RMM - Cyclic Inequalities Marathon 401 - 500

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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401. **If $a,b,c\ge 0$ then:**

$$3\sqrt{3}(a+b)(b+c)(c+a)\le 8\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Andrew Okukura-Romania

$$a^2+ab+b^2\ge\frac{3}{4}(a+b)^2\ \left(a^2+b^2\ge\frac{3}{4}(a^2+b^2)+\frac{1}{2}ab\right)\Rightarrow$$

$$\prod(a^2+ab+b^2)\ge\prod\frac{3}{4}(a+b)^2\Rightarrow$$

$$\Rightarrow 8\sqrt{\prod(a^2+ab+b^2)}\ge 8\sqrt{\prod\frac{3}{4}(a+b)^2}=8\left(\frac{\sqrt{3}}{2}\right)^3\prod(a+b)=$$

$$=3\sqrt{3}(a+b)(b+c)(c+a)$$

Solution 2 by Ravi Prakash-New Delhi-India

$$3(a+b)^2-4(a^2+ab+b^2)=3(a^2+2ab+b^2)-4(a^2+ab+b^2)$$

$$=-(a^2-2ab+b^2)=-(a-b)^2\le 0$$

$$\Rightarrow\sqrt{3}(a+b)\le 2\sqrt{a^2+ab+b^2}\qquad (1)$$

Similarly, $\sqrt{3}(b+c)\le 2\sqrt{b^2+bc+c^2}$ ***(2)***

$\sqrt{3}(c+a)\le 2\sqrt{c^2+ca+a^2}$ ***(3)***

Multiplying (1), (2), (3) we get

$$3\sqrt{3}(a+b)(b+c)(c+a)\le 8\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\prod_{cyc}(a^2+ab+b^2)=\prod_{cyc}\left(\frac{3(a+b)^2}{4}+\frac{(a-b)^2}{4}\right)\ge\frac{27}{64}\prod_{cyc}(a+b)^2$$

$$\Rightarrow\sqrt{\prod_{cyc}(a^2+ab+b^2)}\ge 3\sqrt{3}\prod_{cyc}(a+b)$$

(proved)

402. If $a,b,c>0$ then:

$$\frac{4}{a+b+c}\left(\sum a^2\right)\left(\sum a^4\right)\le 3\sum a^5+\frac{1}{(a+b+c)^3}\left(\sum a^2\right)^4$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4}{\sum a}\left(\sum a^2\right)\left(\sum a^4\right)\overset{(1)}{\le} 3\sum a^5$$

$$\textit{(1)}\Leftrightarrow 3\left(\sum a^5\right)\left(\sum a\right)^3+\left(\sum a^2\right)^4\ge 4\left(\sum a^2\right)\left(\sum a^4\right)\left(\sum a\right)^2$$

$$\Leftrightarrow \sum a^7b+\sum ab^7+5\sum a^6b^2+5\sum a^2b^6+10abc\left(\sum a^5\right)+$$

$$+abc\left(\sum a^4b+\sum ab^4\right)+4a^2b^2c^2\left(\sum a^2\right)\overset{(2)}{\ge} 5\sum a^5b^3+5\sum a^3b^5+2\sum a^4b^4+$$

$$+8abc\left(\sum a^3b^2+\sum a^2b^3\right)$$

Now, $\sum a^7b+\sum ab^7=\sum(a^7+ab^7)\underset{(a)}{\overset{A-G}{\ge}} 2\sum a^4b^4$

Also, $5\sum a^6b^2+5\sum a^2b^6=5\sum a^2b^2(a^4+b^4)\overset{Chebyshev}{\ge} 5\sum\frac{1}{2}a^2b^2(a^2+b^2)(a^2+b^2)$

$$\underset{(b)}{\overset{A-G}{\ge}} 5\sum a^3b^3(a^2+b^2)=5\sum a^5b^3+5\sum a^3b^5$$

Schur $\Rightarrow a^3(a-b)(a-c)+b^3(b-c)(b-a)+c^3(c-a)(c-b)\ge 0$

$$\Rightarrow \sum a^5+abc\left(\sum a^2\right)\ge \sum a^4b+\sum ab^4$$

$$\Rightarrow 4abc\left(\sum a^5\right)+4a^2b^2c^2\left(\sum a^2\right)\overset{(c)}{\ge} 4abc\left(\sum a^4b+\sum ab^4\right)$$

(a)+(b)+(c)$\Rightarrow LHS\ge 2\sum a^4b^4+5\left(\sum a^5b^3+\sum a^3b^5\right)+6abc\left(\sum a^5\right)+$

$$+5abc\left(\sum a^4b+\sum ab^4\right)\overset{?}{\ge} 5\left(\sum a^5b^3+\sum a^3b^5\right)+2\sum a^4b^4+$$

$$+8abc\left(\sum a^3b^2+\sum a^2b^3\right)\Leftrightarrow$$

$$\Leftrightarrow 6abc\left(\sum a^5\right)+5abc\left(\sum a^4b+\sum ab^4\right)\overset{?}{\ge} 8abc\left(\sum a^3b^2+\sum a^2b^3\right)\quad \textit{(3)}$$

Now, $a^5+b^5\overset{Chebyshev}{\ge}\frac{1}{2}(a^2+b^2)(a^3+b^3)\overset{A-G}{\ge} ab(a^3+b^3)\ge a^2b^2(a+b)=$

$$= a^3b^2 + a^2b^3$$

$$\Rightarrow \sum(a^5 + b^5) \geq \sum a^3b^2 + \sum a^2b^3 \Rightarrow 2\sum a^5 \geq \sum a^3b^2 + \sum a^2b^3$$

$$\Rightarrow 6abc\left(\sum a^5\right) \geq 3abc\left(\sum a^3b^2 + \sum a^2b^3\right) \quad \textit{(d)}$$

Again, $\sum a^4b + \sum ab^4 = \sum ab\,(a^3 + b^3) \geq \sum a^2b^2\,(a + b) = \sum a^3b^2 + \sum a^2b^3$

$$\Rightarrow 5abc\left(\sum a^4b + \sum ab^4\right) \geq 5abc\left(\sum a^3b^2 + \sum a^2b^3\right) \quad \textit{(e)}$$

(d)+(e)⇒(3) is true (proved)

403. If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{1}{\sqrt{x + y^2 + z^2}} + \frac{1}{\sqrt{x^2 + y + z^2}} + \frac{1}{\sqrt{x^2 + y^2 + z}} \leq \sqrt{3}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{1}{\sqrt{x + y^2 + z^2}} = \sum_{cyc} \frac{\sqrt{2 + x}}{\sqrt{(x + y^2 + z^2)(2 + x)}}$$

$$\overset{Cauchy-Schwarz}{\geq} \frac{1}{x + y + z}\sum_{cyc}\sqrt{2 + x} \leq \frac{1}{3}\sqrt{3\sum_{cyc}(x + 2)}\left[\begin{matrix}\because \sqrt{x}\ is\ a\\ concave\ function\end{matrix}\right]$$

$$= \frac{\sqrt{27}}{3} = \sqrt{3} \quad \textit{(proved)}$$

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$\frac{1}{\sqrt{x+y^2+z^2}} \leq \frac{1}{\sqrt{x+\frac{(3-x)^2}{2}}} = \frac{\sqrt{2}}{\sqrt{x^2-4x+9}} \leq \frac{x+5}{6\sqrt{3}} \Leftrightarrow (x-1)^2(x^2+8x+9) \geq 0 \ \textit{true}$$

Similarly $\frac{1}{\sqrt{x^2+y+z^2}} \leq \frac{y+5}{6\sqrt{3}}; \frac{1}{\sqrt{x^2+y^2+z}} \leq \frac{z+5}{6\sqrt{3}}. \sum\frac{1}{\sqrt{x+y^2+z^2}} \leq \frac{1}{6\sqrt{3}}\sum(x+5) = \frac{3}{\sqrt{3}} = \sqrt{3}$

404. For $a, b, c > 0$. Prove:

$$\frac{(a + b)a^3}{a^2 + ab + b^2} + \frac{(b + c)b^3}{b^2 + bc + c^2} + \frac{(c + a)c^3}{c^2 + ca + a^2} \geq \frac{2(a + b + c)^2}{9}$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

Solution 1 by Do Quoc Chinh-Vietnam

By Cauchy-Schwarz's inequality, we have:

$$\left(\sum\frac{a^3(a+b)}{a^2+ab+b^2}\right)\left(\sum\frac{a(a^2+ab+b^2)}{a+b}\right)\ge(a^2+b^2+c^2)^2$$

We have: $\sum\frac{a(a^2+ab+b^2)}{a+b}=\sum\frac{a[(a+b)^2-ab]}{a+b}=\sum a(a+b)-\sum\frac{a^2b}{a+b}$

By Cauchy-Schwarz's inequality, we have: $\sum\frac{a^2b}{a+b}=\sum\frac{a^2b^2}{ab+b^2}\ge\frac{(\sum ab)^2}{\sum a^2+\sum ab}$

$$\Rightarrow\sum\frac{a(a^2+ab+b^2)}{a+b}\le\sum a(a+b)-\frac{(\sum ab)^2}{\sum a^2+\sum ab}$$

$$=\sum a^2+\sum ab-\frac{(\sum ab)^2}{\sum a^2+\sum ab}=$$

$$=\frac{(\sum a^2+\sum ab)^2-(\sum ab)^2}{\sum a^2+\sum ab}=\frac{(\sum a^2)^2+2(\sum a^2)(\sum ab)}{\sum a^2+\sum ab}=\frac{(\sum a^2)(\sum a)^2}{\sum a^2+\sum ab}$$

$$\Rightarrow LHS\ge\frac{(a^2+b^2+c^2)^2}{\sum\frac{a(a^2+ab+b^2)}{a+b}}\ge\frac{(\sum a^2+\sum ab)(\sum a^2)}{(\sum a)^2}$$

$$\ge\frac{(\sum a^2+\sum ab)(\sum a)^2}{3(\sum a)^2}=\frac{\sum(a+b)^2}{6}\ge\frac{4(\sum a)^2}{18}=\frac{2(\sum a)^2}{9}$$

The equality holds for $a=b=c$.

Solution 2 by Nguyen Duc Viet-Vietnam

$\frac{(a+b)a^3}{a^2+ab+b^2}-\frac{(a+b)b^3}{a^2+ab+b^2}=a^2-b^2$. ***So,*** $\sum\frac{(a+b)a^3}{a^2+ab+b^2}=\sum\frac{(a+b)b^3}{a^2+ab+b^2}$. ***We will prove that:***

$$2\cdot\sum\frac{(a+b)a^3}{a^2+ab+b^2}=\sum\frac{(a+b)(a^3+b^3)}{a^2+ab+b^2}\ge\frac{4(a+b+c)^2}{9}$$

We have: $\frac{(a+b)(a^3+b^3)}{a^2+ab+b^2}=\frac{(a+b)^2[(a^2+ab+b^2)+2(a-b)^2]}{3(a^2+ab+b^2)}\ge\frac{(a+b)^2(a^2+ab+b^2)}{3(a^2+ab+b^2)}=\frac{(a+b)^2}{3}$

$$\Rightarrow\sum\frac{(a+b)(a^3+b^3)}{a^2+ab+b^2}-\frac{4(a+b+c)^2}{9}\ge\sum\frac{(a+b)^2}{3}-\frac{4(a+b+c)^2}{9}=$$

$=\frac{2(a^2+b^2+c^2-ab-bc-ca)}{9}\ge 0$ ***(true). The equality holds for*** $a=b=c$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

First, we will show that: $\frac{b^3}{a+b}+\frac{c^3}{b+c}+\frac{a^3}{c+a}\ge\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}$

Consider $b^3(b+c)(c+a)+c^3(a+b)(c+a)+a^3(a+b)(b+c)=$

$=a^4b+a^4c+b^4a+b^4c+c^4a+c^4b+a^3bc+ab^2c+abc^3+a^3b^2+b^3c^2+c^3a^2$

$ab^2(b+c)(c+a)+bc^2(a+b)(c+a)+ca^2(a+b)(b+c)=$

$=a^3bc+ab^2c+abc^3+a^3c^2+c^3b^2+b^3a^2+2(a^2b^2c+a^2bc^2+ab^2c^2)$

Because $b^4a+a^3b^2\ge 2a^2b^3, c^4b+b^3c^2\ge 2b^2c^3, a^4c+c^3a^2\ge 2a^3c^2$

and since $(ab^2+bc^2+ca^2)^2\ge(a+b+c)(a^2b^2c+a^2bc^2+ab^2c^2)$ ***(A)***

$(a^2b+b^2c+c^2a)^2\ge(a+b+c)(a^2b^2c+a^2bc^2+ab^2c^2)$ ***(B)***

Hence $(ab^2+bc^2+ca^2)^2+(a^2b+b^2c+c^2a)^2\ge$

$\ge 2(a+b+c)(a^2b^2c+a^2bc^2+ab^2c^2)$

Hence $\frac{b^3}{a+b}+\frac{c^3}{b+c}+\frac{a^3}{c+a}\ge\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}$ ***is to be true***

second, $\frac{(a+b)a^3}{a^2+ab+b^2}+\frac{(b+c)b^3}{b^2+bc+c^2}+\frac{(c+a)c^3}{c^2+ca+a^2}=\frac{a^4}{a^2+\frac{ab^2}{a+b}}+\frac{b^4}{b^2+\frac{bc^2}{b+c}}+\frac{c^4}{c^2+\frac{ca^2}{c+a}}$

$$\ge\frac{(a^2+b^2+c^2)(a+b+c)^2}{3(a^2+b^2+c^2)+\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}}$$

Iff $a^2+b^2+c^2\ge 2\left(\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\right)$

Iff $\frac{a^2(c+a)}{c+a}+\frac{b^2(a+b)}{a+b}+\frac{c^2(b+c)}{b+c}\ge 2\left(\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\right)$

Iff $\frac{a^3}{c+a}+\frac{b^3}{a+b}+\frac{c^3}{b+c}\ge\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}$ ***ist o be true***

Therefore from first and second, if is to be true.

