soumava Chakraborty-Kolkata-India

$$x+y \ge \sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}} \Leftrightarrow (x+y)^2 \ge xy+\frac{x^2+y^2}{2}+2\sqrt{xy}\sqrt{\frac{x^2+y^2}{2}}$$

$$\Leftrightarrow \frac{x^2+y^2}{2}+xy-2\sqrt{\frac{x^2+y^2}{2}}\sqrt{xy} \ge 0 \Leftrightarrow \left(\sqrt{\frac{x^2+y^2}{2}}-\sqrt{xy}\right)^2 \ge 0 \to \textbf{\textit{true}}$$

$$\therefore x+y-\sqrt{xy} \ge \sqrt{\frac{x^2+y^2}{2}} \Leftrightarrow 2\left(x+y-\sqrt{xy}\right)^2 \overset{(1)}{\ge} x^2+y^2$$

Similarly, $2\left(z-\sqrt{zt}+t\right)^2 \overset{(2)}{\geq} z^2+t^2$; *(1).(2)⇒ given inequality is true (proved).*

381. $\alpha = \begin{vmatrix} \frac{1}{x+a} & \frac{1}{x+b} & \frac{1}{x+c} \\ \frac{1}{y+a} & \frac{1}{y+b} & \frac{1}{y+c} \\ \frac{1}{z+a} & \frac{1}{z+b} & \frac{1}{z+c} \end{vmatrix}, \beta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}, \gamma = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

If $a, b, c, x, y, z > 0$ **then:**

$$3^9|\alpha| \geq \frac{|\beta\gamma|}{(a+b+c+x+y+z)^9}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\alpha = \begin{vmatrix} \frac{1}{a+x} & \frac{1}{b+x} & \frac{1}{c+x} \\ \frac{1}{a+y} & \frac{1}{b+y} & \frac{1}{c+y} \\ \frac{1}{a+z} & \frac{1}{b+z} & \frac{1}{c+z} \end{vmatrix}$$

$$C_3 \to C_3 - C_2, C_2 \to C_2 - C_1,$$

$$\alpha = \begin{vmatrix} \frac{1}{a+x} & \frac{a-b}{(a+x)(b+x)} & \frac{b-c}{(b+x)(c+x)} \\ \frac{1}{a+y} & \frac{a-b}{(b+y)(a+y)} & \frac{b-c}{(b+y)(c+y)} \\ \frac{1}{a+z} & \frac{a-b}{(a+z)(b+z)} & \frac{b-c}{(b+z)(c+z)} \end{vmatrix}$$

$$= \frac{(a-b)(b-c)\alpha_1}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)}$$

where $\alpha_1 = \begin{vmatrix} (b+x)(c+x) & c+x & a+x \\ (b+y)(c+y) & c+y & a+y \\ (b+z)(c+z) & c+z & a+z \end{vmatrix}$

$$C_3 \to C_3 - C_2, C_1 \to C_1 - bC_2$$

$$\alpha_1=\begin{vmatrix}x(c+x) & c+x & a-c\\ y(c+y) & c+y & a-c\\ z(c+z) & c+z & a-c\end{vmatrix}$$

Therefore $(a-c)$ *common from* C_3 *and use* $C_2\to C_2-cC_3$

$$\alpha_1=(a-c)\begin{vmatrix}x(c+x) & x & 1\\ y(c+y) & y & 1\\ z(c+z) & z & 1\end{vmatrix}$$

$$C_1\to C_1-cC_2$$

$$\alpha_1=(a-c)\begin{vmatrix}x^2 & x & 1\\ y^2 & y & 1\\ z^2 & z & 1\end{vmatrix}=-(a-c)\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix}$$

$$\therefore |Num\ of\ \alpha|=|\beta\gamma|$$

Denominator of $\alpha=(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)$

$$\le\left(\frac{a+x+b+x+c+x+a+y+b+y+c+y+a+z+b+z+c+z}{9}\right)^9$$

$$=\left(\frac{a+b+c+x+y+z}{3}\right)^9$$

$$\Rightarrow 3^9\ (\text{Denominator of }\alpha)\le(a+b+c+x+y+z)^9$$

Thus, $3^9|\alpha|=\frac{3^9|Num\ of\ \alpha|}{Den\ of\ \alpha}\ge\frac{|\beta\gamma|}{(a+b+c+x+y+z)^9}$

382.

$$a,b,c,d\ge 0,p\ge q\ge r\ge 0$$

$$x=\frac{a+b+c+d}{4}-\sqrt[4]{abcd},y=\frac{a+b+c}{3}-\sqrt[3]{abc},z=\frac{a+b}{2}-\sqrt{ab}$$

Prove that:

$$3(px+3qy+2rz)\ge(4x+3y+2z)(p+q+r)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

We prove that: $4x\ge 3y\ge 2z\ge 0$

$$\because 4x\ge 3y\Leftrightarrow a+b+c+d-\sqrt[4]{abcd}\ge a+b+c-\sqrt[3]{abc}\Leftrightarrow d+3\sqrt[3]{abc}\ge 4\sqrt[4]{abcd}$$

It is true because:

$$d+3\sqrt[3]{abc}=d+\sqrt[3]{abc}+\sqrt[3]{abc}+\sqrt[3]{abc}\overset{AM-GM}{\geq}4\sqrt[4]{d\sqrt[3]{[abc]^3}}=4\sqrt[4]{abcd}$$

$$\because 3y\geq 2z\Leftrightarrow a+b+c-3\sqrt[3]{abc}\geq a+b-2\sqrt{ab}\Leftrightarrow c+2\sqrt{ab}\geq 3\sqrt[3]{abc}$$

It is true because $c+2\sqrt{ab}=c+\sqrt{ab}+\sqrt{ab}\geq 3\sqrt[3]{c\sqrt{(ab)^2}}=3\sqrt[3]{abc}$

$\because 2z\geq 0\Leftrightarrow z\geq 0\Leftrightarrow a+b\geq 2\sqrt{ab}$ ***(true). Similarly:*** $3y,4x\geq 0$

Hence: $4x\geq 3y\geq 2z\geq 0$. ***More,*** $p\geq q\geq r\geq 0$ ***then using Chebyshev's inequality:***

$$4xp+3yq+2zr\geq\frac{1}{3}(4x+3y+2z)(p+q+r)$$

$$\Leftrightarrow 3(4px+3qy+2rz)\geq(4x+3y+2z)(p+q+r)\ \textbf{Proved}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$4x\geq 3y\Leftrightarrow a+b+c+d-4\sqrt[4]{abcd}\geq a+b+c-3\sqrt[3]{abc}\Leftrightarrow d+3\sqrt[3]{abc}\overset{(1)}{\geq}4\sqrt[4]{abcd}$$

It is easy to note that, if , at least one variable equals to 0***, then (1) is true.***

We now consider $a,b,c,d>0$***. Then*** $d+3\sqrt[3]{abc}=d+\sqrt[3]{abc}+\sqrt[3]{abc}+\sqrt[3]{abc}$

$$\overset{A-G}{\geq}4\sqrt[4]{d\cdot abc}=4\sqrt[4]{abcd}\Rightarrow 4x\overset{(a)}{\geq}3y.\ \textbf{Also, } 3y\geq 2z\Leftrightarrow$$

$$a+b+c-3\sqrt[3]{abc}\geq a+b-2\sqrt{ab}\Leftrightarrow c+2\sqrt{ab}\overset{(2)}{\geq}3\sqrt[3]{abc}$$

It is easy to note that, if, at least one variable equals to 0***, then (2) is true.***

We now consider $a,b,c>0$***.***

Then $c+2\sqrt{ab}=c+\sqrt{ab}+\sqrt{ab}\overset{A-G}{\geq}3\sqrt[3]{c\cdot ab}\Rightarrow 3y\overset{(b)}{\geq}2z$

(a), (b)$\Rightarrow 4x\geq 3y\geq 2z$ ***&*** $\because p\geq q\geq r$

$$\therefore 3(4px+3qy+2rz)\overset{Chebyshev}{\geq}\frac{3}{3}(4x+3y+2z)(p+q+r)$$

$$=(4x+3y+2z)(p+q+r)\ \textbf{(proved)}$$

383. If $0<a,b,c<\frac{\pi}{2}$ **then:**

$$\left(\frac{a+b+c}{ab+bc+ca}\sin\left(\frac{ab+bc+ca}{a+b+c}\right)\right)^{a+b+c}\geq\left(\frac{\sin b}{b}\right)^a\left(\frac{\sin c}{c}\right)^b\left(\frac{\sin a}{a}\right)^c$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $f(x)=\frac{\sin x}{x}\left(0<x<\frac{\pi}{2}\right)\Rightarrow f'(x)=\frac{x\cos x-\sin x}{x^2}<0\ \left(0<x<\frac{\pi}{2}\right)\Rightarrow f(x)\searrow\left(0;\frac{\pi}{2}\right)$