Solution 4 by Sarah El-Kenitra-Morocco

$$\sum\frac{(a+b)a^3}{a^2+ab+b^2}=\sum\frac{a^2(a^2+ab+b^2)-a^2b^2}{a^2+ab+b^2}=\sum\left(a^2-\frac{a^2b^2}{a^2+ab+b^2}\right)$$

By AM-GM we have $\sum\left(a^2-\frac{a^2b^2}{a^2+ab+b^2}\right)\ge\sum\left(a^2-\frac{a^2b^2}{3ab}\right)=\frac{3\sum a^2-\sum ab}{3}$

Another AM-GM we get $\frac{3\sum a^2-\sum ab}{3}\ge\frac{(a+b+c)^2-\frac{2}{3}(a+b+c)^2}{3}=\frac{(a+b+c)^2}{9}$

Therefore $\sum\frac{(a+b)a^3}{a^2+ab+b^2}\ge\frac{(a+b+c)^2}{9}$.

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{(a+b)a^3}{a^2+ab+b^2}=\sum\frac{a^4+(a^3b-b^4)+b^4}{a^2+ab+b^2}=\sum\frac{a^4+b^4}{a^2+ab+b^2}+\sum\frac{b(a^3-b^3)}{a^2+ab+b^2}$$

$$=\sum\frac{a^4+b^4+a^2b^2}{a^2+ab+b^2}+\sum\frac{b(a-b)(a^2+ab+b^2)}{a^2+ab+b^2}-\sum\frac{a^2b^2}{a^2+ab+b^2}$$

$$=\sum\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{a^2+ab+b^2}+\sum ab-\sum b^2-\sum\frac{a^2b^2}{a^2+ab+b^2}$$

$$=\sum(a^2-ab+b^2)+\sum ab-\sum a^2-\sum\frac{a^2b^2}{a^2+ab+b^2}=$$

$$=\sum a^2-\sum\frac{a^2b^2}{a^2+ab+b^2}\ge\frac{2(\sum a)^2}{9}\Leftrightarrow\sum\frac{a^2b^2}{a^2+ab+b^2}\le\sum a^2-\frac{2(\sum a)^2}{9}$$

$$\because a^2+ab+b^2\overset{A-G}{\ge}3ab\therefore\sum\frac{a^2b^2}{a^2+ab+b^2}\le\sum\frac{a^2b^2}{3ab}=\frac{\sum ab}{3}\overset{?}{\le}\sum a^2-\frac{2(\sum a)^2}{9}$$

$$\Leftrightarrow\frac{7\sum a^2-4\sum ab}{9}\overset{?}{\ge}\frac{\sum ab}{3}\Leftrightarrow 7\sum a^2\ge 7\sum ab\to \textit{true (hence proved)}$$

405. For $x,y,z>0\wedge xyz=1$. Prove:

$$\frac{x}{x^{12}+2y^4+1}+\frac{y}{y^{12}+2z^4+1}+\frac{z}{z^{12}+2x^4+1}\le\frac{x^8+y^8+z^8}{4}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{x}{x^{12}+2y^4+1}\overset{AM\ge GM}{\le}\sum_{cyc}\frac{x}{2x^6+2y^4}\overset{AM\ge GM}{\le}\sum_{cyc}\frac{x}{4x^3y^2}=$$

$$=\frac{1}{4}\sum_{cyc}\frac{1}{x^2y^2}=\frac{x^2+y^2+z^2}{4}=\frac{27x^2y^2z^2(x^2+y^2+z^2)}{108x^2y^2z^2}\overset{AM\ge GM}{\le}\frac{(x^2+y^2+z^2)^4}{108}$$

$$\overset{\substack{Root-Mean-\\Square\ Inequality}}{\le}\frac{3^3(x^8+y^8+z^8)}{108}=\frac{x^8+y^8+z^8}{4}\ \textit{(proved)}$$

Solution 2 by Marian Ursărescu-Romania

$$x^{12}+2y^4+1=x^{12}+y^4+y^4+1\ge 4\sqrt[4]{x^{12}\cdot y^8\cdot 1}\Rightarrow$$

$$x^{12}+2y^4+1\ge 4x^3y^2\Rightarrow\frac{1}{x^{12}+2y^4+1}\le\frac{1}{4x^3y^2}$$

Inequality becomes: $\sum\frac{x}{x^{12}+2y^4+1}\le\frac{1}{4}\sum\frac{x}{x^3y^2}=\frac{1}{4}\sum\frac{1}{x^2y^2}=\frac{1}{4}\sum\frac{x^2y^2z^2}{x^2y^2}=\frac{1}{4}\sum z^2$

We must show this: $x^2+y^2+z^2\le x^8+y^8+z^8$ ***(1)***

Now, we use: $a^2+b^2+c^2\ge ab+bc+ac\Rightarrow$

$$x^8+y^8+z^8\ge x^4y^4+y^4z^4+z^4x^4\ge x^2y^2z^4+x^2y^4z^2+x^4y^2z^2$$

$=x^2y^2z^2(x^2+y^2+z^2)=x^2+y^2+z^2$. ***So, (1) is true***

406. **Prove that for all non-negative numbers x,y,z satisfying $x+y+z=1$**

$$1\le\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy}\le\frac{9}{8}$$

Germany NMO

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$x,y,z\ge 0\Rightarrow\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy}\ge\frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1\quad (1)$$

We prove: $\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy}\le\frac{9}{8}$

$$\sum\frac{x}{1-yz}\le\sum\frac{x}{1-\frac{(y+z)^2}{4}}=\sum\frac{x}{1-\frac{(1-x)^2}{4}}=4\sum\frac{x}{3+2x-x^2}$$

$$\frac{x}{3+2x-x^2}\le\frac{63x+3}{256}\Leftrightarrow(7x-9)(3x-1)^2\le 0\ \textit{(true, } x\le 1\textit{)}$$

$$\Rightarrow\sum\frac{x}{1-yz}\le 4\sum\frac{63x+3}{156}=\frac{63\sum x+9}{64}=\frac{63\cdot 1+9}{64}=\frac{9}{8}$$

$$\Rightarrow\sum\frac{x}{1-yz}\le\frac{9}{8}\Leftrightarrow x=y=z=\frac{1}{3}\quad (2)$$

$$(1),(2)\Rightarrow 1\le\frac{x}{1-yz}+\frac{y}{1-xz}+\frac{z}{1-xy}\le\frac{9}{8}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$\forall$ ***non – negative*** $x,y,z|\sum x=1$, ***we have:*** $1\overset{(a)}{\le}\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy}\overset{(b)}{\le}\frac{9}{8}$

$y=1-z-x\le 1$ ($\because z,x\ge 0$). ***Similarly,*** $z,x\le 1$. ***But,*** $\because$ ***no 2 among*** x,y,z ***can be*** $=1$ ***simultaneously,*** $\therefore yz<1\Rightarrow 1-yz>0$. ***Similarly,*** $1-zx>0$ ***and*** $1-xy>0$

$$\therefore (b) \Leftrightarrow 9(1-yz)(1-zx)(1-xy) -$$

$$8\{x(1-zx)(1-xy) + y(1-xy)(1-yz) + z(1-yz)(1-zx)\} \geq 0$$

$$\Leftrightarrow 9xyz(\sum x) + 8(\sum x^2y + \sum xy^2) + 9 \geq 9x^2y^2z^2 + 8xyz(\sum x^2) + 9\sum xy + 8(\sum x)$$

$$\Leftrightarrow 9xyz + 8\left(\sum x^2 y + \sum xy^2\right) + 9\left(\sum x\right)^3 \geq 9x^2y^2z^2 + 8xyz\left(\sum x^2\right)\left(\sum x\right) +$$

$$+9\sum xy\left(\sum x\right) + 8\left(\sum x\right)^3$$

$$\Leftrightarrow 9xyz + 8(\sum x^2y + \sum xy^2) + (\sum x)^3 \geq 9x^2y^2z^2 + 8xyz(\sum x^2)(\sum x) + 9\sum xy(\sum x) \quad \textbf{(1)}$$

Now, $1 = \sum x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \Rightarrow \frac{1}{27} \geq xyz \Rightarrow xyz \overset{(2)}{\leq} \frac{1}{27}$

$$\therefore 9x^2y^2z^2 + 8xyz(\sum x^2)(\sum x) + 9\sum xy(\sum x) \overset{by\ (2)}{\leq} \frac{xyz}{3} + \frac{8}{27}(\sum x^2)(\sum x) + 9\sum xy(\sum x) \textbf{(3)}$$

(1), (3) $\Rightarrow$ ***in order to prove (b), it suffices to prove:*** $9xyz + 8(\sum x^2y + \sum xy^2) + (\sum x)^3 \geq$

$$\geq \frac{xyz}{3} + \frac{8}{27}\left(\sum x^2\right)\left(\sum x\right) + 9\sum xy\left(\sum x\right) \Leftrightarrow 234xyz + 216\left(\sum x^2y + \sum xy^2\right) +$$

$$+27\left(\sum x\right)^3 \geq 8\left(\sum x^2\right)\left(\sum x\right) + 243\sum xy\left(\sum x\right) \Leftrightarrow$$

$\Leftrightarrow 19\sum x^3 + 46(\sum x^2y + \sum xy^2) \geq 333xyz$ ***(4). Now,*** $19\sum x^3 + 46(\sum x^2y + \sum xy^2) \overset{A-G}{\geq}$

$\geq 57xyz + 46 \cdot 6xyz \geq 333xyz \Rightarrow$ ***(4) is true*** $\Rightarrow$ ***(b) is true. Also,*** $\sum \frac{x}{1-yz} = \sum \frac{x^2}{x-xyz} \geq$

$\overset{Bergstrom}{\geq} \frac{(\sum x)^2}{\sum x - 3xyz} = \frac{1}{1-3xyz}$ ***(5). Now,*** $\because xyz \leq \frac{1}{27}, \therefore 1 - 3xyz \geq \frac{8}{9} > 0 \therefore \frac{1}{1-3xyz} \underset{(6)}{\overset{?}{\geq}} 1 \Leftrightarrow$

$\Leftrightarrow 1 \overset{?}{\geq} 1 - 3xyz \Leftrightarrow xyz \overset{?}{\geq} 0 \rightarrow$ ***true*** $\therefore x, y, z \geq 0, \therefore$ ***(5), (6)*** $\Rightarrow \sum \frac{x}{1-yz} \geq 1 \Rightarrow$ ***(a) is true.***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $a + b + c = p = 1, ab + bc + ca = q$ ***and*** $r = abc$***. Now,***

$\prod_{cyc}(1-xy) = 1 - q + pr - r^2$ ***and*** $\sum_{cyc}(1-xy)(1-yz) = 3 - 2q + pr$

$$\sum_{cyc} \frac{x}{1-yz} = \sum_{cyc} \frac{x^2}{x - xyz} \geq \frac{(x+y+z)^2}{x+y+z-3xyz} = \frac{1}{1-3xyz}$$

We need to prove, $\frac{1}{1-3xyz} \geq 1 \Leftrightarrow xyz \geq 0$***, which is true***

$$\sum_{cyc} \frac{x}{1-yz} = \sum_{cyc} \frac{x(yz + 1 - yz)}{1-yz} = xyz\sum_{cyc}\frac{1}{1-yz} + \sum_{cyc} x$$

$\le \frac{1}{27}\sum_{cyc}\frac{1}{1-yz}+1$ *we need to prove,* $\frac{9}{8}\ge 1+\frac{1}{27}\sum_{cyc}\frac{1}{1-yz}$

$$\Leftrightarrow \frac{27}{8}\ge \frac{3-2q+pr}{1-q+pr-r^2}\Leftrightarrow 3-11q+19r-27r^2\ge 0$$

$$\Leftrightarrow 3-11q+18r\ge 0\begin{bmatrix} we\ know\ 1\ge 27r\Rightarrow r\ge 27r^2 \\ 3-11q+19r-27r^2\ge 3-11q+18r\end{bmatrix}$$

$$\Leftrightarrow 3-11(ab+bc+ca)+18abc\ge 0\Leftrightarrow 3\ge 11(ab+bc+ca)-18abc$$

$$\Leftrightarrow 3\ge xy(11-18z)+11z(x+y)\Leftrightarrow 3\ge \frac{(1-z)^2}{4}(11-18z)+11z(1-z)$$

$\Leftrightarrow 18z^3-3z^2-4z+1\ge 0\Leftrightarrow (3z-1)^2(2z+1)\ge 0$*, which is true. Hence proved*

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

First, we will show that $\frac{1}{1-xy}+\frac{1}{1-yz}+\frac{1}{1-zx}\le \frac{1}{8xyz}, x, y, z>0$

Because $xy+yz+zx\le \frac{1}{3}\le \frac{1}{3^7(xyz)^2}\Rightarrow 3^7(x^3y^3z^2+x^2y^3z^3+x^3y^2z^3)\le 1\Rightarrow$

$$\Rightarrow \sqrt[8]{3^7(x^3y^2z^2+x^2y^3z^3+x^3y^2z^2)}\le 1\Rightarrow \sqrt[8]{x^3y^3z^2}+\sqrt[8]{x^2y^3z^3}+\sqrt[8]{x^3y^2z^3}\le 1$$

$$\Rightarrow \sqrt[8]{\frac{(xyz)^8}{x^5y^5z^6}}+\sqrt[8]{\frac{x^8y^8z^8}{x^2y^5z^5}}+\sqrt[8]{\frac{x^8y^8z^8}{x^5y^6z^5}}\le 1$$

$$\Rightarrow 8\left(\frac{1}{\sqrt[8]{x^5y^5z^6}}+\frac{1}{\sqrt[8]{x^6y^5z^5}}+\frac{1}{\sqrt[8]{x^5y^6z^5}}\right)\le \frac{1}{8xyz}$$

$$\Rightarrow \frac{1}{3xy+3yz+2zx}+\frac{1}{3yz+3zx+2xy}+\frac{1}{3zx+3xy+2yz}\le \frac{1}{8xyz}$$

$$\Rightarrow \frac{1}{\frac{3}{z}+\frac{3}{x}+\frac{2}{y}}+\frac{1}{\frac{3}{x}+\frac{3}{y}+\frac{2}{z}}+\frac{1}{\frac{3}{y}+\frac{3}{z}+\frac{2}{x}}\le \frac{1}{8}$$

$$\Rightarrow \frac{1}{\frac{3}{z}+\frac{2}{x}+\frac{3}{y}-\frac{1}{y}}+\frac{1}{\frac{3}{x}+\frac{3}{y}+\frac{3}{z}-\frac{1}{2}}+\frac{1}{\frac{3}{y}+\frac{3}{z}+\frac{3}{x}-\frac{1}{x}}\le \frac{1}{8}$$

$\Rightarrow \frac{1}{\frac{1}{xyz}-\frac{1}{y}}+\frac{1}{\frac{1}{xyz}-\frac{1}{z}}+\frac{1}{\frac{1}{xyz}-\frac{1}{x}}\le \frac{1}{8}\Rightarrow \frac{1}{1-xy}+\frac{1}{1-yz}+\frac{1}{1-zx}\le \frac{1}{8xyz}$ *is to be true*

second from $x+y+z=1$*, we get:* $\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy}\le \frac{9}{8}$

Iff $\frac{x}{1-yz}-x+\frac{y}{1-2x}-y+\frac{1}{1-xy}-z\le \frac{1}{8}$

Iff $\frac{xyz}{1-yz}+\frac{xyz}{1-zx}+\frac{xyz}{1-xy}\le\frac{1}{8}$

Iff $\frac{1}{1-xy}+\frac{1}{1-yz}+\frac{1}{1-2x}\le\frac{1}{8xyz}$ *is to be true. Therefore, it is true.*

407. **If x,y and z are positive real numbers such that $xyz\ge 7+5\sqrt{2}$, then:**

$$x^2+y^2+z^2-2(x+y+z)\ge 3.$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Firstly $\left(\sqrt{2}+1\right)^3=2\sqrt{2}+1+3\left(\sqrt{2}\right)^2\cdot 1+3\sqrt{2}=7+5\sqrt{2}\Rightarrow\sqrt[3]{7+5\sqrt{2}}=\sqrt{2}+1$

$\therefore xyz\ge 7+5\sqrt{2}\Rightarrow\sqrt[3]{xyz}\ge\sqrt{2}+1$. *Now,* $t=\sum x\overset{A-G}{\ge}3\sqrt[3]{xyz}=3\left(\sqrt{2}+1\right)$

Now, $t^2-6t-9\ge 0\Leftrightarrow t\le\frac{6-\sqrt{72}}{2}=3-3\sqrt{2}$ *or* $t\ge\frac{6+\sqrt{72}}{2}=3+3\sqrt{2}$

$\because t>0,\therefore t^2-6t-9\ge 0\Leftrightarrow t\ge 3\left(\sqrt{2}+1\right)$, *that is,*

$t\ge 3\left(\sqrt{2}+1\right)\Rightarrow t^2-6t-9\ge 0\Rightarrow\Rightarrow 6t+9\le t^2\Rightarrow$

$\Rightarrow 6\sum x+9\le\left(\sum x\right)^2$ ***(1)***

But $\left(\sum x\right)^2\le 3\sum x^2$ ***(2)***

(1), (2) $\Rightarrow 3\sum x^2\ge 6\sum x+9\Rightarrow\sum x^2-2\sum x\ge 3$

Thus, it is established that $\forall x,y,z>0$ *such that* $xyz\ge 7+5\sqrt{2}$, *we have*

$x^2+y^2+z^2-2(x+y+z)\ge 3$. ***(Done)***

408. **If a,b and c are positive real numbers, then prove that,**

$$\frac{a(b-c)}{c(a+b)}+\frac{b(c-a)}{a(b+c)}+\frac{c(a-b)}{b(c+a)}\ge 0$$

Proposed by Neculai Stanciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Numerator of LHS $=a^2b\left(b^2-c^2\right)(c+a)+b^2c\left(c^2-a^2\right)(a+b)+c^2a\left(a^2-b^2\right)(b+c)$

$=\sum a^3b^3-abc\left(\sum a^2b\right)\rightarrow$ ***(a)***

Let us consider $x,y,z>0$

$$x^3+y^3+y^3\overset{A-G}{\geq}3xy^2\to \textbf{(1)}$$

$$y^3+z^3+z^3\overset{A-G}{\geq}3yz^2\to \textbf{(2)}$$

$$z^3+x^3+x^3\overset{A-G}{\geq}3zx^2\to \textbf{(3)}$$

$$\textbf{(1)}+\textbf{(2)}+\textbf{(3)}\Rightarrow \sum x^3\geq \sum xy^2\to \textbf{(i)}$$

Putting $x=ab, y=bc, z=ca$ ***and applying (i), we get***

$$\sum a^3b^3\geq abc\left(\sum a^2b\right)\Rightarrow \sum a^3b^3-abc\left(\sum a^2b\right)\geq 0\Rightarrow \textbf{numerator of LHS}\geq 0\ \textbf{(by(a))}$$

Also, denominator of LHS $=abc\prod(a+b)>0\ \therefore LHS\geq 0$ ***(Proved)***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $\frac{(ab+bc)}{(ac+bc)}+\frac{(bc+ca)}{(ab+ac)}+\frac{(ab+ac)}{(ab+bc)}\geq 3, a,b,c>0$

Hence $\frac{(ab+bc)}{(ac+bc)}+\frac{(bc+ca)}{(ab+ac)}+\frac{(ab+ac)}{(ab+bc)}\geq \frac{(ac+bc)}{(ac+bc)}+\frac{(ab+ac)}{(ab+ac)}+\frac{(ab+bc)}{(ab+bc)}$

That is, $\frac{ab}{ac+bc}+\frac{bc}{ab+ac}+\frac{ca}{ab+bc}\geq \frac{ab}{ab+ac}+\frac{bc}{bc+ab}+\frac{ca}{ca+bc}$

$$\Rightarrow \frac{ab-ac}{ac+bc}+\frac{bc-ab}{ab+ac}+\frac{ca-bc}{ab+bc}\geq 0\Rightarrow \frac{a}{c}\left(\frac{b-c}{a+b}\right)+\frac{b}{a}\left(\frac{c-a}{b+c}\right)+\frac{c}{b}\left(\frac{a-b}{a+c}\right)\geq 0$$

Therefore it is to be true.

409. **If x, y and z are positive real numbers, then prove that**

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)}\geq \frac{8}{9}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Christos Eythimiou-Greece

$$x,y,z>0\Rightarrow \frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)}\geq \frac{8}{9}$$

$$x,y,z>0\Rightarrow \frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)}=$$

$$=\frac{(x+y+z)(xy+yz+zx)-\sqrt[3]{xyz}\sqrt[3]{xyyzzx}}{(x+y+z)(xy+yz+zx)}\geq$$

$$\geq \frac{(x+y+z)(xy+yz+zx) - \frac{x+y+z}{3} \cdot \frac{xy+yz+zx}{3}}{(x+y+z)(xy+yz+zx)} = \frac{8}{9}$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9} \Leftrightarrow$$

$$\textbf{(1)}\ \frac{(x+y)(y+z)(z+x)}{(x+y)(y+z)(z+x)+xyz} \geq \frac{8}{9} \qquad xyz \leq \frac{(x+y)(y+z)(z+x)}{8}$$

$$LHS = \frac{(x+y)(y+z)(z+x)}{(x+y)(y+z)(z+x)+xyz} \geq \frac{\prod(x+y)}{\prod(x+y) + \frac{\prod(x+y)}{8}}$$

$$= \frac{\prod(x+y)}{\prod(x+y)\left(1+\frac{1}{8}\right)} = \frac{1}{\frac{9}{8}} = \frac{8}{9}$$

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$p = x+y+z; q = xy+yz+zx; r = xyz$$

$$LHS = \frac{pq-r}{pq} \geq \frac{8}{9} \Leftrightarrow pq - r \geq \frac{8}{9} \Leftrightarrow pq \geq 9r \ \textit{(true)}$$

410. **If $0 < a, b, c \leq 1$ then:**

$$\frac{1}{a+a^a} + \frac{1}{b+b^b} + \frac{1}{c+c^c} \geq \frac{9}{3+a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdul Aziz-Semarang-Indonesia

$$\sum \frac{1}{a^a+a} = \sum \frac{1}{(1+a-1)^a+a} \overset{Bernoulli}{\geq} \sum \frac{1}{1+a(a-1)+a} = \sum \frac{1}{a^2+1}$$

$$\overset{CS}{\geq} \frac{(1+1+1)^2}{a^1+1+b^2+1+c^2+1} = \frac{9}{a^2+b^2+c^2+3}$$

Equality holds when $a = b = c = 1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{1}{a+a^2} \underset{(1)}{\overset{Bergstrom}{\geq}} \frac{9}{\sum a + \sum a^a}$$

Bernoulli's inequality $\Rightarrow \forall r \in [0,1]$ ***and*** $x \geq -1, (1+x)^r \leq 1+rx$

$\therefore as(a-1) > -1$ **and** $a \le 1, a^a = \{1+(a-1)\}^a \overset{Bernoulli}{\le} 1 + a(a-1)$

$= a^2 - a + 1 \Rightarrow a^a \overset{(i)}{\le} a^2 - a + 1$. **Similarly,** $b^2 \overset{(ii)}{\le} b^2 - b + 1$ **and** $c^c \overset{(iii)}{\le} c^2 - c + 1$

(i)+(ii)+(iii)$\Rightarrow \sum a^2 \overset{(2)}{\le} \sum a^2 - \sum a + 3 \Rightarrow \sum a + \sum a^a \le \sum a^2 + 3 \Rightarrow \frac{9}{\sum a + \sum a^a} \overset{(2)}{\ge} \frac{9}{3+\sum a^2}$

(1),(2)$\Rightarrow \sum \frac{1}{a+a^a} \ge \frac{9}{3+\sum a^2}$ **(proved)**

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x \in \mathbb{R}^+, 0 < x \le 1$, **we have**

1. $0 < x \le 1; 0 < x^2 \le 1 \Rightarrow 1 < x^2 + 1 \le 2$

2. $0 < x \le 1; 0 < x \le x^x \le 1 \Rightarrow 0 < x + x \le x^x + x \le 1 + x \le 2$

Hence $(x^2+1) \in (1,2]$ **and** $(x^x + x) \in (0,2]$ **and since** $x^2 + 1$ **and** $x + x^x$ **an**

increasing functions on $(0,1]$. **So,** $x^2 + 1 \ge x + x^x, x \in (0,1] \Rightarrow \frac{1}{x+x^x} \ge \frac{1}{x^2+1}$.

That is $\frac{1}{a+a^a} + \frac{1}{b+b^b} + \frac{1}{c+c^c} \ge \frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1}$, **for** $0 < a,b,c \le 1 \ge \frac{9}{3+a^2+b^2+c^2}$.

Therefore it is to be true.

411. **If** $a,b,c > 0, ab+bc+ca = 6abc$ **then:**

$$\frac{1}{\sqrt{ab(a+b)}} + \frac{1}{\sqrt{bc(b+c)}} + \frac{1}{\sqrt{ca(c+a)}} \le 3 + \frac{a+b+c}{4abc}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Antonis Anastasiadis-Greece

For $x,y > 0: \sqrt{x+y}^3 + \sqrt{x+y} \ge 2(x+y) \ge 4\sqrt{xy} \Rightarrow \sqrt{x+y}\left(\sqrt{x+y}^2 + 1\right) \ge 4\sqrt{xy}$

$$\Leftrightarrow \frac{x+y+1}{4} \ge \sqrt{\frac{xy}{x+y}} \Leftrightarrow \frac{x+y+1}{4xy} \ge \sqrt{\frac{1}{xy(x+y)}} \Leftrightarrow \frac{1}{4y} + \frac{1}{4x} + \frac{1}{4xy} \ge \frac{1}{\sqrt{xy(x+y)}}$$

So: $\left.\begin{array}{l} \frac{1}{4a} + \frac{1}{4b} + \frac{1}{4ab} \ge \frac{1}{\sqrt{ab(a+b)}} \\ \frac{1}{4b} + \frac{1}{4c} + \frac{1}{4bc} \ge \frac{1}{\sqrt{bc(b+c)}} \\ \frac{1}{4a} + \frac{1}{4c} + \frac{1}{4ac} \ge \frac{1}{\sqrt{ac(a+c)}} \end{array}\right\} (+) \Rightarrow$

$$\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}+\frac{a+b+c}{4abc}\ge\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ac(a+c)}}\Leftrightarrow$$

$$\frac{bc+ac+ab}{2abc}+\frac{a+b+c}{4abc}\ge\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ac(a+c)}}\Leftrightarrow$$

$$3+\frac{a+b+c}{4abc}\ge\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ac(a+c)}}$$

Solution 2 by Marian Ursărescu-Romania

$$ab+bc+ca=6abc\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=6$$

Let $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\Rightarrow x+y+z=6,x,y,z>0$

$\frac{1}{\sqrt{ab(a+b)}}=\frac{1}{\sqrt{\frac{1}{xy}\left(\frac{1}{x}+\frac{1}{y}\right)}}=\frac{xy}{\sqrt{x+y}}$*. We must show this:*