(Because: $g(x)=x\cos x-\sin x\left(0<x<\frac{\pi}{2}\right)$

$\Rightarrow g'(x)=-x\sin x<0\Rightarrow g(x)\searrow\left(0,\frac{\pi}{2}\right)\Rightarrow g(x)<g(0)=0$)

$$\Rightarrow f''(x)=-\frac{(x^2-2)\sin x+2x\cos x}{x^3}<0\ \left(0<x<\frac{\pi}{2}\right)$$

(Because: $h(x)=-[(x^2-2)\sin x+2x\cos x]$

$\Rightarrow h'(x)=-x^2\cos x<0\left(0<x<\frac{\pi}{2}\right)\Rightarrow h(x)\searrow\left(0;\frac{\pi}{2}\right)\Rightarrow h(x)<h(0)=0$)

Now, inequality $\Leftrightarrow (a+b+c)\log u\ge a\log v+b\log w+c\log t$

$$\left[u=\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\left(\frac{ab+bc+ca}{a+b+c}\right)},v=\frac{\sin b}{b},w=\frac{\sin c}{c},t=\frac{\sin a}{a}\right]$$

Using Jensen's inequality with $\varphi(x)=\log x\ (x>0)$

$$a\varphi(v)+b\varphi(w)+c\varphi(t)\le(a+b+c)\cdot\varphi\left(\frac{av+bw+ct}{a+b+c}\right)$$

$=(a+b+c)\cdot\log\frac{av+bw+ct}{a+b+c}$. We must show that

$$u\ge\frac{av+bw+ct}{a+b+c}\Leftrightarrow(a+b+c)u\ge av+bw+ct$$

$$\Rightarrow av+bw+ct=a\cdot\frac{\sin b}{b}+b\cdot\frac{\sin c}{c}+c\cdot\frac{\sin a}{a}$$

$\overset{(Jensen)}{\le}(a+b+c)\cdot\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\left(\frac{ab+bc+ca}{a+b+c}\right)}$. Proved.

Solution 2 by Michael Sterghiou-Greece

$$\left(\frac{a+b+c}{ab+bc+ca}\sin\left(\frac{ab+bc+ca}{a+b+c}\right)\right)^{a+b+c}\ge\left(\frac{\sin b}{b}\right)^a\left(\frac{\sin c}{c}\right)^b\left(\frac{\sin a}{a}\right)^c\quad (1)$$

For simplicity, let $p=\sum_{cyc}a$, $q=\sum_{cyc}ab$. By weighted AM-GM we have:

$\sum_{cyc}a\frac{\sin b}{b}\ge p\sqrt[p]{\prod_{cyc}\left(\frac{\sin b}{b}\right)^a}$ as all terms are >0 on $\left(0,\frac{\pi}{2}\right)$

RHS of (1) $\le\left[\frac{1}{p}\cdot\sum_{cyc}a\frac{\sin b}{b}\right]^p$ which suffices to be $\le\left(\frac{p}{q}\cdot\sin\frac{q}{p}\right)^p=LHS$ of (1)

This leads to $\sum_{cyc} a\frac{\sin b}{b} \le \frac{p^2}{q}\cdot\sin\frac{q}{p}$ *(2). Consider the function*

$f(t) = \frac{\sin t}{t}$ *over* $\left(0,\frac{\pi}{2}\right)$ *with* $f''(t) = -\frac{1}{t^3}(t^2-2)\sin t + 2t\cos t$

Consider further the function $g(t) = (t^2-2)\sin t + 2t\cos t$ *with*

$g'(t) = t^2\cos t > 0$ *on* $\left(0,\frac{\pi}{2}\right)$ *hence* $g(t)\uparrow$ *and* $g(t) > g(0) = 0$

Therefore $f''(t) < 0 \to f(t)$ *concave. Applying now the generalized Jensen inequality, we get:*

$$\sum_{cyc} a\cdot\frac{\sin b}{b} \le p\cdot\frac{\sin\frac{\sum_{cyc} ab}{\sum_{cyc} a}}{\frac{\sum_{cyc} ab}{\sum_{cyc} a}} \to p\cdot\frac{\sin\frac{a}{p}}{\frac{a}{p}} = \frac{p^2}{q}\sin\frac{q}{p} = RHS \text{ of (2). We are done!}$$

384. $x = \frac{(a+b+c+d)^4}{256\,abcd}, y = \frac{(a+b+c)^3}{27abc}, z = \frac{(a+b)^2}{4ab}, a,b,c,d \ge 1$

Prove that:

$$ab(1+c+cd)(x+y+z) \le 3(abcdx + abcy + abz)$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$a,b,c,d \ge 1 \Rightarrow ab \le abc \le abcd; a+b-2\sqrt{ab} \ge 0 \Rightarrow z \ge 1$, *similarly:* $x,y \ge 1$

We have: $z \le y \le x$. *In fact:*

$$\frac{(a+b)^2}{4ab} \le \frac{(a+b+c)^3}{27abc} \Leftrightarrow 27c(a+b)^2 \le 4(a+b+c)^3 \quad (1)$$

$$\because 2c(a+b)(a+b) \overset{AM-GM}{\le} \frac{(2c+2a+2b)^3}{27} \Leftrightarrow 2\cdot 27c(a+b)^2 \le 8(a+b+c)^3 \Leftrightarrow \text{(1) true.}$$

$$\frac{(a+b+c)^3}{27abc} \le \frac{(a+b+c+d)^4}{256abcd} \quad (2)$$

$$\Leftrightarrow 256d(a+b+c)^3 \le 27(a+b+c+d)^4$$

$$\therefore 3d(a+b+c)(a+b+c)(a+b+c) \overset{AM-GM}{\le} \frac{(3d+3a+3b+3c)^4}{256}$$

$$\Leftrightarrow 3\cdot 256d(a+b+c)^3 \le 3^4(a+b+c+d)^4 \Leftrightarrow \text{(2) true.}$$

Now, using Chebyshev's inequality:

$$(abcdx + abcy + abz) \ge \frac{1}{3}(abdd + abc + ab)(x+y+z)$$

$$\Leftrightarrow 3(abcdx + abcy + abz) \ge ab(1+c+cd)(x+y+z) \text{ Proved}$$

385. If $0 < z < y < x < \frac{\pi}{2}$ then:

$$\frac{\sin x}{\sin y} + \frac{\sin x + \sin y}{\sin z} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$\frac{\sin x}{\sin y} + \frac{\sin x+\sin y}{\sin z} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2} \quad (1)$$

$$(1) \to \frac{\sin x}{\sin y} + \frac{\sin x}{\sin z} + \frac{\sin y}{\sin z} \overset{AM-GM}{\geq} 3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2} \quad (2)$$

RHS of (2) suffices to be $\geq \frac{6}{\pi} \cdot \sqrt[3]{\left(\frac{x}{z}\right)^2}$ *which reduces to*

$\theta = \frac{\frac{\sin x}{x}}{\frac{\sin z}{z}} \geq \sqrt{\frac{8}{\pi^3}}$ *(3). Consider the function* $f(t) = \frac{\sin t}{t}$ *over* $\left(0, \frac{\pi}{2}\right)$.