$\sum\frac{xy}{\sqrt{x+y}}\le 3+\frac{1}{4}(xy+xz+yz)$ *with* $x+y+z=6$ *(1)*

We show this: $\frac{xy}{\sqrt{x+y}}\le\frac{x+y+xy}{4}\Leftrightarrow$ *(1')*

$(x+y+z)\sqrt{x+y}\ge 4xy$ *(2)*

Let $x+y=S,xy=p.S=x+y\ge 2\sqrt{xy}\Rightarrow S\ge 2\sqrt{p}$

(2) $\Leftrightarrow(S+p)\sqrt{S}\ge 4p$ *(3)*

But $(S+p)\sqrt{S}\ge 2S\sqrt{p}\ge 4\sqrt{p}\cdot\sqrt{p}=4p\Rightarrow$ *then (3) its true.*

From (1') $\Rightarrow\sum\frac{xy}{\sqrt{x+y}}\le\frac{2(x+y+z)}{4}+\frac{1}{4}(xy+xz+yz)=3+\frac{1}{4}(xy+xz+yz).$

Solution 3 by Abdul Aziz-Semarang-Indonesia

A fact $a+b\ge 2\sqrt{ab}\Leftrightarrow a+b\ge\frac{2ab}{\sqrt{ab}}\Leftrightarrow\frac{2\sqrt{a+b}}{ab}\ge\frac{4}{\sqrt{ab(a+b)}}$*, by AM-GM*

$$\frac{4}{\sqrt{ab(a+b)}}\le\frac{2\sqrt{a+b}}{ab}\le\frac{1}{ab}+\frac{a+b}{ab}=\frac{1}{ab}+\frac{1}{a}+\frac{1}{b}\Leftrightarrow\frac{1}{\sqrt{ab(a+b)}}\le\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{a}+\frac{1}{b}\right)$$

Analogue, $\frac{1}{\sqrt{bc(b+c)}}\le\frac{1}{4}\left(\frac{1}{bc}+\frac{1}{b}+\frac{1}{c}\right)$

$$\frac{1}{\sqrt{ca(c+a)}}\le\frac{1}{4}\left(\frac{1}{ca}+\frac{1}{c}+\frac{1}{a}\right)$$

$$-------------+$$

$$\sum \frac{1}{\sqrt{ab(a+b)}} \le \frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt{ab(a+b)}} \le \frac{a+b+c}{4abc}+\frac{1}{2}\cdot\frac{(ab+bc+ca)}{abc} \Leftrightarrow \sum \frac{1}{\sqrt{ab(a+b)}} \le \frac{a+b+c}{4abc}+3$$

Equality holds when $a=b=c=\frac{1}{2}$

Solution 4 by Boris Colakovic-Belgrade-Serbia

$$ab+bc+ca=6abc \Leftrightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=6 \quad (1)$$

$$\frac{1}{a}+\frac{1}{b} \overset{AM-GM}{\ge} \frac{2}{\sqrt{ab}} \Rightarrow \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) \ge \frac{1}{\sqrt{ab}} \quad (2)$$

$$\frac{1}{a+b}+1 \overset{AM-GM}{\ge} \frac{2}{\sqrt{a+b}} \Rightarrow \frac{1}{2}\left(\frac{1}{a+b}+1\right) \ge \frac{1}{\sqrt{a+b}} \quad (3)$$

$$(2)*(3) \Rightarrow \frac{1}{\sqrt{ab(a+b)}} \le \frac{1}{4}\cdot\frac{a+b}{ab}\left(\frac{1}{a+b}+1\right)=\frac{1}{4ab}+\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right) \quad (4)$$

$$\text{Similarly } \frac{1}{\sqrt{bc(b+c)}} \le \frac{1}{4bc}+\frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right) \quad (5)$$

$$\frac{1}{\sqrt{ca(c+a)}} \le \frac{1}{4ca}+\frac{1}{4}\left(\frac{1}{c}+\frac{1}{a}\right) \quad (6)$$

$$(4)+(5)+(6) \Rightarrow LHS \le \frac{1}{4}\cdot 2\underbrace{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}_{from\ (1)}+\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=$$

$$=\frac{1}{2}\cdot 6+\frac{a+b+c}{4abc}=3+\frac{a+b+c}{4abc}.$$ ***sign „ = „ holds for*** $a=b=c=\frac{1}{2}$

Solution 5 by Theodoros Sampas-Greece

$$\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{bc(b+c)}+\frac{1}{\sqrt{ca(c+a)}} \le 3+\frac{a+b+c}{4abc} \quad (1)$$

$$\frac{1}{4ab}+\frac{1}{a+b} \ge \frac{2}{2\sqrt{ab(a+b)}} \ge \frac{1}{\sqrt{ab(a+b)}}, \frac{1}{4bc}+\frac{1}{b+c} \ge \frac{1}{\sqrt{bc(b+c)}}, \frac{1}{4ca}+\frac{1}{c+a} \ge \frac{1}{\sqrt{ca(c+a)}} \quad (2)$$

$$(2) \Rightarrow \frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ca(c+a)}} \le \frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \quad (3)$$

$$3+\frac{a+b+c}{4abc}=\frac{ab+bc+ca}{2abc}+\frac{a+b+c}{4abc}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \quad (4)$$

From (1), (3), (4) it suffices to prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \le \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad (5)$$

$$\frac{1}{a+b} \le \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right), \frac{1}{b+c} \le \frac{1}{4}\left(\frac{1}{b}+\frac{1}{c}\right), \frac{1}{c+a} \le \frac{1}{4}\left(\frac{1}{c}+\frac{1}{a}\right) \overset{+}{\Rightarrow} (5)$$

Solution 6 by Lazaros Zachariadis-Thessaloniki-Greece

$$LHS=\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ca(c+a)}}=\frac{1}{\sqrt{4ab\cdot\frac{(a+b)}{4}}}+\frac{1}{\sqrt{4bc\cdot\frac{(b+c)}{4}}}+\frac{1}{\sqrt{4ca\cdot\frac{(c+a)}{4}}}$$

$$\frac{a+b}{4}\geq\frac{ab}{a+b}$$

$$\leq\frac{\sqrt{a+b}}{2ab}+\frac{\sqrt{b+c}}{2bc}+\frac{\sqrt{c+a}}{2ca}\overset{AM-GM}{\leq}\frac{a+b+1}{4ab}+\frac{b+c+1}{4bc}+\frac{c+a+1}{4ca}$$

$$=\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\frac{1}{4}\left(\frac{1}{b}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}+\frac{1}{a}\right)=\frac{c+a+b}{4abc}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$=\frac{a+b+c}{4abc}+\frac{1}{2}\cdot\frac{ab+bc+ca}{abc}=\frac{a+b+c}{4abc}+\frac{1}{2}\cdot 6=\frac{a+b+c}{4abc}+3=RHS$$

$$„=„\ a=b=c=\tfrac{1}{2}$$

Generalization by Nguyen Thanh Nho-Vietnam

$a,b,c,k>0\wedge ab+bc+ca=kabc$. **Prove that:**

$$\frac{1}{\sqrt{ab(a+b)}}+\frac{1}{\sqrt{bc(b+c)}}+\frac{1}{\sqrt{ca(c+a)}}\leq\sqrt{\frac{k}{6}\left(\frac{k}{2}+\frac{3(a+b+c)}{2kabc}\right)}$$

412. **If $x,y>0$ then:**

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\leq\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2$$

Proposed by Mihalcea Andrei Stefan-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Consider $\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2-4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)=$

$$=4+(x+y)^2+\left(\frac{x+y}{xy}\right)^2+4(x+y)+4\left(\frac{x+y}{xy}\right)+\frac{2(x+y)^2}{xy}-$$

$$-4\left[xy+x+1+y+1+\frac{(x+1)(y+1)}{xy}\right]=$$

$$=(x+y)^2-4xy+\left(\frac{x+y}{xy}\right)^2-4\left(\frac{1}{x}+\frac{1}{y}\right)+4\left(\frac{1}{x}+\frac{1}{y}\right)-\frac{4}{xy}+4(x+y)-4(x+y)+$$

$$+4-12+\frac{2(x+y)^2}{xy}=(x-y)^2+\frac{(x-y)^2}{x^2y^2}+\frac{2(x-y)^2}{xy}\geq 0$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\leq\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2$$

Iff $4(xy+x+1)(xy+y+1)\leq(2xy+x^2y+y^2x+x+y)\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)$

Iff $4(xy^2+xy+y)(x^2y+xy+x)\leq$

$$\leq(2xy+x^2y+y^2x+x+y)(2xy+x^2y+y^2x+x+y)=(2xy+x^2y+y^2x+x+y)^2$$

and it ist o be true.

Because $4(xy^2+xy+y)(x^2y+xy+x)\leq(xy^2+x^2y+2xy+x+y)^2$

Therefore it is to be true.

413. **If** $a,b,c>1, ab+bc+ca=abc$ **then:**

$$abc^c+bca^a+cab^b\geq a^2b^2c^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{c^c(ab)+a^a(bc)+b^b(ca)}{\sum ab}\overset{weighted\ A-G}{\geq}\sqrt[(\sum ab)]{(c^c)^{ab}(a^a)^{bc}(b^b)^{ca}}$$

$$=\sqrt[\sum ab]{(abc)^{abc}}=\sqrt[abc]{(abc)^{abc}}\quad\left(\because\sum ab=abc\right)=(abc)$$

$$\Rightarrow\sum c^c\cdot ab\geq\left(\sum ab\right)(abc)=a^2b^2c^2\quad\left(\because\sum ab=abc\right)$$

(proved)

414. **If** $a,b,c,d,e>0, c+d+e=1$ **then:**

$$\left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4\geq 3(a+3b)^4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

From Hölder inequality $\Rightarrow$

$$\Rightarrow \left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4 \geq \frac{\left(a+\frac{b}{c}+a+\frac{b}{d}+a+\frac{b}{c}\right)^4}{27} \quad (1)$$

From (1) we must show this:

$$\frac{\left(3a+\frac{b}{c}+\frac{b}{d}+\frac{b}{e}\right)^4}{27} \geq 3(a+3b)^4 \Leftrightarrow \left[3a+b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\right]^4 \geq 81(a+3b)^4 \Leftrightarrow$$

$$\Leftrightarrow 3a+b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \geq 3(a+3b) \Leftrightarrow b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \geq 9b \Leftrightarrow \frac{1}{c}+\frac{1}{d}+\frac{1}{e} \geq 9 \text{ true}$$

because $(c+d+e)\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \geq 9$***, but*** $c+d+e=1 \Rightarrow \frac{1}{c}+\frac{1}{d}+\frac{1}{e} \geq 9$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $a, b, c, d, e > 0$ ***and*** $c+d+e=1$***, we get that*** $\sqrt[3]{cde} \leq \frac{1}{3} \Rightarrow \frac{1}{\sqrt[3]{cde}} \geq 3 \Rightarrow$

$$\Rightarrow \frac{b}{\sqrt[3]{cde}} \geq 3b \Rightarrow a+\frac{b}{\sqrt[3]{cde}} \geq a+3b \Rightarrow \left(a+\sqrt[3]{\frac{b}{c}\cdot\frac{b}{d}\cdot\frac{b}{e}}\right)^3 \geq (a+3b)^3 \Rightarrow$$

$$\Rightarrow \left(a+\frac{b}{c}\right)\left(a+\frac{b}{d}\right)\left(a+\frac{b}{d}\right) \geq (a+3b)^3 \Rightarrow \left(a+\frac{b}{c}\right)^4\left(a+\frac{b}{d}\right)^4\left(a+\frac{b}{e}\right)^4 \geq (a+3b)^2$$

$$\Rightarrow \sqrt[3]{\left(a+\frac{b}{c}\right)^4\left(a+\frac{b}{d}\right)^4\left(a+\frac{b}{c}\right)^4} \geq (a+3b)^3 \Rightarrow$$

$$\Rightarrow 3\sqrt[3]{\left(a+\frac{b}{c}\right)^4\left(a+\frac{b}{d}\right)^4\left(a+\frac{b}{e}\right)^4} \geq 3(a+3b)^3 \Rightarrow$$

$\Rightarrow \left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4 \geq 3(a+3b)^4$***. Therefore if is to be true.***

Solution 3 by Amit Dutta-Jamshedpur-India

AM ***of*** m^{th} ***power*** $\geq m^{th}$ ***power of AM*** $\forall m \in R-(0,1)$***. i.e.***

$\left(\frac{x_1^m+x_2^m+\cdots+x_n^m}{n}\right) \geq \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^m$***. Using this,***

$$\frac{\left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4}{3} \geq \left\{\frac{3a+b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)}{3}\right\}^4$$

Also, $\frac{c^{-1}+d^{-1}+e^{-1}}{3} \geq \left(\frac{c+d+e}{3}\right)^{-1} \geq \left(\frac{1}{3}\right)^{-1} \Rightarrow c^{-1}+d^{-1}+e^{-1} \geq 3\left(\frac{1}{3}\right)^{-1} \geq 9 \Rightarrow$

$$\Rightarrow \frac{1}{c}+\frac{1}{d}+\frac{1}{e}\geq 9\Rightarrow \left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4\geq 3\left\{\frac{3a+9b}{3}\right\}^4\geq 3(a+3b)^4$$

(proved)

415. **If $x,y,z,t>0$ then:**

$$\sum\frac{yzt}{\left(\sqrt[3]{ztx}+\sqrt[3]{txy}+\sqrt[3]{xyz}\right)^3}\geq\frac{4}{27}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu-Romania

$$\textit{Inequality}\Leftrightarrow\sum\left(\frac{\sqrt[3]{yzt}}{\sqrt[3]{ztx}+\sqrt[3]{txy}+\sqrt[3]{xyz}}\right)^3\geq\frac{4}{27}\quad(1)$$

Let $\sqrt[3]{yzt}=x_1,\ \sqrt[3]{ztx}=x_2,\ \sqrt[3]{txy}=x_3,\ \sqrt[3]{xyz}=x_4$

$$(1)\ \textit{becomes}\ \sum\left(\frac{x_1}{x_2+x_3+x_4}\right)^3\geq\frac{4}{27}\quad(2)$$

$$\textit{From Holder we have: }\sum\left(\frac{x_1}{x_2+x_3+x_4}\right)^3\geq\frac{\left(\sum\frac{x_1}{x_2+x_3+x_4}\right)^3}{16}\quad(3)$$

$$\textit{From (2)+(3) we must show }\left(\sum\frac{x_1}{x_2+x_3+x_4}\right)^3\geq\frac{64}{27}\Leftrightarrow$$

$$\sum\frac{x_1}{x_2+x_3+x_4}\geq\frac{4}{3}\quad(4)$$

$$\textit{Let}\left.\begin{array}{l}x_2+x_3+x_4=y_1\\x_1+x_3+x_4=y_2\\x_1+x_2+x_4=y_3\\x_1+x_2+x_3=y_4\end{array}\right\}\Rightarrow\begin{array}{l}x_1=\frac{-2y_1+y_2+y_3+y_4}{3}\\x_2=\frac{y_1-2y_2+y_3+y_4}{3}\\x_3=\frac{y_1+y_2-2y_3+y_4}{3}\\x_4=\frac{y_1+y_2+y_3-2y_4}{3}\end{array}$$

$$\textit{Inequality (4) becomes: }\frac{1}{3}\sum(-2y_1+y_2+y_3+y_4)\geq\frac{4}{3}\Leftrightarrow$$

$$\sum(-2y_1+y_2+y_3+y_4)\geq 4\quad(5)$$

But $\sum(-2y_1+y_2+y_3+y_4)=-8+\sum\left(\frac{y_1}{y_2}+\frac{y_2}{y_1}\right)\geq -8+12=4\Rightarrow$ ***(5) is true.***

416. If $a,b,c>1$ then:

$$\frac{1}{2\sqrt{2}}\sin\frac{\pi}{3a}\sin\frac{\pi}{3b}\sin\frac{\pi}{3c}>\frac{1}{\sqrt{(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)}}$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian

It is known that if $x>3$ *then* $\sqrt{x^2+9}\sin\frac{\pi}{x}>3, x\to 3a\Rightarrow \sin\frac{\pi}{3a}>\frac{1}{\sqrt{a^2+1}}$

$LHS>\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{(a^2+1)}}\cdot\frac{1}{\sqrt{(b^2+1)}}\cdot\frac{1}{\sqrt{(c^2+1)}}$. *Now, we prove that:*

$$\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{(a^2+1)(b^2+1)(c^2+1)}}>\frac{1}{\sqrt{(a^2+b^2+2)(b^2+c^2+1)(c^2+a^2+2)}}$$

$$(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)>8(a^2+1)(b^2+1)(c^2+1)$$

$$a^2+b^2+2=(a^2+1)+(b^2+1)>2\sqrt{(a^2+1)(b^2+1)}$$

$$b^2+c^2+2=(b^2+1)+(c^2+1)>2\sqrt{(b^2+1)(c^2+1)}$$

$$c^2+a^2+2=(c^2+1)+(a^2+1)>2\sqrt{(c^2+1)(a^2+1)}$$

417. For $a,b,c\in(0,+\infty)\wedge x,y\in[1;+\infty)$. Prove:

$$\frac{a^x}{(b+c)^y}+\frac{b^x}{(a+c)^y}+\frac{c^x}{(a+b)^y}\ge\frac{(a+b+c)^{x-y}}{2^y3^{x-y-1}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

Let $a\ge b\ge c\Rightarrow\frac{1}{b+c}\ge\frac{1}{c+a}\ge\frac{1}{a+b}$

$$\sum_{cyc}\frac{a^x}{(b+c)^y}\ge\frac{1}{3}\left(\sum_{cyc}a^x\right)\left(\sum_{cyc}\frac{1}{(a+b)^y}\right)\ge\left(\frac{a+b+c}{3}\right)^x\cdot 3\left(\frac{3}{2(a+b+c)}\right)^y$$

$[\because x,y\in[1,+\infty)]$

$=\frac{(a+b+c)^{x-y}}{2^y3^{x-y-1}}$ *(proved)*

418. If $a,b,c\in(0,1), a^2+b^2+c^2=\sqrt{3}$ then:

$$(1-a^2)^{\frac{1}{a}}\cdot(1-b^2)^{\frac{1}{b}}\cdot(1-c^2)^{\frac{1}{c}}<\frac{1}{e^3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c\in(0,1)\Big|\sum a^2=3, (1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}}<\frac{1}{e^3}$$

$$\because a,b,c\in(0,1), 0<(1-a^2),(1-b^2)(1-c^2)<1$$

Now, $(1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}}<\frac{1}{e^3}\Leftrightarrow \ln\left((1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}}\right)<\ln\left(\frac{1}{e^3}\right)$

$$\Leftrightarrow\left(\frac{1}{a}\right)\ln(1-a^2)+\left(\frac{1}{b}\right)\ln(1-b^2)+\left(\frac{1}{c}\right)\ln(1-c^2)<-3=-\sqrt{3}\left(\sum a^2\right)$$

$\Leftrightarrow\sum\left[\left(\frac{1}{a}\right)\ln(1-a^2)+\sqrt{3}a^2\right]\overset{(1)}{<}0$. *Let* $f(x)=\ln(1-x^2)+\sqrt{3}x^3\ \forall x\in[0,1)$

We have $f'(x)=x\left(3\sqrt{3}x-\frac{2}{1-x^2}\right)=\left(\frac{x}{1-x^2}\right)\left(3\sqrt{3}x(1-x^2)-2\right)\overset{?}{\le}0$

$$\Leftrightarrow 3\sqrt{3}x-3\sqrt{3}x^3-2\overset{?}{\le}0\Leftrightarrow 9x^3-9x+2\sqrt{3}\overset{?}{\ge}0$$

$\Leftrightarrow(3x+2\sqrt{3})(\sqrt{3}x-1)^2\overset{?}{\ge}0\to$ *true* $\Rightarrow f'(x)\le 0\ \forall x\in[0,1)$

$\Rightarrow f(x)\le f(0)=0,\forall x\in[0,1)\Rightarrow\forall x\in[0,1), f(x)\le 0$, *equality at* $x=0$

$$\therefore x\in(0,1), f(x)<0\Rightarrow\ln(1-x^2)+\sqrt{3}x^3<0$$

$$\Rightarrow\left(\frac{1}{x}\right)\ln(1-x^2)+\sqrt{3}x^2\overset{(a)}{<}0\ \forall x\in(0,1)$$

$$\therefore\left(\frac{1}{a}\right)\ln(1-a^2)+\sqrt{3}a^2\underset{(i)}{\overset{by\,(a)}{<}}0,\left(\frac{1}{b}\right)\ln(1-b^2)+\sqrt{3}b^2\underset{(ii)}{\overset{by\,(a)}{<}}0\ \&$$

$\left(\frac{1}{c}\right)\ln(1-c^2)+\sqrt{3}c^2\underset{(iii)}{\overset{by\,(a)}{<}}0$; *(i)+(ii)+(iii)⇒ (1) is true (Proved)*

419. If $a,b,c\ge 0$ then:

$$3^e\cdot(a^e+b^e+c^e)^\pi\le 3^\pi\cdot(a^\pi+b^\pi+c^\pi)^e$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Omran Kouba-Damascus-Syria

Let a,b,c be positive numbers and suppose that $1 \le p < q$,

by Hölder's inequality we have

$$a^p+b^p+c^p \le \left((a^p)^{\frac{q}{p}}+(b^p)^{\frac{q}{p}}+(c^p)^{\frac{q}{p}}\right)^{\frac{p}{q}}(1+1+1)^{1-\frac{p}{q}} \le 3(a^q+b^q+c^q)^{\frac{p}{q}}3^{-\frac{p}{q}}$$

Equivalently: $3^p(a^p+b^p+c^p)^q \le 3^q(a^q+b^q+c^q)^p$

And the desired inequality follows by taking $(p,q)=(e,\pi)$, since $e<\pi$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c \ge 1$, we have: $3\cdot 3^{\left(\frac{1}{e}-1\right)}\left(a^{(e\pi)\frac{1}{e}}+b^{(e\pi)\frac{1}{e}}+c^{(e\pi)\frac{1}{e}}\right) \le 3^{\frac{1}{e}}(a^\pi+b^\pi+c^\pi)$

$$\Rightarrow 3(a^{e\pi}+b^{e\pi}+c^{e\pi})^{\frac{1}{e}} \le 3^{\frac{1}{e}}(a^\pi+b^\pi+c^\pi)$$

$$\Rightarrow 3^e(a^{e\pi}+b^{e\pi}+c^{e\pi}) \le 3(a^\pi+b^\pi+c^\pi)^e$$

$$\Rightarrow \frac{3^{\pi-1}}{2\pi}(a^{e\pi}+b^{e\pi}+c^{e\pi}) \le \frac{1}{3^e}(a^\pi+b^\pi+c^\pi)^e$$

$$\Rightarrow \left(\frac{a^e+b^e+c^e}{3}\right)\pi \le \left(\frac{a^\pi+b^\pi+c\text{^}\pi}{3}\right)^e$$

$\Rightarrow 3^e(a^e+b^e+c^e)^\pi \le 3^\pi(a^\pi+b^\pi+c^\pi)^e$***. Therefore it's true.***

420. If $a,b,c \ge 0$ then:

$$a^2\sqrt{b^2+c^2}+b^2\sqrt{c^2+a^2}+c^2\sqrt{a^2+b^2} \ge \sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michail Stergoiu-Greece

$$a,b,c>0 \Rightarrow \sum\nolimits_{cyc} a^2\sqrt{b^2+c^2} \ge \sqrt{\prod\nolimits_{cyc}(a^2+b^2)} \quad \textbf{(1)}$$

After rewritting LHS of (1) we use Minkowski's inequality

$\sum_{cyc}\sqrt{a^4b^2+a^4c^2} \ge \sqrt{(\sum a^2b)^2+(\sum a^2c)^2}$ ***which reduces to***

$$\sum_{cyc} a^2\sqrt{b^2+c\text{^}2} \ge \sqrt{\prod_{cyc}(a^2+b^2)+2abc\left(\sum_{cyc}ab^2+\sum_{cyc}ac^2\right)-2a^2b^2c^2}$$

It suffices to show that $\sum_{cyc} ab^2 + \sum_{cyc} ac^2 \geq abc$ which is obvious true by AM-GM

(in fact it is $\geq 6abc$)

Solution 2 by Soumava Chakraborty-Kolkata-India

Upon squaring, given inequality becomes

$$\sum a^4b^2 + \sum a^2b^4 + 2a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)} + 2b^2c^2\sqrt{(c^2+a^2)(a^2+b^2)} +$$

$$+2c^2a^2\sqrt{(a^2+b^2)(b^2+c^2)} \overset{(1)}{\geq} 2a^2b^2c^2 + \sum a^4b^2 + \sum a^2b^4$$

Now, $b^2 + c^2 \geq \frac{1}{2}(b+c)^2$ & $c^2 + a^2 \geq \frac{1}{2}(b+c)(c+a)$

$$\Rightarrow \sqrt{(b^2+c^2)(c^2+a^2)} \geq \frac{1}{2}(b+c)(c+a)$$

$$\therefore 2a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)} \overset{(a)}{\geq} a^2b^2(b+c)(c+a) = a^2b^2\left(\sum ab\right) + a^2b^2c^2$$

$(\because a, b, c \geq 0)$. *Similarly,* $2b^2c^2\sqrt{(c^2+a^2)(a^2+b^2)} \overset{(b)}{\geq} b^2c^2(\sum ab) + a^2b^2c^2$ &

$$2c^a a^2\sqrt{(a^2+b^2)(b^2+c^2)} \overset{(c)}{\geq} c^2a^2\left(\sum ab\right) + a^2b^2c^2$$

(a)+(b)+(c)$\Rightarrow 2\sum a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)} \geq (\sum ab)(\sum a^2b^2) + 3a^2b^2c^2 \geq$

$$\geq 2a^2b^2c^2 (\because a, b, c \geq 0)$$

$$\Leftrightarrow \sum a^4b^2 + \sum a^2b^4 + 2\sum a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)} \geq$$

$\geq \sum a^4b^2 + \sum a^2b^4 + 2a^2b^2c^2 \Rightarrow$ *(1) is true (Done)*

421. **If $x, y, z > 0, x + y + z = 3$ then:**

$$\sum \sqrt{(x+y+1)(y+z+1)} \leq 6 + \sum \frac{x^3+z^3}{x^2+z^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$6 + \sum \frac{x^3+z^3}{x^2+z^2} \overset{Chebyshev}{\underset{(1)}{\geq}} 6 + \sum \frac{(x+z)(x^2+z^2)}{2(x^2+z^2)} = 6 + \frac{\sum(x+z)}{2} = 6 + \sum x = 6 + 3 = 9$$

$$\sum\sqrt{(x+y+1)(y+z+1)}\overset{C-B-S}{\leq}\sqrt{\sum(x+y)+3}\sqrt{\sum(x+y)+3}=$$

$$=2\sum x+3=2\cdot 3+3=9\overset{by\ (1)}{\leq}6+\sum\frac{x^3+z^3}{x^2+z^2}\quad \textit{(proved)}$$

Solution 2 by Amit Dutta-Jamshedpur-India

We have to prove $\sum\left(\frac{x^3+z^3}{x^2+z^2}\right)+6\geq\sum\sqrt{(x+y+1)(y+z+1)}\Rightarrow LHS=\sum\left(\frac{x^3+z^3}{x^2+z^2}\right)+6.$

By Cauchy's Scwarz inequality: $(x^3+z^3)(x+z)\geq(x^2+z^2)^2$

$$\left(\frac{x^3+z^3}{x^2+z^2}\right)\geq\left(\frac{x^2+z^2}{x+z}\right)\qquad (1)$$

Again, by Cauchy's Schwarz inequality: $(x^2+z^2)(1^2+1^2)\geq(x+z)^2$

$$\left(\frac{x^2+z^2}{x+z}\right)\geq\left(\frac{x+z}{2}\right)\quad (2)$$

From (1) & (2): $\left(\frac{x^3+z^3}{x^2+z^2}\right)\geq\left(\frac{x+z}{z}\right)\Rightarrow\sum\frac{x^3+z^3}{x^2+z^2}\geq\sum\left(\frac{x+z}{z}\right)\geq(x+y+z)$

$$\therefore LHS=\sum\left(\frac{x^3+z^3}{x^2+z^2}\right)+6; LHS\geq(x+y+z)+6$$

$$LHS\geq(x+y+z)+(x+y+z)+3\ \{\because x+y+z=3\}$$

$$LHS\geq 2(x+y+z)+3\quad (3)$$

By AM-GM: $\frac{(x+y+1)+(y+z+1)}{2}\geq\sqrt{(x+y+1)(y+z+1)}\Rightarrow$

$$\Rightarrow\sum\frac{(x+y+1)+(y+z+1)}{2}\geq\sum\sqrt{(x+y+1)(y+z+1)}\Rightarrow$$

$$\Rightarrow\frac{4(x+y+z)+6}{2}\geq\sum\sqrt{(x+y+1)(y+z+1)}\Rightarrow$$

$$\Rightarrow 2(x+y+z)+3\geq\sum\sqrt{(x+y+1)(y+z+1)}\quad (4)$$

From (3) & (4): $LHS\geq\sum\sqrt{(x+y+1)(y+z+1)}\Rightarrow$

$$\Rightarrow\sum\left(\frac{x^3+z^3}{x^2+z^2}\right)+6\geq\sum\sqrt{(x+y+1)(y+z+1)}\quad \textit{(Proved)}$$

422. If $x,y,z>0$ then:

$$\frac{(x^4+y^4)^2+(y^4+z^4)^2+(z^4+x^4)^2}{\sqrt{x^4+y^4+z^4}}\geq 4\sqrt{3}x^2y^2z^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $x^4+y^4=a, y^4+z^4=b, z^4+x^4=c$.