$f'(t) = \frac{1}{t^2}(t\cos t - \sin t) < 0$ *because* $t < \tan t$ *on* $\left(0, \frac{\pi}{2}\right)$ *so* $f(t)$ *is decreasing.*

Because $x > z$ *we observe that the ratio* θ *varies between* $\frac{2}{\pi}$

(when $x \to \frac{\pi}{2}$ *and* $z \to 0$ *as* $\lim_{u\to 0}\frac{\sin u}{u} = 1$*)*

and 1 *when* $x \to z$, *therefore* $\frac{\frac{\sin x}{x}}{\frac{\sin z}{z}} > \frac{2}{\pi} > \left(\frac{2}{\pi}\right)^3$ *as* $\frac{2}{\pi} < 1$. *Done.*

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{\sin x}{\sin y} + \frac{\sin x + \sin y}{\sin z} = \frac{\sin x}{\sin y} + \frac{\sin x}{\sin z} + \frac{\sin y}{\sin z} \overset{(AM-GM)}{\geq} 3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2}$$

We must show that: $3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2} \Leftrightarrow \left(\frac{\sin x}{\sin z}\right)^2 > \frac{8}{\pi^3}\left(\frac{x}{z}\right)^2 \Leftrightarrow \frac{\sin x}{\sin z} > \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \cdot \frac{x}{z}$

$$\Leftrightarrow \pi\sqrt{\pi} \cdot \frac{\sin x}{x} > 2\sqrt{2} \cdot \frac{\sin z}{z} \quad (*)$$

Because: $\sin z < z$ *and* $\sin x > \frac{2x}{\pi}$ *for* $x, \in \left(0, \frac{\pi}{2}\right)$

$$\pi\sqrt{\pi} \cdot \frac{\sin x}{x} > \pi\sqrt{\pi} \cdot \frac{2}{\pi} = 2\sqrt{\pi};\ 2\sqrt{2}\,\frac{\sin z}{z} < 2\sqrt{2} \cdot 1 = 2\sqrt{2}$$

We have: $2\sqrt{\pi} > 2\sqrt{2} \Rightarrow$ ***(*) true. Proved.***

386. **If** $0 < a \le b < \frac{\pi}{6}$ **then:**

$$\sin(5\sqrt{ab})\cdot\sin\left(\frac{12ab}{a+b}\right) \ge \sin(6\sqrt{ab})\cdot\sin\left(\frac{10ab}{a+b}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sin(5\sqrt{ab})\cdot\sin\left(\frac{12ab}{a+b}\right) = \frac{1}{2}\left[\cos\left(5\sqrt{ab}-\frac{12ab}{a+b}\right)-\cos\left(5\sqrt{ab}+\frac{12ab}{a+b}\right)\right]$$

$$\sin(6\sqrt{ab})\cdot\sin\left(\frac{10ab}{a+b}\right) = \frac{1}{2}\left[\cos\left(6\sqrt{ab}-\frac{10ab}{a+b}\right)-\cos\left(6\sqrt{ab}+\frac{10ab}{a+b}\right)\right]$$

Must show that:

$$\cos\left(5\sqrt{ab}-\frac{12ab}{a+b}\right)+\cos\left(6\sqrt{ab}+\frac{10ab}{a+b}\right) \ge \cos\left(6\sqrt{ab}-\frac{10ab}{a+b}\right)+\cos\left(5\sqrt{ab}+\frac{12ab}{a+b}\right)$$

$$\Leftrightarrow 2\left\{\cos\left[\frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}\right]\cdot\cos\left[\frac{\sqrt{ab}}{2}+\frac{11ab}{a+b}\right]\right\} \ge 2\left\{\cos\left[\frac{11\sqrt{ab}}{2}+\frac{ab}{a+b}\right]\cdot\cos\left[\frac{\sqrt{ab}}{2}-\frac{11ab}{a+b}\right]\right\} \quad \textbf{(*)}$$

$$\cos\left[\frac{11ab}{a+b}+\frac{\sqrt{ab}}{2}\right] \overset{(1)}{\ge} \cos\left[\frac{11\sqrt{ab}}{2}+\frac{ab}{a+b}\right] \Leftrightarrow \frac{11\sqrt{ab}}{2}+\frac{ab}{a+b} \ge \frac{11ab}{a+b}+\frac{\sqrt{ab}}{2}$$

$$\Leftrightarrow 5\sqrt{ab} \ge 10\cdot\frac{ab}{a+b} \Leftrightarrow a+b \ge 2\sqrt{ab} \quad \textit{(true)}$$

$$\cos\left(\frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}\right) \overset{(2)}{\ge} \cos\left(\frac{\sqrt{ab}}{2}-\frac{11ab}{a+b}\right) \Leftrightarrow \frac{\sqrt{ab}}{2}-\frac{11ab}{a+b} \ge \frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}$$

$\Leftrightarrow 6\sqrt{ab} \ge \frac{12ab}{a+b} \Leftrightarrow a+b \ge 2\sqrt{ab}$ ***(true). From (1) and (2) we have: (*) true. Proved.***

387. **If** $x, y \in \left(0, \frac{\pi}{2}\right)$ **then:**

$$\frac{3}{\sin x}-\frac{1}{2\sin y\cos x}+\frac{1}{2\cos x\cos y} < \frac{6}{\sin 2x\sin 2y\cos x}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\frac{3}{\sin x}-\frac{1}{2\sin y\cos x}+\frac{1}{2\cos x\cos y} < \frac{6}{\sin 2x\sin 2y\cos x}$$

$$\Leftrightarrow \frac{6\cos^2 x\sin 2y+\sin 2x\,(\sin y-\cos y)}{\sin 2x\sin 2y\cos x}<\frac{6}{\sin 2x\sin 2y\cos x}$$

Because: $0<x,y<\frac{\pi}{2}\Rightarrow \sin 2x,\sin 2y,\cos x>0$. ***We need to prove:***

$$6\cos^2 x\sin 2y+\sin 2x\,(\sin y-\cos y)<6$$

$$\Leftrightarrow 3(1+\cos 2x)\sin 2y+\sin 2x\,(\sin y-\cos y)<6$$

$$\Leftrightarrow 3\cos 2x\sin 2y+\sin 2x\,(\sin y-\cos y)<3$$ ***(*) We have:***

$$LHS_{(*)}\le 3\left|\cos 2x\,|\sin 2y|+|\sin 2x||\sin y-\cos y|\right|\overset{(BCS)}{\le}\sqrt{9\sin^2 2y+1-\sin 2y}\overset{(1)}{\le}3$$

(1) $\Leftrightarrow 9\sin^2 2y-\sin 2y<8\quad (t=\sin 2y,\,0<t<1)$

$$\Leftrightarrow 9t^2-t-8<0\Leftrightarrow 9(t-1)\left(t+\frac{8}{9}\right)<0;$$ ***(True)*** $\Rightarrow$ ***(1) true*** $\Rightarrow$ ***(*) true.***

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$f_1=\frac{3}{\sin x}-\frac{1}{2\sin y\cos x}+\frac{1}{2\cos x\cdot\cos y}\le\frac{6}{\sin 2x\cdot\sin 2y\cdot\cos x}$$

$$f_1=\frac{3}{\sin x}+\frac{1}{2\cos x}\cdot\left[\frac{1}{\cos y}-\frac{1}{\sin y}\right];\ f_1=\frac{3}{\sin x}+\frac{1}{2\cos x}\cdot\frac{\sin y-\cos y}{\cos y\cdot\sin y}$$

$$f_1=\frac{3}{\sin x}+\frac{\sin y-\cos y}{\cos x\cdot\sin y}=\frac{6\cos x}{\sin 2x}+\frac{\sin y\cdot\cos y}{\sin y\cdot\cos x};\ f_1=\frac{6\cos^2 x\cdot\sin y+\sin x\cdot(\sin y-\cos y)}{\sin 2x\cdot\sin y\cdot\cos x}$$

Let us prove that: $6\cos^2 x\cdot\sin 2y+\sin x\,(\sin y-\cos y)\overset{?}{<}6$

$$6\cos^2 x\cdot\sin 2y-\sin x\,(\cos y-\sin y)\overset{?}{<}6$$

$$6\cos^2 x\sin 2y-\sqrt{2}\cdot\sin 2x\cdot\cos\left(y-\frac{\pi}{x}\right)\overset{?}{<}6$$

Suppose:

$$f(y)=6\cos^2 x\cdot\sin 2y-\sqrt{2}\cdot\sin 2x\cdot\cos\left(y-\frac{\pi}{4}\right),\ y\in\left]0,\frac{\pi}{2}\right[$$

$$\lim_{\substack{y\to 0\\ >}}\big(f(y_1)\big)=-\sin x,\ \lim_{\substack{y\to\frac{\pi}{2}\\ <}}\big(f(y)\big)=-\sin 2x$$

$$'f(y)=6\cos^2 x\cdot 2\cos 2y-\sqrt{2}\sin 2x\left(-\sin\left(y-\frac{\pi}{4}\right)\right)$$

$$'f(y)=12\cos^2 x\cdot\cos 2y+\sqrt{2}\cdot\sin x\cdot\sin\left(u-\frac{\pi}{4}\right)$$

y	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
f'(y)	+ + + + + + + + + + + +	0	− − − − − − − − − − − −
$f(y)$	$-\sin 2x$ ↗	$6\cos^2 x - \sqrt{2}\sin 2x$	↘ $-\sin 2x$

$$\forall x \in \left]0, \frac{\pi}{2}\right[: 6\cos^2 x \cdot \sin 2y - \sqrt{2}\sin 2x \cdot \cos\left(y - \frac{\pi}{4}\right) \le 6\cos^3 x - \sqrt{2}\sin 2x$$