Then $a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$

are three sides of a triangle with circumradius, inradius & semi-perimeter $=R,r,s$

(say). Now, $2\sum x^4=\sum a=2s \Rightarrow \sum x^4=s \therefore z^4=s-a, x^4=s-b, y^4=s-c$. Using the above transformation, given inequality becomes

$$\frac{\sum a^2}{\sqrt{s}} \ge 4\sqrt{3}\sqrt{(s-a)(s-b)(s-c)} \Leftrightarrow$$

$$\Leftrightarrow \sum a^2 \ge 4\sqrt{3}S \rightarrow \text{true (Ionescu – Weitzenbock) (proved)}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Since } \frac{\left((x^4+y^4)^2+(y^4+z^4)^2+(z^4+x^4)^2\right)^2}{(x^4+y^4+z^4)} \ge \frac{\left(\frac{(x^4+y^4+y^4+z^4+z^4+x^4)^2}{3}\right)^2}{(x^4+y^4+z^4)} = \frac{\left(\frac{4}{3}(x^4+y^4+z^4)^2\right)^2}{x^4+y^4+z^4} =$$

$$= \frac{16}{9}(x^4+y^4+z^4)^3 \ge \frac{16}{9}\left(3\sqrt[3]{x^4y^4z^4}\right)^3 = 16 \times 3x^4y^4z^4$$

$$\text{Hence } \frac{(x^4+y^4)^2+(y^4+z^4)^2+(z^4+x^4)^2}{\sqrt{x^4+y^4+z^4}} \ge 4\sqrt{3}x^2y^2z^2$$

423. For $a,b,c,d \ge 1$. Prove:

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4} \ge \frac{4}{1+abcd}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Chris Kyriazis-Athens-Greece

The range of the function $e^{\frac{t}{4}}, t \ge 0$ is $[1,+\infty)$. So, we can find t_1,t_2,t_3,t_4 such that $e^{\frac{t_1}{4}}=a, e^{\frac{t_2}{4}}=b, e^{t_3}=c, e^{t_4}=d$. Considering the function $g(t)=\frac{1}{1+e^{\frac{t}{4}}}$ when $t \ge 0$, it's

easy to check that is convex in $[0,+\infty)$. $\left(g'(1)=\frac{e^{\frac{t}{2}}-e^{\frac{4}{4}}}{16\left(e^{\frac{t}{4}}+1\right)^3}>0, \text{ when } t>0\right)$

Then, Jensen inequality gives us that

$$g(t_1)+g(t_2)+g(t_3)+g(t_4)\ge 4g\left(\frac{t_1+t_2+t_3+t_4}{4}\right)\Rightarrow$$

$$\Rightarrow \frac{1}{1+e^{\frac{t_1}{4}}}+\frac{1}{1+e^{\frac{t_2}{4}}}+\frac{1}{1+e^{\frac{t_3}{4}}}+\frac{1}{1+e^{\frac{t_4}{4}}}\ge \frac{4}{1+e^{\frac{t_1+t_2+t_3+t_4}{4}}}\Rightarrow$$

$$\Rightarrow \frac{4}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\ge \frac{4}{1+abcd}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=\frac{1}{\sqrt{1+e^x}}$ *for all* $0<e^x\le 1$ *or* $x\in[0,-\infty)$.

$$f'(x)=\frac{\frac{e^x}{2}}{(1+e^x)^{\frac{3}{2}}}\Rightarrow f''(x)=-\frac{\frac{e^x}{2}}{(1+e^x)^{\frac{3}{2}}}+\frac{3e^{2x}}{4}\cdot\frac{1}{(1+e^x)^{\frac{5}{2}}}=\frac{\frac{e^x}{2}(e^x-2)}{(1+e^x)^{\frac{5}{2}}}<0,$$ *hence* f *is concave*

Let $a^m=e^x, b^m=e^y$ *and* $e^z=c^m$ *then*

$$\sum_{cyc}\frac{1}{1+a^m}=\sum_{cyc}\frac{1}{\sqrt{1+e^x}}\le\frac{3}{\sqrt{1+e^{\frac{x+y+z}{3}}}}=\frac{3}{\sqrt{1+\sqrt[3]{(abc)^m}}}\le\frac{3\sqrt{2}}{1+(abc)^{\frac{m}{6}}}$$

$\left[\because \sqrt{2(x^2+y^2)}\ge x+y\right]$ *(proved)*

424. **If $x,y,z,t\ge 1$ then:**

$$x^x\cdot y^y\cdot z^z\cdot t^t\ge x^{\sqrt[3]{yzt}}\cdot y^{\sqrt[3]{ztx}}\cdot z^{\sqrt[3]{txy}}\cdot t^{\sqrt[3]{xyz}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

If $x,y,z,t\ge 1$ *then:* $x^x\cdot y^y\cdot z^z\cdot t^t\ge x^{\sqrt[3]{yzt}}\cdot y^{\sqrt[3]{ztx}}\cdot z^{\sqrt[3]{txy}}\cdot t^{\sqrt[3]{xyt}}$

$$\sqrt[3]{yzt}\le\frac{y+z+t}{3}\Rightarrow x^{\sqrt[3]{yzt}}\le x^{\frac{y+z+t}{3}}\quad (1)$$

From (1) and similarly we must show: $x^x\cdot y^y\cdot z^z\cdot t^t\ge x^{\frac{y+z+t}{3}}\cdot y^{\frac{z+t+x}{3}}\cdot z^{\frac{t+x+y}{3}}\cdot t^{\frac{x+y+t}{3}}$

$$\Leftrightarrow x^{3x}\cdot y^{3y}\cdot z^{3z}\cdot t^{3t}\ge x^{x+z+t}\cdot y^{z+t+x}\cdot z^{t+x+y}\cdot t^{x+y+z}\Leftrightarrow$$

$$\Leftrightarrow x^{4x}\cdot y^{4y}\cdot z^{4z}\cdot t^{4t}\ge (xyzt)^{x+y+z+t}\Leftrightarrow$$

$$\Leftrightarrow x^xy^yz^zt^t\ge\sqrt[4]{xyzt}^{\,x+y+z+t}\quad (2)$$

$$\text{But } \sqrt[4]{xyzt}\le\frac{x+y+z+t}{4}\quad (3)$$

From (2)+(3) we must show: $x^x y^y z^z t^t \geq \left(\frac{x+y+z+t}{4}\right)^{x+y+z+t} \Leftrightarrow$

$$\Leftrightarrow \ln(x^x y^y z^z t^t) \geq \ln\left(\frac{x+y+z+t}{4}\right)^{x+y+z+t} \Leftrightarrow$$

$$\Leftrightarrow x\ln x + y\ln y + z\ln z + \ln t \geq (x+y+z+t)\ln\left(\frac{x+y+z+t}{4}\right) \quad (4)$$

Let $f(\alpha) = \alpha\ln\alpha$

$f'(\alpha) = \ln\alpha + 1, f''(\alpha) = \frac{1}{\alpha} > 0, \forall\alpha > 1 \Rightarrow f$ ***convex from Jensen's inequality*** $\Rightarrow$

$$\Rightarrow f\left(\frac{x+y+z+t}{4}\right) \leq \frac{f(x)+f(y)+f(z)+f(t)}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{x\ln x + y\ln y + z\ln z + t\ln t}{4} \geq \frac{x+y+z+t}{4}\ln\left(\frac{x+y+z+t}{4}\right) \Leftrightarrow$$ ***4 its true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{yzt} \overset{G\leq A}{\leq} \frac{y+z+t}{3} \therefore x^{\sqrt[3]{yzt}} \overset{(1)}{\leq} x^{\frac{y+z+t}{3}} \ (\because x \geq 1)$$

Similarly, $y^{\sqrt[3]{ztx}} \overset{(2)}{\leq} y^{\frac{z+t+x}{3}}; z^{\sqrt[3]{txy}} \overset{(3)}{\leq} z^{\frac{t+x+y}{3}}\ \&\ t^{\sqrt[3]{xyz}} \overset{(4)}{\leq} t^{\frac{x+y+z}{3}}$

(1)·(2)·(3) ·(4) $\Rightarrow RHS \overset{(a)}{\leq} x^{\frac{y+z+t}{3}} \cdot y^{\frac{z+t+x}{3}} \cdot z^{\frac{t+x+y}{3}} \cdot t^{\frac{x+y+z}{3}}$

weighted GM $\leq$ ***weighted AM*** $\Rightarrow$

$$\sqrt[\frac{y+z+t}{3}+\frac{z+t+x}{3}+\frac{t+x+y}{3}+\frac{x+y+z}{3}]{x^{\frac{y+z+t}{3}} \cdot y^{\frac{z+t+x}{3}} \cdot z^{\frac{t+x+y}{3}} \cdot t^{\frac{x+y+z}{3}}} \leq$$

$$\leq \frac{\frac{1}{3}\{x(y+z+t) + y(z+t+x) + z(t+x+y) + t(x+y+z)\}}{\sum x} =$$

$$= \frac{\frac{2}{3}(xy+xz+xt+yz+yt+zt)}{\sum x} \Rightarrow \sqrt[\sum x]{x^{\frac{y+z+t}{3}} \cdot y^{\frac{z+t+x}{3}} \cdot z^{\frac{t+x+y}{3}} \cdot t^{\frac{x+y+z}{3}}}$$

$$\overset{(i)}{\leq} \frac{\frac{2}{3}(xy+xz+xt+yz+yt+zt)}{\sum x}$$

Now, $(\sum x)^2 = \sum x^2 + 2(xy+xz+xt+yz+yt+zt) =$

$$= \frac{x^2+y^2+z^2}{3} + \frac{x^2+y^2+t^2}{3} + \frac{x^2+z^2+t^2}{3} + \frac{y^2+z^2+t^2}{3} +$$

$$+2(xy+xz+xt+yz+yt+zt) \geq$$

$$\geq \frac{xy+yz+zx+xy+yt+tx+xz+zt+tx+yz+zt+ty}{3}+$$

$$+2(xy+xz+xt+yz+yt+zt)=\frac{8}{3}(xy+xz+xt+yz+yt+zt)\Rightarrow$$

$$\Rightarrow \frac{2}{3}(xy+xz+xt+yz+yt+zt)\overset{(ii)}{\leq}\frac{(\sum x)^2}{4}$$

(a), (i), (ii) $\Rightarrow RHS \overset{(5)}{\leq} \left(\frac{\sum x}{4}\right)^{\sum x}$

Again, weighted GM ≥ weighted HM $\Rightarrow \sqrt[x+y+z+t]{x^x y^y z^z t^t} \geq \frac{\sum x}{\sum \frac{x}{x}} = \frac{\sum x}{4} \Rightarrow LHS \overset{(6)}{\geq} \left(\frac{\sum x}{4}\right)^{\sum x}$

(5), (6) $\Rightarrow LHS \geq RHS$ ***(Proved)***

425. If $x,y,z>0$ then:

$$\frac{1}{x^2+y^2+2z^2}+\frac{1}{y^2+z^2+2x^2}+\frac{1}{z^2+x^2+2y^2}\geq \frac{2xyz\sqrt{3(x^2+y^2+z^2)}}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Le Van-Ho Chi Minh-Vietnam

By geometrizing, we may transform $(a;b;c)=(x^2+y^2;y^2+z^2;z^2+x^2)$ ***of which*** a,b ***and*** c ***are three sides of triangle, namely*** ΔABC***. Hence, the to-prove problem becomes:*** $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq \frac{2\sqrt{3s(s-a)(s-b)(s-c)}}{abc}=\frac{2\sqrt{3}s}{abc}=\frac{\sqrt{3}}{2R}$

By $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$***, it is enough to prove that:***

$$\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\geq \sqrt{3}$$

Indeed, applying Schwarz's inequality: $\sum \frac{1}{\sin A+\sin B}\geq \frac{9}{2(\sin A+\sin B+\sin C)}\geq \sqrt{3}$

Note that in any given triangle ABC, $\sin A+\sin B+\sin C\leq \frac{3\sqrt{3}}{2}$.