Note: $6\cos^2 x - \sqrt{2}\sin 2x \overset{?}{<} 6 \Rightarrow 6\cos^2 x - 6 \overset{?}{<} \sqrt{2}\sin 2x \Rightarrow 6(\cos^2 x - 1) \overset{?}{<} \sqrt{2}\sin 2x$

$-6\sin^2 x \overset{?}{<} \sqrt{2}\sin 2x \Rightarrow \sqrt{2}\sin 2x + 6\sin^2 x \overset{?}{>} 0$ ***it's true for*** $x \in]0, \pi[$

So, $f_1 \le \frac{6}{\sin 2x \cdot \sin 2y \cdot \cos x}$

388. If $0 < a < b < \frac{\pi}{2}$ then:

$$\frac{e^{\sin b} - e^{\sin a}}{\sin b - \sin a} > 1 + \frac{\sin(a + b)}{2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

For $0 < \alpha < \beta < 1$: $\frac{e^\beta - e^\alpha}{\beta - \alpha} = \frac{1}{\beta - \alpha}\left[(\beta - \alpha) + \frac{1}{2!}(\beta^2 - \alpha^2) + \frac{1}{3!}(\beta^3 - \alpha^3) + \cdots\right]$

$$= 1 + \frac{1}{2!}(\beta + \alpha) + \frac{1}{3!}(\beta^2 + \beta\alpha + \alpha^2) + \cdots > 1 + \frac{1}{2}(\beta + \alpha)$$

$$\Rightarrow \frac{e^{\sin b} - e^{\sin a}}{\sin b - \sin a} > 1 + \frac{1}{2}(\sin b + \sin a)$$

$$\ge 1 + \frac{1}{2}(\sin b\cos a + \sin a\cos b) = 1 + \frac{1}{2}\sin(b + a)$$

Solution 2 by Sagar Kumar-Patna Bihar-India

$0 < a < b < \frac{\pi}{2}$. ***Let*** $f(x) = e^x$. ***Consider the interval*** $[\sin a, \sin b]$

$f(x)$ ***is continuous and differentiable*** $\therefore$ ***By LMVT there exist*** $c \in (\sin a, \sin b)$

s.t $\frac{e^{\sin b} - e^{\sin a}}{\sin b - \sin a} = e^c \Rightarrow \frac{e^{\sin b} - e^{\sin a}}{\sin b - \sin a} \ge 1 + c + \frac{c^2}{2!} \ldots > 1 + \frac{(\sin a + \sin b)}{2}$

$$\frac{e^{\sin b}-e^{\sin a}}{\sin b-\sin a}>1+\frac{\sin(a+b)}{2}$$

as $\sin(a+b)=\sin a\cos b+\cos a\sin b$ *which is clearly less than*

$$\sin(a)+\sin(b)$$

Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

$$\left.\begin{array}{c}0<a<b<\frac{\pi}{2}\\ \sin x\nearrow, x\in\left[0,\frac{\pi}{2}\right]\end{array}\right\}\text{ so, }\sin a<\sin b$$

$f(x)=e^x$ *convex. Hermite Hadamard:* $\frac{\int_{\sin a}^{\sin b}f(x)dx}{\sin b-\sin a}\geq f\left(\frac{\sin a+\sin b}{2}\right)$

$$\Leftrightarrow\frac{[e^x]_{\sin x}^{\sin b}}{\sin b-\sin a}\geq e^{\frac{\sin a+\sin b}{2}}>\frac{\sin a+\sin b}{2}+1$$

$$\Leftrightarrow\frac{e^{\sin b}-e^{\sin a}}{\sin b-\sin b}>\frac{\sin a\cdot 1+\sin b\cdot 1}{2}+1\geq\frac{\sin a\cdot\cos b+\sin b\cdot\cos a}{2}+1$$

$$=\frac{\sin(a+b)}{2}+1$$

Solution 4 by Soumava Chakraborty-Kolkata-India

If $0<a<b<\frac{\pi}{2}$, *then* $\frac{e^{\sin b}-e^{\sin a}}{\sin b-\sin a}>1+\frac{\sin(a+b)}{2}$

$$1+\frac{\sin(a+b)}{2}=1+\frac{\sin a\cos b+\cos a\sin b}{2}<1+\frac{\sin a+\sin b}{2}$$

$$(\because\cos a,\cos b<1\ and\ \sin a,\cos a,\sin b,\cos b>0)\overset{?}{<}\frac{e^{\sin b}-e^{\sin a}}{\sin b-\sin a}$$

$$\Leftrightarrow e^{\sin b}-b-\frac{1}{2}\sin^2 b\overset{(1)}{>}e^{\sin a}-a-\frac{1}{2}\sin^2 a$$

Let $f(x)=e^{\sin x}-x-\frac{1}{2}\sin^2 x\ \forall x\in\left(0,\frac{\pi}{2}\right)$

$$f'(x)=\cos x\left(e^{\sin x}-\sin x-1\right)=(\cos x)\left(e^{\sin x}\right)-\sin x\cos x-\cos x$$

$$>\cos x(1+\sin x)-\sin x\cos x-\cos(\because e^{\sin x}>1+\sin x)=0$$

$\Rightarrow f'(x)>0\Rightarrow f(x)$ *is an increasing function on* $\left(0,\frac{\pi}{2}\right)$

$\therefore$ *as* $b>a,\therefore e^{\sin b}-b-\frac{1}{2}\sin^2 b>e^{\sin a}-a-\frac{1}{2}\sin^2 a\Rightarrow$ ***(1) is true (proved)***

389. If $0 < x, y, z < \frac{\pi}{6}$ then:

$$(\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)} + (\sin^2 y)^{\sin\left(\frac{z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)} + (\sin^2 z)^{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} > 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Rightarrow (\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cdot\cos\left(\frac{y-z}{2}\right)} = (\sin x)^{(\sin y+\sin z)}$$

Need to prove $\Rightarrow \sum_{cyc}(\sin x)^{(\sin y+\sin z)} > 1$

$$\Leftrightarrow (1 + \sin x - 1)^{(\sin y+\sin z)} \overset{Bernoulli}{>} 1 + (\sin x - 1)(\sin y + \sin z)$$

$$= 1 + \sin x \cdot \sin y + \sin x \cdot \sin y - \sin z$$

$$\Rightarrow \sum_{cyc}(\sin x)^{(\sin y+\sin z)} > 3 + 2\sum_{cyc}\sin x \sin y - 2(\sin x + \sin y + \sin z)$$

$$> 3 + 2\sum_{cyc}\sin x\,(\sin y - 1) > 3 - 2\sum_{cyc}\sin x\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2$$

$$\because x < \frac{\pi}{6} \Rightarrow \sin x < \frac{1}{2} \Rightarrow \left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2 < \frac{1}{2}$$

$$\Leftrightarrow \sum_{cyc}\sin x\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2 < \frac{3}{4} \Rightarrow 2\sum_{cyc}\sin x\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2 < \frac{3}{2}$$

Surely $\Rightarrow \sum_{cyc}(\sin x)^{\sin y+\sin z} > 1$ *(proved)*

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right) = \frac{1}{2}[\sin y + \sin z]$$

$$\sin\left(\frac{x+z}{2}\right)\cos\left(\frac{z-x}{2}\right) = \frac{1}{2}[\sin z + \sin x]$$

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}[\sin x + \sin y]$$

$$\Rightarrow LHS \geq (\sin x)^{(\sin y+\sin z)} + (\sin y)^{[\sin z+\sin x]} + (\sin z)^{[\sin x+\sin y]}$$

$$= (\sin x)^{\sin y}(\sin x)^{\sin z} + (\sin y)^{\sin z}\cdot(\sin y)^{\sin x} + (\sin z)^{\sin x}\cdot(\sin z)^{\sin y}$$

Let $X = \sin x\,; Y = \sin y\,; Z = \sin z\,; \left(X, Y, Z \in \left(0; \frac{1}{2}\right)\right)$

We prove that: $X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X \cdot Z^Y > 1$*. Using AM-GM:*

$$X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X\cdot Z^Y\geq 3\sqrt[3]{(X^Y\cdot Y^Z\cdot Z^X)(Y^X\cdot X^Z\cdot Z^X)}$$

But $X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}};\ Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$