Q.E.D. Equality holds when triangle ABC ***is equilateral, in other words*** $x=y=z$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{x^2+y^2+2z^2}+\frac{1}{y^2+z^2+2x^2}+\frac{1}{z^2+x^2+2y^2}\overset{(1)}{\geq} \frac{2xyz\sqrt{3(x^2+y^2+z^2)}}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}$$

Let $x^2+y^2=a, y^2+z^2=b, z^2+x^2=c$. Then $a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ are sides of a triangle with circumradius, inradius, semi-perimeter $=R,r,s$ (say).

Now, $2\sum x^2=\sum a=2s \Rightarrow \sum x^2=s \Rightarrow y^2=s-c, x^2=s-b, z^2=s-a$

Using above transformation, (1) becomes:

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\ge\frac{2\sqrt{\pi(s-a)}}{abc}\sqrt{3s}\Leftrightarrow\frac{3\sum ab+\sum a^2}{2abc+\sum ab\,(2s-c)}\ge\frac{2rs\cdot\sqrt{3}}{4Rs}\Leftrightarrow$$

$$\Leftrightarrow\frac{4s^2+s^2+4Rr+r^2}{2abc+2s(s^2+4Rr+r^2)-3abc}\ge\frac{\sqrt{3}}{2R}\Leftrightarrow\frac{5s^2+4Rr+r^2}{2s(s^2+4Rr+r^2)-4Rrs}\ge\frac{\sqrt{3}}{2R}\Leftrightarrow$$

$$\Leftrightarrow\frac{5s^2+4Rr+r^2}{s(s^2+2Rr+r^2)}\ge\frac{\sqrt{3}}{R}\Leftrightarrow\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\overset{(2)}{\ge}\frac{\sqrt{3}s}{R}.$$ *Now, RHS of (2)* $\overset{Mitrinovic}{\le}\frac{\sqrt{3}}{R}\cdot\frac{3\sqrt{3}R}{2}=\frac{9}{2}\le$

$$\overset{?}{\le}\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\Leftrightarrow 10s^2+8Rr+2r^2\overset{?}{\ge}9s^2+18Rr+9r^2\Leftrightarrow s^2\underset{(3)}{\overset{?}{\ge}}10Rr+7r^2$$

Now, LHS of (3) $\overset{Gerretsen}{\ge}16Rr-5r^2\overset{?}{\ge}10Rr+7r^2\Leftrightarrow 6Rr\overset{?}{\ge}12R^2\Leftrightarrow R\overset{?}{\ge}2r\rightarrow$

true (Euler) (Proved)

426. If $x,y,z>0$ then:

$$8(x+y+z)^9\sum\left(\frac{yz}{xy+xz}\right)^3\ge 3^{10}x^3y^3z^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$x+y+z\ge 3\sqrt[3]{xyz}\Rightarrow(x+y+z)^9\ge 3^9x^3y^3z^3\quad (1)$$

From (1) inequality becomes: $8\sum\left(\frac{yz}{xy+xz}\right)^3\ge 3\Leftrightarrow\sum\left(\frac{yz}{xy+xz}\right)^3\ge\frac{3}{8}$ *(2)*

From Hölder's inequality we have: $\sum\left(\frac{yz}{xy+xz}\right)^3\ge\frac{1}{9}\left(\sum\frac{yz}{xy+xz}\right)^3$ *(3)*

From (2) + (3) we must show: $\left(\sum\frac{yz}{xy+xz}\right)^3\ge\frac{27}{8}\Leftrightarrow\sum\frac{yz}{xy+xz}\ge\frac{3}{2}$ *(4)*

Let $yz=a, xy=b, xz=c, a,b,c>0$.

(4) $\Leftrightarrow\sum\frac{a}{b+c}\ge\frac{3}{2}$ *(true because is Nesbitt's inequality)*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{yz}{xy+xz}\right)^3 \overset{Chebyshev}{\geq} \frac{1}{9}\left(\sum\frac{yz}{xy+xz}\right)^3 =$$

$$=\frac{1}{9}\left(\sum\frac{y^2z^2}{xy^2z+xyz^2}\right)^3 \overset{Bergstrom}{\geq} \frac{1}{9}\cdot\frac{(\sum xy)^6}{\{2xyz(\sum x)\}^3} \geq$$

$$\geq \frac{1}{9}\cdot\frac{\{3xyz(\sum x)\}^3}{\{2xyz(\sum x)\}^3}\left(\because \left(\sum xy\right)^2 \geq 3xyz\left(\sum x\right)\right) = \frac{1}{9}\cdot\frac{27}{8} = \frac{3}{8}$$

$$\therefore LHS \geq 8\cdot\frac{3}{8}\left(\sum x\right)^9 \overset{?}{\geq} 3^{10}(xyz)^3$$

$$\Leftrightarrow (\sum x)^3 \overset{?}{\geq} 27xyz \Leftrightarrow \sum x \overset{?}{\geq} 3\sqrt[3]{xyz} \rightarrow \textit{true by AM} \geq \textit{GM (Proved)}$$

427. **If $0 < x,y,z \leq 2, \sqrt{xy}+\sqrt{yz}+\sqrt{zx} = 3$ then:**

$$\sqrt{2} < \frac{3+\sqrt{y(2-x)}+\sqrt{z(2-y)}+\sqrt{x(2-z)}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \leq 2$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

First: $3+\sqrt{y(2-x)}+\sqrt{z(2-y)}+\sqrt{x(2-z)} \leq 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$ ***(1)***

But $\left.\begin{array}{l}\sqrt{y(2-x)} \leq \frac{y+2-x}{2}\\ \sqrt{z(2-y)} \leq \frac{z+2-y}{2}\\ \sqrt{x(2-z)} \leq \frac{x+2-t}{2}\end{array}\right\} \Rightarrow \sum\sqrt{y(2-x)} \leq 3$ ***(2)***

From (1) + (2) we must show: $6 \leq 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \Leftrightarrow$

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq 3 \quad \textbf{(3)}$$

From $\sqrt{xy}+\sqrt{yz}+\sqrt{xz} = 3 \Rightarrow \exists a,b,c > 0$ ***such that***

$$x = \frac{3bc}{a(a+b+c)}, y = \frac{3ac}{b(a+b+c)}, z = \frac{3ab}{c(a+b+c)} \quad \textbf{(4)}$$

From (3) + (4) we must show: $\sqrt{\frac{3bc}{a(a+b+c)}}+\sqrt{\frac{3ac}{b(a+b+c)}}+\sqrt{\frac{3ab}{c(a+b+c)}} \geq 3 \Leftrightarrow$

$\Leftrightarrow ab+bc+ac \geq \sqrt{3abc(a+b+c)} \Leftrightarrow (ab+ac+bc)^2 \geq 3abc(a+b+c)$ *which its true because* $(m+n+p)^2 \geq 3(ma+mp+np)$

Second: $\sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) < 3+\sqrt{y(2-x)}+\sqrt{z(2-y)}+\sqrt{x(2-z)}$

$$\Leftrightarrow \sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) < \sqrt{xy}+\sqrt{yz}+\sqrt{zx}+\sqrt{y(2-x)}+\sqrt{z(2-y)}+\sqrt{x(2-z)} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{y}(\sqrt{x}+\sqrt{2-x})+\sqrt{z}(\sqrt{y}+\sqrt{2-y})+\sqrt{x}(\sqrt{z}+\sqrt{2-z}) > \sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \quad \textbf{(5)}$$

Let $f:[0,2]\to\mathbb{R}, f(x)=\sqrt{x}+\sqrt{2-x}; f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{2-x}}=\frac{\sqrt{2-x}-\sqrt{x}}{2\sqrt{x(2-x)}}$

$$f'(x)=0 \Rightarrow x=1$$

x	0		1		2
$f'(x)$	1	+ + + + + + + + +	0	− − − − − − − − −	−1
$f(x)$	$\sqrt{2}$ m	↗	2 M	↘	$\sqrt{2}$ m

$$\Rightarrow \sqrt{x}+\sqrt{2-x} \geq \sqrt{2} \quad \textbf{(6)}$$

From (6) $\Rightarrow \sqrt{y}(\sqrt{x}+\sqrt{2-x}) \geq \sqrt{2}y$ *and similarly (7)*

From (7) $\Rightarrow \sum\sqrt{y}\left(\sqrt{x}+\sqrt{2-x}\right) > \sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z})$ *(strictly)*

428. For $a,b,c \in (0;1] \wedge m \in \mathbb{N}^*$. Prove:

$$\frac{1}{\sqrt{1+a^m}}+\frac{1}{\sqrt{1+b^m}}+\frac{1}{\sqrt{1+c^m}} \leq \frac{3\sqrt{2}}{1+(abc)^{\frac{m}{6}}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Chris Kyriazis-Athens-Greece

The range of the function $g(t)=e^{-\frac{t}{2}}$ *is* $(0,1]$ *when* $t\geq 0$*. So, there are unique* t_1,t_2,t_3 *such that* $e^{-\frac{t_1}{2}}=a^m, e^{-\frac{t_2}{2}}=b^m, e^{-\frac{t_3}{2}}=c^m$ *(since* $0<a^m,b^m,c^m\leq 1$*).*

Considering the function: $f(x)=\frac{1}{\sqrt{1+e^{-\frac{x}{2}}}}, x\geq 0$ *we have that* $f''(x)<0$ *for every*

$x > 0$ *because* $2e^x - 1 > 0$ *when* $x > 0$. *So* t *is concave in* $[0, +\infty)$. *By Jensen's inequality, we have that* $\frac{f(t_1)+f(t_2)+f(t_3)}{3} \le f\left(\frac{t_1+t_2+t_3}{3}\right) \Rightarrow \frac{1}{\sqrt{1+e^{-\frac{t_1}{2}}}} + \frac{1}{\sqrt{1+e^{-\frac{t_2}{2}}}} + \frac{1}{\sqrt{1+e^{-\frac{t_3}{2}}}} \le$

$\frac{3}{\sqrt{1+e^{-\frac{t_1+t_2+t_3}{6}}}}$ *or* $\frac{1}{\sqrt{1+a^m}} + \frac{1}{\sqrt{1+b^m}} + \frac{1}{\sqrt{1+c^m}} \le \frac{3}{\sqrt{1+\sqrt[3]{a^m b^m c^m}}}$. *It suffices to prove that:*

$$\frac{3}{\sqrt{1+\sqrt[3]{a^m b^m c^m}}} \le \frac{3\sqrt{2}}{1+\sqrt[6]{a^m b^m c^m}} \text{ or } 1 + \sqrt[6]{a^m b^m c^m} \le \sqrt{2\left(1 + \sqrt[3]{a^m b^m c^m}\right)} \text{ or}$$

$$1 + \sqrt[3]{a^m b^m c^m} + 2\sqrt[6]{a^m b^m c^m} \le 2 + 2\sqrt[3]{a^m b^m c^m} \text{ or } \left(\sqrt[3]{a^m b^m c^m}\right)^2 \ge 0 \text{ which holds!!!}$$

429. **Let $a, b, c \in (0; +\infty) \wedge ab + bc + ca = 3$. Prove:**

$$\frac{1}{\sqrt[6]{a^2+3}} + \frac{1}{\sqrt[6]{b^2+3}} + \frac{1}{\sqrt[6]{c^2+3}} \le \frac{\sqrt[3]{36}}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{\frac{1}{3}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Marian Ursarescu-Romania

Because $ab + bc + ac = 3 \Rightarrow \exists x, y, z > 0$ *such that:*

$a = \frac{\sqrt{3}x}{\sqrt{xy+xz+yt}}, b = \frac{\sqrt{3}y}{\sqrt{xy+xt+yt}}, c = \frac{\sqrt{3}z}{\sqrt{xy+xz+yt}}$. *Inequality becomes:*

$$\sum \frac{\sqrt[6]{xy+xz+yt}}{\sqrt[6]{3}\sqrt[6]{x^2+xy+xz+yt}} \le \frac{\sqrt[3]{36}}{2} \cdot \frac{\sqrt[6]{xy+xz+yz}}{3}\left(\sqrt[3]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt[6]{(x+y)(x+z)}} \le \frac{\sqrt[3]{36}}{2}\left(\sqrt[6]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \quad (1)$$

$$(1) \Leftrightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le \frac{9}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \quad (2)$$

From Hölder's inequality $\Rightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le 9\sum \frac{1}{\sqrt{(x+y)(x+z)}}$ *(3)*

From (2)+(3) we must show: $\sum \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ *(4)*

But $\begin{matrix} x+y \ge 2\sqrt{xy} \\ x+z \ge 2\sqrt{xz} \end{matrix} \Rightarrow (x+y)(x+z) \ge 4x\sqrt{yz} \Rightarrow \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2\sqrt{x\sqrt{yz}}} \Rightarrow$

$$\Rightarrow \sum \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2}\sum \frac{1}{\sqrt{x\sqrt{yz}}} \quad (5)$$

From (4)+(5) we must show: $\sum \frac{1}{\sqrt{x\sqrt{yz}}} \le \sum \frac{1}{x}$ ***(6)***

Now use $\alpha^2+\beta^2+\gamma^2 \ge \alpha\beta+\alpha\gamma+\beta\gamma \Rightarrow \sum \frac{1}{x} = \sum \frac{1}{(\sqrt{x})^2} \ge \sum \frac{1}{\sqrt{xy}} = \sum \frac{1}{(\sqrt{\sqrt{xy}})^2} \ge$

$\ge \sum \frac{1}{\sqrt{x\sqrt{yz}}} \Rightarrow$ ***(6) its true.***

Solution 2 by Soumava Chakraborty-Kolkata-India

Firstly, $\sum x^3 \overset{Chebyshev}{\ge} \frac{1}{9}\left(\sum x\right)^3 \Rightarrow \left(\sum x\right)^3 \le 9\sum x^3 \Rightarrow \sum x \overset{(1)}{\le} \sqrt[3]{9}\sqrt[3]{\sum x^3}$

Now, $a^2+3 = a^2+ab+bc+ca \overset{(2)}{=} (c+a)(a+b)$.