Now, we want to prove: $X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$ *(Similarly:* $Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$*)*

$$\Leftrightarrow Y\ln X+Z\ln Y+X\ln Z>-\ln\left(3\sqrt{3}\right)$$

Using Jensen's inequality with $f(t)=\ln(\sin t),\ t\in\left(0,\frac{\pi}{2}\right)$

$$Y\ln X+Z\ln Y+X\ln Z\geq (X+Y+Z)\ln\left(\frac{XY+YZ+ZX}{X+Y+Z}\right)$$

$$>\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\ln\left(\frac{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}\right)=\frac{3}{2}\ln\frac{1}{2}=-\frac{3}{2}\ln 2>-\ln\left(3\sqrt{3}\right)$$

(Because $g(n)=n\ln\frac{\alpha}{n}\searrow\left(0;\frac{1}{2}\right)$*)*

Hence, $X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X+Z^X\cdot Z^Y>1$ *(Proved)*

390. If $x\in\mathbb{R}$ then:

$$\sin^{-1}(\sin^2 x)+\sin^{-1}(\cos^2 x)<\frac{49-9\cos 4x}{24}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Tran Hong-Dong Thap-Vietnam

$$\sin^{-1}(\cos^2 x)+\sin^{-1}(\sin^2 x)\leq\frac{49-9[8\cos^4 x-8\cos^2 x+1]}{24}\quad (*)$$

Let $u=\sin^{-1}(\cos^2 x),\ v=\sin^{-1}(\sin^2 x)\ \left(0\leq u,v\leq\frac{\pi}{2}\right)$

$$\Rightarrow \cos^2 x=\sin u;\ \sin^2 x=\sin v\Rightarrow \sin u+\sin v=1\Rightarrow 0<u+v\leq\frac{\pi}{2}$$

$$(*)\Leftrightarrow 24(u+v)\leq 40-72\sin^2 u+72\sin u$$

$$\Leftrightarrow 9\sin^2 u-9\sin u+3(u+v)-5\leq 0\quad (**)$$

(**) true because: $-9\sin u\sin v+3(u+v)-5\leq 0+3\cdot\frac{\pi}{2}-5<0$ *(proved)*

391. If $A\in M_3(\mathbb{R}),\ Tr\ A=\det A=1$. Prove that:

$$\det(A^2+A+I_3)\geq 3Tr\left(A^{-1}\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Serban George Florin-Romania

$$P(\lambda) = \lambda^3 - (Tr\ A)\lambda^2 + (Tr\ A^*)\lambda - \det A = -\det(A - \lambda I_3)$$

$$\varepsilon^3 = 1, \varepsilon^2 + \varepsilon + 1 = 0, \varepsilon = -\frac{1 + i\sqrt{3}}{3}$$

$$\det(A^2 + A + I_3) = \det(A - \varepsilon I_3) \cdot \det(A - \varepsilon^2 I_3) = P(\varepsilon) \cdot P(\varepsilon^2) =$$

$$= (1 - (Tr\ A)\varepsilon^2 + (Tr\ A^*)\varepsilon - \det A)(1 - (Tr\ A)\varepsilon + (Tr\ A^*)\varepsilon^2 - \det A) =$$

$$= (1 - \varepsilon^2 + (Tr\ A^*)\varepsilon - 1) \cdot (1 - \varepsilon + (Tr\ A^*)\varepsilon^2 - 1) =$$

$$= [-\varepsilon^2 + (Tr\ A^*)\varepsilon] \cdot [-\varepsilon + (Tr\ A^*)\varepsilon^2] = \varepsilon^3 - (Tr\ A^*)\varepsilon^4 -$$

$$-(Tr\ A^*)\varepsilon^2 + (Tr\ A^*)(Tr\ A^*)\varepsilon^3 = 1 - \varepsilon \cdot (Tr\ A^*) - (Tr\ A^*)\varepsilon^2 +$$

$$+(Tr\ A^*)^2 = 1 - (Tr\ A^*)(\varepsilon + \varepsilon^2) + (Tr\ A^*)^2 = (Tr\ A^*)^2 + Tr\ A^* + 1$$

$$A^* = (\det A) \cdot A^{-1} \Rightarrow A^* = A^{-1} \Rightarrow Tr\ A^* = Tr\ A^{-1}$$

$$\Rightarrow \det(A^2 + A + I_3) = \left(Tr\ (A^{-1})\right)^2 + Tr\ (A^{-1}) + 1 \geq 3Tr\ (A^{-1})$$

$$\Leftrightarrow \left(Tr(A^{-1})\right)^2 - 2Tr(A^{-1}) + 1 \geq 0 \Leftrightarrow (Tr\ (A^{-1}) - 1)^2 \geq 0.$$ ***True.***

Solution 2 by Ravi Prakash-New Delhi-India

Let $P(\lambda) = \det(\lambda I - A)$ ***be the characteristic polynomial of*** A***, then:***

$$P(\lambda) = \lambda^3 - (Tr\ A)\lambda^2 + Tr\ (A^*)\lambda - \det(A) = \lambda^3 - \lambda^2 + \alpha\lambda - 1$$

To show: $\det(A^2 + A + I_3) \geq 3\ Tr(A^{-1})$ ***(1)***

$LHS = \det\left((A - \omega I_3)(A - \overline{\omega} I_3)\right)$ ***[***$\omega \neq 1$***, cube root of unity]***

$$= \det(A - \omega I_3)\det(A - \overline{\omega} I_3) = (\det(A - \omega I_3))^2 \geq 0$$

If $(A^{-1}) \leq 0$ ***, (1) immediately follows. Suppose*** $Tr(A^{-1}) = \alpha > 0$

But $A^{-1} = \frac{1}{|A|} \cdot A^* = A^*\ [|A| = 1]$

Now, $|\det(A - \omega I_3)|^2 = \left|(-1)^3 \det|\omega I_3 - A|\right|^2 = |P(\omega)|^2$

$$= \left|\omega^3 - \omega^2 + \alpha\omega - 1\right|^2 = |\omega||\alpha - \omega|^2 = \left|\left(\alpha - \frac{1}{2}\right) + \frac{\sqrt{3}}{2}i\right| = \left(\alpha + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \alpha^2 + \alpha + 1 \geq 3\alpha\ [\because \alpha > 0]$$

Thus, $\det(A^2 + A + I_3) \geq 3\alpha = 3Tr\ (A^*) = 3Tr\ (A^{-1})$

392. If $a,b,c\in(0,\pi)$ then:

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+(\sin a+\sin b+\sin c)^2>1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $x=\sin a;\ y=\sin b;\ z=\sin c$. *Because:* $0<a,b,c<\pi\Rightarrow 0<x,y,z<1$

$$\prod\cos^2 a+\left(\sum\sin a\right)^2>1\Leftrightarrow\prod(1-\sin^2 a)+\left(\sum\sin a\right)^2>1$$

$$\Leftrightarrow(1-x^2)(1-y^2)(1-z^2)+(x+y+z)^2>1$$

$$\Leftrightarrow 1-(x^2+y^2+z^2)+(x^2y^2+y^2z^2+z^2x^2)-(xyz)^2+x^2+y^2+z^2+$$
$$+2(xy+xz+yz)>1$$

$$\Leftrightarrow(x^2y^2+y^2z^2+z^2x^2)+2(xy+yz+zx)-(xyz)^2>0$$

It is true because:

$$(x^2y^2+y^2z^2+z^2x^2)+2xy+2yz+2zx)\ge 6\sqrt[6]{2^3(xyz)^6}=6\sqrt{2}xyz$$

and: $6\sqrt{2}xyz>xyz>(xyz)^2\ \ (\because 0<xyz<1)$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+(\sin a+\sin b+\sin c)^2>1,a,b,c\in(0,\pi)$$

$$\left((\sin a+\sin b)+\sin c\right)^2=(\sin a+\sin b)^2+2\sin c(\sin a+\sin b)+\sin^2 c$$
$$=\sin^2 a+\sin^2 b+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b+\sin^2 c$$
$$=\sin^2 a+\sin^2 b+\sin^2 c+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b$$

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+\sin^2 a+\sin^2 b+\sin^2 c+2\sin a\sin b+$$
$$+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

But: $(1-\sin^2 a)(1-\sin^2 b)(1-\sin^2 c)=\cos^2 a\cdot\cos^2 b\cdot\cos^2 c$

$$(1-\sin^2 b-\sin^2 a+\sin^2 b\sin^2 a)(1-\sin^2 c)=\cos^2 a\cdot\cos^2 b\cdot\cos^2 c$$

$$1-\sin^2 c-\sin^2 b+\sin^2 b\sin^2 c-\sin^2 a+\sin^2 a\cdot\sin^2 c+\sin^2 b\sin^2 a-$$
$$-\sin^2 a\sin^2 b\sin^2 c+\sin^2 a+\sin^2 b+\sin^2 c+$$
$$+2\sin a\cdot\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

$$1+\sin^2 b\sin^2 c+\sin^2 a\sin^2 c+\sin^2 b\sin^2 a-\sin^2 a\sin^2 b\sin^2 c+$$

$$+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}1$$

$$\sin^2 b\sin^2 c+\sin^2 a\cdot\sin^2 c+\sin^2 b\cdot\sin^2 a-\sin^2 a\sin^2 b\sin^2 c+$$

$$+2\sin a\sin b+2\sin c\sin a+2\sin c\sin b\overset{?}{>}0$$

$$\sin b\sin c(\sin b\sin c+2)+\sin a\sin c(\sin a\sin c+2)+$$

$$+\sin b\sin a(\sin b\sin a+2)\overset{?}{>}\sin^2 a\sin^2 b\sin^2 c$$

$$\frac{\sin b\sin c+2}{\sin^2 a\sin b\sin c}+\frac{\sin a\sin c+2}{\sin^2 b\sin a\sin c}+\frac{\sin b\sin a+2}{\sin^2 c\sin a\sin b}\overset{?}{>}0$$

because $a,b,c\in(0,\pi)$: $\sin a>0,\sin b>0$ ***and*** $\sin c>0$***. So, the inequality is true.***

Solution 3 by Amit Dutta-Jamshedpur-India

Let $P=\cos^2 a\cdot\cos^2 b\cdot c+(\sin a+\sin b+\sin c)^2$***. Put*** $\sin a=p,\sin b=q,\sin c=r$

$$P=(1-p^2)(1-q^2)(1-r^2)+(p+q+r)^2\because a,b,c\in(0,\pi)\Rightarrow p,q,r\in(0,1)$$

$$P=1-p^2-q^2-r^2+(pq)^2+(qr)^2+(pr)^2-(pqr)^2+$$

$$+p^2+q^2+r^2+2pq+2qr+2pr$$

$$P=1+(pq)^2+(qr)^2+(pr)^2-(pqr)^2+2(pq+qr+pr)$$

$$P=1+p^2q^2(1-r^2)+q^2r^2+p^2r^2+2(pq+qr+pr)$$

$$\because 0<r<1\Rightarrow 0<r^2<1\Rightarrow(1-r^2)>0$$

$$\therefore P>1\{\because p^2q^2(1-r^2)+q^2r^2+p^2q^2+2(pq+qr+pr)>0\}$$

$$\therefore P>1.$$ ***Proved.***

393. If $0<a,b,c\le 16$ **then:**

$$27\exp\left(\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right)\le\frac{(a+2b)(b+2c)(c+2a)}{abc}$$

Proposed by Daniel Sitaru – Romania

Solution by Michael Sterghiou-Greece

$$27\exp\left[\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right]\le\frac{\prod_{cyc}(a+2b)}{abc}\quad (1)$$

$$(1)\to\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\le\left(\sum_{cyc}\ln\left(\frac{2a+b}{3}\right)\right)-\sum_{cyc}\ln a\ \textit{or}$$

$$\sum_{cyc}(\sqrt{a}-\ln a)\ge\sum_{cyc}\left(\sqrt{\frac{2a+b}{3}}-\ln\left(\frac{2a+b}{3}\right)\right)\quad (2)$$

The function $f(t)=\sqrt{t}-\ln t$ *on* $(0,16]$ *is convex as*

$f''(t)=\frac{4-\sqrt{t}}{4t^2}\ge 0$ *for* $0<t\le 16$. *Assume WLOG that* $a\ge b\ge c$.

Case I: $b\ge\frac{c+a}{2}$. *The triad* (a,b,c) *majorizes the triad* $\left(\frac{a+2b}{3}\ge\frac{c+2d}{3}\ge\frac{b+2c}{3}\right)$ *as:*

$a\ge\frac{a+2b}{3}$; $a+b\ge\frac{a+2b}{3}+\frac{c+2a}{3}\leftrightarrow b\ge c$ *and* $a+b+c=\sum_{cyc}\frac{a+2b}{3}$

Case II: $b\le\frac{c+a}{2}$. *The triad* $(a\ge b\ge c)$ *majorizes the triad* $\left(\frac{a+2a}{3}\ge\frac{a+2b}{3}\ge\frac{b+2c}{3}\right)$ *as*

$a\ge\frac{c+2a}{3}$ *and* $a+b\ge\frac{c+2a}{3}+\frac{a+2b}{3}\leftrightarrow b\ge c$. *Applying Karamata's inequality for the convex function* $f(t)=\sqrt{t}-\ln t$ *on* $(0,16]$ *for the above triads for either case I or II we obtain (2). Done!*

394. **If** $-\frac{\pi}{2}\le x,y,z\le\frac{\pi}{2}$ **then, prove that:**

$$\sin^2 x\cdot\sin^2 y\cdot\sin^2 z+(\cos x+\cos y+\cos z)^2\ge 1$$

Proposed by Sudhir Jha-Kolkata-India

(Inspired by Prof. Daniel Sitaru)

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $\cos x=a,\cos y=b,\cos z=c\because -\frac{\pi}{2}\le x,y,z\le\frac{\pi}{2}$

$$\therefore 0\le\cos x,\cos y,\cos z\le 1\Rightarrow a,b,c\in[0,1]$$

Given inequality $\Leftrightarrow\prod(1-a^2)+(\sum a)^2\ge 1$

$\Leftrightarrow\sum a^2b^2+2\sum ab\overset{(1)}{\ge}a^2b^2c^2$ *(after simplification)*

$$\because 0\le c^2\le 1,\therefore\frac{a^2b^2c^2}{3}\le\frac{a^2b^2}{3}\overset{?}{\le}a^2b^2+2ab$$

$\Leftrightarrow 2a^2b^2+6ab\ge 0\to$ *true* $\because a,b\ge 0\therefore\frac{a^2b^2c^2}{3}\overset{(a)}{\le}a^2b^2+2ab$

Similarly, $\frac{a^2b^2c^2}{3}\overset{(b)}{\le}b^2c^2+2bc$ & $\frac{a^2b^2c^2}{3}\overset{(c)}{\le}c^2a^2+2ca$

(a)+(b)+(c)$\Rightarrow\sum a^2b^2+2\sum ab\ge a^2b^2c^2\Rightarrow$ *(1)* $\Rightarrow$ *given inequality is true (proved)*

Solution 2 by Tran Hong-Dong Thap-Vietnam

Putting: $u = \cos x\,;v = \cos y\,;w = \cos z$

$$\left(x;y;z \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right] \Rightarrow 0 \le u,v,w \le 1 \Rightarrow 0 \le uvw \le 1\right)$$

$$\sin^2 x \sin^2 y \sin^2 z + (\cos x + \cos y + \cos z)^2 \ge 1$$

$$\Leftrightarrow (1-\cos^2 x)(1-\cos^2 y)(1-\cos^2 z) + (\cos x + \cos y + \cos z)^2 \ge 1$$

$$\Leftrightarrow (1-u^2)(1-v^2)(1-z^2) + (u+v+w)^2 \ge 1$$

$$\Leftrightarrow u^2v^2 + v^2w^2 + w^2u^2 + 2(uv+vw+wu) \ge (uvw)^2$$

$$\because u^2v^2 + v^2w^2 + w^2u^2 + 2(uv+vw+wu) \overset{AM-GM}{\ge} 6\sqrt[6]{2(uvw)^6} = 6\sqrt[6]{2}uvw$$

$\because 6\sqrt[6]{2}uvw \ge uvw \ge (uvw)^2$ ***(Because:*** $0 \le uvw \le 1$***) Proved.***

395. If $0 < d < c < b < a < \frac{\pi}{2}$ then:

$$\csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right) < \frac{\sin a}{\sin d}$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\csc\left(\frac{\pi a}{2b}\right) = \frac{1}{\sin\left(\frac{\pi a}{2b}\right)} \le \frac{1}{\frac{2}{\pi}\left(\frac{\pi a}{2b}\right)} = \frac{b}{a};\ \csc\left(\frac{\pi b}{2c}\right) = \frac{1}{\sin\left(\frac{\pi b}{2c}\right)} \le \frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi}{2c}\right)} = \frac{c}{b}$$

$$\csc\left(\frac{\pi c}{2d}\right) = \frac{1}{\sin\left(\frac{\pi c}{2d}\right)} \le \frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi c}{2d}\right)} = \frac{d}{c} \to \csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right) \le \frac{b}{a}\cdot\frac{c}{b}\cdot\frac{d}{c} = \frac{d}{a}$$

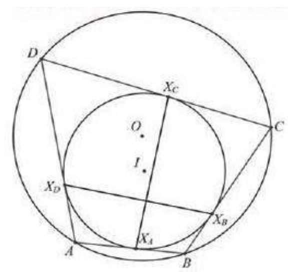
Now, we must show that: $\frac{d}{a} < \frac{\sin a}{\sin d} \leftrightarrow d\sin d < a\sin a$ ***(*)***

Let $f(x) = x\sin x \quad \left(0 < x < \frac{\pi}{2}\right) \to f'(x) = \sin x + x\cos x > 0 \quad \left(0 < x < \frac{\pi}{2}\right)$

Hence, $f(x) \nearrow \left(0;\frac{\pi}{2}\right)$

Because $0 < d < a < \frac{\pi}{2} \to f(d) < f(a) \to d\sin d < a\sin a \to$ ***(*) true. Proved.***

396.



I – incentre, R – circumradii

$$IA\cdot IB\cdot IC\cdot IC\le R^4$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution by Soumava Chakraborty-Kolkata-India

Let $AB=a, BC=b, CD=c, DA=d, AC=p, BD=q, AI=u, BI=v, CI=x, DI=y$

Let s, r be the semi-perimeter, inradius respectively.

Now, $ux+vy\overset{(1)}{=}\frac{4Rr^2(p+q)}{pq}$. Again, $4Rrs=\sqrt{pq}\sqrt{(ab+cd)(bc+ad)}$

$\overset{G\le A}{\le}\sqrt{pq}\left(\frac{ab+cd+bc+ad}{2}\right)=\sqrt{pq}\frac{(c+a)(b+d)}{2}=\frac{\sqrt{pq}}{2}s^2$ ($\because$ by Pitot's theorem for tangential quadrilaterals, $a+c=b+d=s$)

$$\Rightarrow\sqrt{pq}\ge\frac{8Rr}{s}\Rightarrow pq\ge\frac{64R^2r^2}{s^2}\Rightarrow\frac{1}{pq}\overset{(2)}{\le}\frac{s^2}{64R^2r^2}$$

(1), (2) $\Rightarrow ux+vy\le\frac{4Rr^2s^2}{64R^2r^2}(p+q)\overset{?}{\le}2R^2\Leftrightarrow p+q\overset{?}{\le}\frac{32R^3}{s^2}\Leftrightarrow(p+q)^2\underset{(3)}{\overset{?}{\le}}\frac{1024R^6}{s^4}$

$$(p+q)^2=p^2+q^2+2pq$$

$$=\frac{(ac+bd)(ad+bc)}{ab+cd}+\frac{(ac+bd)(ab+cd)}{ad+bc}+2pq$$

$$\overset{Ptolemy}{=}pq\left(\frac{ad+bc}{ab+cd}+1+\frac{ab+cd}{ad+bc}+1\right)=pq\left[\frac{(a+c)(b+d)}{ab+cd}+\frac{(a+c)(b+d)}{ad+bc}\right]$$

$$\overset{Pitot}{=}pqs^2\left(\frac{1}{ab+cd}+\frac{1}{ad+bc}\right)=pqs^2\frac{(a+c)(b+d)}{(ab+cd)(ad+bc)}\underset{(4)}{\overset{Pitot}{=}}\frac{pqs^4}{(ab+cd)(ad+bc)}$$

By Brahmagupta & Parameshara: $4\Delta R=\sqrt{(ab+cd)(ad+bc)(ac+bd)}$

$$(\Delta=area)\Rightarrow 16R^2abcd=(ab+cd)(ad+bc)pq\ (\because\Delta=\sqrt{abcd})$$

$$\Rightarrow 16R^2(r^2s^2) = (ab+cd)(ad+bc)\cdot 2r\left(r+\sqrt{4R^2+r^2}\right)$$

$$\Rightarrow (ab+cd)(ad+bc) \overset{(5)}{=} \frac{8R^2rs^2}{r+\sqrt{4R^2+r^2}}$$

(4), (5)$\Rightarrow (p+q)^2 = \frac{pqs^4\left(r+\sqrt{4R^2+r^2}\right)}{8R^2rs^2}$

$$= \frac{2r\left(r+\sqrt{4R^2+r^2}\right)^2 s^4}{8R^2rs^2} = \frac{\left(r+\sqrt{4R^2+r^2}\right)^2 s^2}{4R^2} \overset{?}{\leq} \frac{2^{10}R^6}{s^4}$$

$\Leftrightarrow \left(r+\sqrt{4R^2+r^2}\right)^2 s^6 \underset{(6)}{\overset{?}{\leq}} 2^{10}R^8\cdot 4 = 2^{12}R^8$. ***But*** $\because s \leq r+\sqrt{4R^2+r^2}$

$\therefore$ ***LHS of (6)*** $\leq \left(r+\sqrt{4R^2+r^2}\right)^8 \overset{?}{\leq} 2^{12}R^8$

$$\Leftrightarrow 2\sqrt{2}R \overset{?}{\geq} r+\sqrt{4R^2+r^2} \Leftrightarrow 8R^2 \overset{?}{\geq} r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}$$

$$\Leftrightarrow 4R^2-2r^2 \overset{?}{\geq} 2r\sqrt{4R^2+r^2} \Leftrightarrow 2R^2-r^2 \overset{?}{\geq} r\sqrt{4R^2+r^2}$$

$$\Leftrightarrow (2R^2-r^2)^2 \overset{?}{\geq} r^2(4R^2+r^2)$$

$$\left(\because \frac{R^2}{r^2} \geq 2\ (L.Fejes\ Toth) > \frac{1}{2} \Rightarrow 2R^2-r^2 > 0\right)$$

$\Leftrightarrow 4R^4-8R^2r^2 \overset{?}{\geq} 0 \Leftrightarrow R^2 \overset{?}{\geq} 2r^2 \rightarrow$ ***true, by L. Fejes Toth***

$\Rightarrow$ ***(6)*** $\Rightarrow$ ***(3) is true***$\Rightarrow ux+vy \leq 2R^2$

$\therefore 2R^2 \geq AI\cdot CI+BI\cdot DI \overset{A-G}{\geq} 2\sqrt{AI\cdot BI\cdot CI\cdot DI} \Rightarrow AI\cdot BI\cdot CI\cdot DI \leq R^4$ ***(proved)***

397. $ABCD$ **– tetrahedron,** r **– inradii,** r_A, r_B, r_C, r_D **– exradii. Prove that:**

$$\frac{1}{\sqrt{r_A}}+\frac{1}{\sqrt{r_B}}+\frac{1}{\sqrt{r_C}}+\frac{1}{\sqrt{r_D}} \leq \frac{4}{\sqrt{2r}}$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $\alpha,\beta,\gamma,\delta$ ***be areas of sides opposite to vertices*** A,B,C,D ***respectively.***

Then, $r_A \overset{(1)}{=} \frac{3V}{-\beta+\gamma+\delta-\alpha}, r_B \overset{(2)}{=} \frac{3V}{\gamma+\delta+\alpha-\beta}, r_C \overset{(3)}{=} \frac{3V}{\delta+\alpha+\beta-\gamma}, r_D \overset{(4)}{=} \frac{3V}{\alpha+\beta+\gamma-\delta}$ ***&***

$r \overset{(5)}{=} \frac{3V}{\alpha+\beta+\gamma+\delta}$, *where* $V \to$ *volume. Now, LHS* $\overset{CBS}{\leq} \sqrt{4}\sqrt{\frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C}+\frac{1}{r_D}}$

$$\overset{by\ (1),(2),(3),(4)}{=} 2\sqrt{\frac{\beta+\gamma+\delta-\alpha+\gamma+\delta+\alpha-\beta+\delta+\alpha+\beta-\gamma+\alpha+\beta+\gamma-\delta}{3V}}$$

$$= 2\sqrt{\frac{2(\alpha+\beta+\gamma+\delta)}{3V}} = \frac{2\sqrt{2}\sqrt{2}}{\sqrt{2}}\sqrt{\frac{\alpha+\beta+\gamma+\delta}{3V}} \overset{using\ (5)}{=} \frac{4}{\sqrt{2r}}$$ ***(Proved)***

398. $A(a,0,0), B(0,b,0), C(0,0,c), O(0,0,0),$

$a,b,c > 0, H$ **– orthocenter,** G **– centroid of** ΔABC**. Prove that:**

$$3OH^2 \leq \sqrt[3]{a^2b^2c^2} \leq 3OG^2$$

Proposed by Daniel Sitaru – Romania

Solution by Le Van-Hanoi-Vietnam

$$G\left(\frac{a}{3};\frac{b}{3};\frac{c}{3}\right) \to 3OG^2 = \frac{1}{3}(a^2+b^2+c^2)$$

$MHS \Leftarrow RHS$ *due to AM-GM of 3 numbers* a^2, b^2 *and* c^2

$$\frac{1}{OH^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$LHS \Leftarrow MHS$ *due to AM-GM of 3 numbers* $(ab)^2, (bc)^2$ *and* $(ca)^2$

QED. Equality holds when $a=b=c$*. The proof of* $\frac{1}{OH^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$:

$AK \perp BC$ *at* K

$$OA \perp OB, OA \perp OC \to OA \perp (OBC) \to OA \perp BC \to BC \perp (OAK) \to BC \perp OK$$

$\to \frac{1}{OK^2} = \frac{1}{b^2}+\frac{1}{c^2}$; $OH \perp AK$; $BC \perp (OAK)$, *stated above* $\to BC \perp OH \to OH \perp (ABC)$

$$\to \frac{1}{OH^2} = \frac{1}{OK^2}+\frac{1}{OA^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

399. Let x,y,z **be positive real numbers such that:** $x+y=z=3$**. Find the minimum of expression:**

$$Q = \frac{1}{x(2y^2-yz+2z^2)}+\frac{1}{y(2z^2-zx+2x^2)}+\frac{1}{z(2x^2-xy+2y^2)}+2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Nguyen Van Nho-Nghe An-Vietnam

$$\textit{We have: } \sum xy\sum\frac{x}{y}\ge\left(\sum x^2\right)^2=9\to\sum\frac{x}{y}\ge\frac{9}{\sum xy}\ \text{and}\ \sum xy\le\frac{(\sum x)^2}{3}=3.$$

$$Q=\sum\left(\frac{1}{x(2y^2-yz+2z^2)}+\frac{x}{3y}\right)+\frac{5}{3}\sum\frac{x}{y}\overset{AM-GM}{\ge}\sum\frac{2}{3y(2y^2-yz+2z^2)}+\frac{15}{\sum xy}$$

$$\overset{AM-GM}{\ge}\sum\frac{4}{3y+2y^2-yz+2z^2}+\frac{15}{\sum xy}\ge\frac{36}{3\sum y+\sum(2y^2-yz+2z^2)}+\frac{15}{\sum xy}$$

$$=\sum\frac{36}{9+4(\sum x)^2-9\sum xy}+\frac{15}{\sum xy}=\sum\frac{4}{5-\sum xy}+\frac{9}{\sum xy}+\frac{6}{\sum xy}$$

$$\ge\frac{(2+3)^2}{5-\sum xy+\sum xy}+\frac{6}{\sum xy}\ge 5+\frac{6}{3}=7;\ Q=7\leftrightarrow x=y=z=1.\ \textit{So: } \min Q=7.$$

400. x,y,z – real numbers different in pairs $x+y+z=0$. Find $\min\Omega$

$$\Omega=(x^2+y^2+z^2)\left(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\right)$$

Proposed by Le Ngo Duc-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$x+y+z=0\Leftrightarrow z=-x-y;\ (x\ne y;x\ne z;y\ne z).\ \textit{We must show that: } \Omega\ge\frac{9}{2}$$

$$\Leftrightarrow[x^2+y^2+(x+y)^2]\left[\frac{1}{(x-y)^2}+\frac{1}{(2y+x)^2}+\frac{1}{(x+2y)^2}\right]\ge\frac{9}{2}$$

$$\Leftrightarrow 2[2x^2+2y^2+2xy][(2y+x)^2(x+2y)^2+(x-y)^2(x+2y)^2+(x-y)^2(y+2x)^2]\ge$$

$$\ge 9\{(x-y)(x+2y)(y+2x)\}^2\Leftrightarrow 243x^4y^2+486x^3y^3+243x^2y^4\ge 0$$

$$\Leftrightarrow 243(x^2y^2)(x^2+2xy+y^2)\ge 0\Leftrightarrow 243(xy)^2(x+y)^2\ge 0\ \textit{(true for } x,y\textit{). Equality:}$$

$$x=0\Rightarrow z=-y\ne 0;\ y=0\Leftrightarrow z=-x\ne 0;\ x=-y(x,y\ne 0)\Rightarrow z=-(-y)-y=0$$

$$\textit{Hence, } \Omega_{\min}=\frac{9}{2}\Leftrightarrow(x;y;z)=(0;t;-t)\ \textit{or}\ (x;y;z)=(t;-t;0)$$

$$\textit{or}\ (t;0;-t)\quad\textit{(with } t\ne 0\textit{)}$$

Solution 2 by Michael Sterghiou-Greece

$$\textit{If } n\in\mathbb{N}^*, a,b,c\in\mathbb{R}:\ \prod_{cyc}(a-b)^2\ne 0\wedge\sum_{cyc}a=0\ \textit{find the min of:}$$

$$A=\left(\sum_{cyc}a^{2n}\right)\left(\sum_{cyc}\frac{1}{(a-b)^{2n}}\right)\quad (T)$$

WLOG assume $a > b > 0 > c$. ***We have*** $c = -(a+b)$. $A \cdot b^{2n} \cdot \frac{1}{b^{2n}}$

gives $A = [x^{2n} + 1 + (x+1)^{2n}] \cdot \left[\frac{1}{(x-1)^{2n}} + \frac{1}{(x+2)^{2n}} + \frac{1}{(2x+1)^{2n}}\right]$ ***with***

$x = \frac{a}{b} > 1$ ***[If*** $a > 0 > b > c$ ***we eliminate*** a ***and with*** $x = \frac{c}{b} > 1$ ***we are at the same situation]. (T) expands to:***

$$\left(\frac{x}{x-1}\right)^{2n} + \left(\frac{1}{x-1}\right)^{2n} + \left(\frac{x+1}{x-1}\right)^{2n} + \left(\frac{x}{x+2}\right)^{2n} + \left(\frac{1}{x+2}\right)^{2n} + \left(\frac{x+1}{x+2}\right)^{2n} +$$

$$+\left(\frac{1}{2x+1}\right)^{2n} + \left(\frac{x}{2x+1}\right)^{2n} + \left(\frac{x+1}{2x+1}\right)^{2n} \quad (1)$$

The terms $\left(\frac{x}{x-1}\right)^{2n} + \left(\frac{x+1}{x+2}\right)^{2n} \underset{AM-GM}{\geq} 2\left[\frac{x^2+x}{x^2+x-2}\right]^n \geq 2^*$ ***and also the terms*** $\left(\frac{x+1}{x-1}\right)^{2n} + \left(\frac{x}{x+2}\right)^{2n} \geq 2^*$. ***In addition:*** $\left(\frac{1}{x-1}\right)^{2n} + \left(\frac{1}{x+2}\right)^{2n} + \left(\frac{1}{2x+1}\right)^{2n} \geq 0^*$. ***Now, the function,*** $f(x) = \left(\frac{x}{2x+1}\right)^{2n} + \left(\frac{x+1}{2x+1}\right)^{2n}$ ***is decreasing because:***

$$f'(x) = 2n \cdot \frac{1}{(2x+1)^2}\left[\left(\frac{x}{2x+1}\right)^{2n} - \left(\frac{x+1}{2x+1}\right)^{2n}\right] < 0, \textit{ therefore}$$

$$f(x) \geq \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\left(\frac{x}{2x+1}\right)^{2n} + \left(\frac{x+1}{2x+1}\right)^{2n}\right] = 2\left(\frac{1}{2}\right)^{2n} = \left(\frac{1}{2}\right)^{2n-1}$$

Summing up the terms of (1) we get $A \geq 2 + 2 + 0 + \left(\frac{1}{2}\right)^{2n-1}$

or A ***minimum*** $= 4 + \left(\frac{1}{2}\right)^{2n-1}$

*** We observe that these*** $\min$ ***are achieved when*** $x \to +\infty$ ***therefore it is legitimate to write at the limit*** $x \to \infty$ ***that*** $A \geq 4 + \left(\frac{1}{2}\right)^{2n-1}$

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To be continued!

Daniel Sitaru