Similarly, $b^2+3 \overset{(3)}{=} (b+c)(a+b)$

& $c^2+3 \overset{(4)}{=} (b+c)(c+a)$***; (2), (3), (4)*** $\Rightarrow LHS = \sum \frac{1}{\sqrt[6]{(a+b)(b+c)}} \overset{CBS}{\le} \sqrt{\sum \frac{1}{\sqrt[3]{a+b}}}\sqrt{\sum \frac{1}{\sqrt[3]{b+c}}} =$

$$= \sum \frac{1}{\sqrt[3]{a+b}} \overset{A-G}{\le} \sum \frac{1}{\sqrt[3]{2\sqrt{ab}}} = \frac{1}{\sqrt[3]{2}}\sum \frac{1}{\sqrt[6]{ab}} \overset{CBS}{\le} \frac{1}{\sqrt[3]{2}}\sqrt{\sum \frac{1}{\sqrt[3]{a}}}\sqrt{\sum \frac{1}{\sqrt[3]{a}}} = \frac{1}{\sqrt[3]{2}}\sum \frac{1}{\sqrt[3]{a}} =$$

$$= \frac{1}{\sqrt[3]{2}\sqrt[3]{abc}}\left(\sum \sqrt[3]{ab}\right) \overset{by\ (1)}{\le} \frac{1}{\sqrt[3]{2}\sqrt[3]{abc}}\sqrt[3]{9}\sqrt[3]{\sum ab} = \sqrt[3]{\frac{9}{2}}\frac{1}{\sqrt[3]{abc}}\left(\sum ab\right)^{\frac{1}{3}} =$$

$= \sqrt[3]{\frac{36}{8}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{36}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}}$ ***(Proved)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$ab+bc+ca = 3$ **and** $\prod_{cyc}(a+b) \ge \frac{9}{8}(a+b+c)(ab+bc+ca)$

$$\sum_{cyc} \frac{1}{\sqrt[6]{a^2+3}} = \sum_{cyc} \frac{1}{\sqrt[6]{a^2+ab+bc+ca}} = \sum_{cyc} \frac{1}{\sqrt[6]{(a+b)(a+c)}} =$$

$$= \frac{1}{\sqrt[6]{\prod_{cyc}(a+b)}}\sum_{cyc}\sqrt[6]{a+b} \le \frac{1}{\sqrt[6]{\prod_{cyc}(a+b)}}\sqrt{3^5\cdot 2(a+b+c)}$$

$$\leq \frac{\sqrt[6]{3^5\cdot 2(a+b+c)}}{\sqrt[6]{\frac{8}{9}(a+b+c)(ab+bc+ca)}} = \sqrt[6]{\frac{3^7}{4(ab+bc+ca)}} \leq \sqrt[6]{\frac{3^9}{4}\sum_{cyc}\frac{1}{ab}} \leq$$

$$\leq \sqrt[6]{\frac{3^8}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2} = \frac{\sqrt[3]{36}}{2}\sqrt[3]{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \text{ (proved)}$$

430. If $a, b, c \in \mathbb{R}$, then:

$$(a^3+b^3+c^3-3abc)^2 \leq (a^2+b^2+c^2)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michail Stergioiu-Greece

$$(a^3+b^3+c^3-3abc)^2 \leq (a^2+b^2+c^2)^3 \quad (1)$$

(1) is homogeneous so we can assume WLOG that $a+b+c=1$*. Then, as*

$q \leq \frac{p^2}{3} \to q \leq \frac{1}{3}$ *or* $-\infty < q \leq \frac{1}{3}$*, where* $p=\sum_{cyc} a$, $q=\sum_{cyc} ab$, $r=abc$.

(1) $\to (p^3-3pq+3r-3r)^2 \leq (p^2-2q)^3 \to (1-3q)^2 \leq (1-2q)^3 \to$

$\to q^2(8q-3) \leq 0 \to q \leq \frac{3}{8}$ *which holds as* $q \leq \frac{1}{3} = \frac{3}{9} < \frac{3}{8}$*. We are done.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$\sum a^3 - 3abc = (\sum a)(\sum a^2 - \sum ab) \therefore$ *given inequality becomes:*

$$\left(\sum a\right)^2\left(\sum a^2-\sum ab\right)^2 \leq \left(\sum a^2\right)^3 \Leftrightarrow \left(\sum a^2+2\sum ab\right)\left(\sum a^2-\sum ab\right)^2 \leq$$

$\leq (\sum a^2)^3 \Leftrightarrow (x+2y)(x^2+y^2-2xy) \leq x^3$ *(where* $\sum a^2 = x, \sum ab = y$*)* $\Leftrightarrow$

$\Leftrightarrow y^2(2y-3x) \leq 0 \Leftrightarrow 3x \geq 2y \Leftrightarrow x+2(x-y) \geq 0 \Leftrightarrow$

$\Leftrightarrow \sum a^2 + 2(\sum a^2 - \sum ab) \geq 0 \Leftrightarrow \sum a^2 + \sum(a-b)^2 \geq 0 \to$ *true (Hence proved)*

431. If $a, b, c > 0, a+b+c=3$ then:

$$a^4+b^4+c^4 \geq 3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$f(a,b,c)=a^4+b^4+c^4+\lambda(a+b+c-3)$$

$$\begin{cases} f'_a=4a^3+\lambda=0 \\ f'_b=4b^3+\lambda=0 \\ f'_c=4c^3+\lambda=0 \\ f'_\lambda=a+b+c-3=0\end{cases} \to a=b=c=-\sqrt[3]{\frac{\lambda}{4}} \to -3\sqrt[3]{\frac{\lambda}{4}}-3=0 \to \lambda=-4 \to \begin{cases} a=1 \\ b=1 \\ c=1\end{cases}$$

$$H_f(1,1,1,-4)=\begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} \to \Delta_1=12>0, \Delta_2=144>0, \Delta_3=12^3>0$$

$$minf(a,b,c,\lambda)=f(1,1,1-4)=3 \to a^4+b^4+c^4 \ge 3$$

432. If $a,b,c>0, abc=1$ then:

$$\frac{1}{a^4+b^4+c}+\frac{1}{b^4+c^4+a}+\frac{1}{c^4+a^4+b} \le 1$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a^4+b^4+c=a^4+b^4+abc\cdot c \overset{Chebyshev}{\ge} \frac{1}{2}(a^2+b^2)^2+abc^2 \ge$$

$$\overset{A-G}{\ge} ab(a^2+b^2)+abc^2=ab(a^2+b^2)+abc^2=ab\left(\sum a^2\right) \Rightarrow$$

$$\Rightarrow \frac{1}{a^4+b^4+c} \overset{(1)}{\le} \frac{1}{ab\sum a^2}. \text{ Similarly, } \frac{1}{b^4+c^4+a} \overset{(2)}{\le} \frac{1}{bc\sum a^2} \text{ and, } \frac{1}{c^4+a^4+b} \overset{(2)}{\le} \frac{1}{ca\sum a^2}$$

$$(1)+(2)+(3) \Rightarrow LHS \le \left(\frac{1}{\sum a^2}\right)\left(\sum \frac{1}{ab}\right) \overset{abc=1}{=} \left(\frac{1}{\sum a^2}\right)\left(\sum a\right) \overset{?}{\le} 1 \Leftrightarrow \sum a \underset{(4)}{\overset{?}{\le}} \sum a^2. \text{ Now,}$$

$$\sum a \overset{CBS}{\le} \sqrt{3\sum a^2} \overset{?}{\le} \sum a^2 \Leftrightarrow \sum a^2 \underset{(5)}{\overset{?}{\ge}} 3. \text{ Now, } \sum a^2 \overset{Chebyshev}{\ge} \frac{(\sum a)^2}{3} \overset{?}{\ge} 3 \Leftrightarrow \sum a \underset{(6)}{\overset{?}{\le}} 3.$$

$$\text{Now, } \sum a \overset{A-G}{\ge} 3\sqrt[3]{abc}=3 \Rightarrow (6) \text{ is true (proved)}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $abc=1, a,b,c>0$ we have

$$ab+bc+ca \ge 3 \Rightarrow (ab)^2+(bc)^2+(ca)^2 \ge 3 \Rightarrow 2((ab)^2+(bc)^2+(ca)^2) \ge 6 \text{ and}$$

$a^4+b^4+c^4 \geq \frac{(a^3+b^3+c^3)(a+b+c)}{3} \geq a^3+b^3+c^3$ ***and since for*** $x,y,z>0$***, we get***

$\frac{1}{x^4+y^4+z} \leq \frac{1+1+z^3}{(x^2+y^2+z^2)^2}$***. Hence*** $\frac{1}{a^4+b^4+c}+\frac{1}{b^4+c^4+a}+\frac{1}{c^4+a^4+b} \leq \frac{1+1+1+1+1+1+a^3+b^3+c^3}{(a^2+b^2+c^2)^2}=$

$= \frac{a^3+b^3+c^3+6}{a^4+b^4+c^4+2((ab)^2+(bc)^2+(ca)^2)} \leq \frac{a^3+b^3+c^3+6}{a^3+b^3+c^3+6} = 1$***. Therefore it is to be true.***

Because $abc=1, a,b,c>0$***, we have:*** $\frac{1}{a^4+b^4+c}+\frac{1}{b^4+c^4+a}+\frac{1}{c^4+a^4+b} \leq$

$$\leq \frac{1}{a^3b+b^2a+abc^2}+\frac{1}{b^3c+c^3b+a^2bc}+\frac{1}{c^3a+a^3c+ab^2c}=$$

$$=\frac{1}{ab(a^2+b^2+c^2)}+\frac{1}{bc(a^2+b^2+c^2)}+\frac{1}{ca(a^2+b^2+c^2)}=$$

$$=\frac{c}{a^2+b^2+c^2}+\frac{a}{a^2+b^2+c^2}+\frac{b}{a^2+b^2+c^2}=\frac{a+b+c}{a^2+b^2+c^2} \leq$$

$\leq \frac{a+b+c}{\frac{(a+b+c)^2}{3}} = \frac{3}{a+b+c} \leq 1$***. Therefore it is to be true.***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know, $x^4+y^4 \geq \frac{(x^2+y^2)^2}{2} \geq xy(x^2+y^2)$

$$\sum_{cyc}\frac{1}{a^4+b^4+c} \leq \sum_{cyc}\frac{1}{ab(a^2+b^2)+c} = \sum_{cyc}\frac{1}{\frac{a^2+b^2}{c}+c} = \frac{a+b+c}{a^2+b^2+c^2} \leq 1$$

Now, $\sum_{cyc}a^2 \geq \frac{(a+b+c)^2}{3} \geq a+b+c$***. Hence proved.***

433. **If $a,b,c>0, abc=1$ then:**

$$\sum\frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} \geq \frac{1}{a^{21}}+\frac{1}{b^{21}}+\frac{1}{c^{21}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$RHS = \frac{\sum a^{21}b^{21}}{a^{21}b^{21}c^{21}} = \frac{\sum a^{21}b^{21}}{(abc)^{21}} = \sum a^{21}b^{21} \ (\because abc=21)$$

$\therefore$ ***given inequality*** $\Leftrightarrow \sum\frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} \overset{(i)}{\geq} \sum a^{21}b^{21}$***. We shall now prove:***

$$\frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} \overset{(1)}{\geq} a^{21}b^{21}.$$

Now, LHS of (1) $\overset{Chebyshev}{\geq} \frac{\left(\frac{1}{2}(a^2+b^2)(a^{14}+b^{14})\right)\left(\frac{1}{2}(a^4+b^4)(a^{28}+b^{28})\right)}{(a^2+b^2)(a^4+b^4)} \overset{?}{\geq} a^{21}b^{21} \Leftrightarrow$

$$\Leftrightarrow (a^{14}+b^{14})(a^{28}+b^{28}) \underset{(2)}{\overset{?}{\geq}} 4a^{21}b^{21}.$$

Now, LHS of (2) $\overset{A-G}{\geq} (2a^7b^7)(2a^{14}b^{14}) = 4a^{21}b^{21} \Rightarrow$ *(2) is true* $\Rightarrow$ *(1) is true.*

Similalry, $\frac{(b^{16}+c^{16})(b^{32}+c^{32})}{(b^2+c^2)(b^4+c^4)} \overset{(2)}{\geq} b^{21}c^{21}$ & $\frac{(c^{16}+a^{16})(c^{32}+a^{32})}{(c^2+a^2)(c^4+a^4)} \overset{(1)}{\geq} c^{21}a^{21}$

(1)+(2)+(3)⇒ (i) is true (Proved)

Solution 2 by Amit Dutta-Jamshedpur-India

Using AM of m^th power ≥ m^th power of AM, i.e. $\frac{a_1^n+a_2^n}{2} \geq \left(\frac{a_1+a_2}{2}\right)^m, \forall m \in R-(0,1)$

Put $a_1 = a^2, a_2 = b^2, m = 8 \Rightarrow \frac{(a^2)^8+(b^2)^8}{2} \geq \left(\frac{a^2+b^2}{2}\right)^8 \Rightarrow \frac{a^{16}+b^{16}}{2} \geq \frac{(a^2+b^2)^8}{2^8} \Rightarrow$

$$\Rightarrow \frac{a^{16}+b^{16}}{a^2+b^2} \geq \left(\frac{a^2+b^2}{2}\right)^2 \quad (1)$$

Also, putting $a_1 = a^4, a_2 = b^4, m = 8 \Rightarrow \frac{(a^4)^8+(b^4)^8}{2} \geq \left(\frac{a^4+b^4}{2}\right)^8 \Rightarrow \frac{a^{32}+b^{32}}{a^4+b^4} \geq \left(\frac{a^4+b^4}{2}\right)^7 \quad (2)$

From (1) & (2): $LHS = \sum \frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} \geq \sum \left[\frac{(a^2+b^2)(a^4+b^4)}{4}\right]^7$

$$LHS \geq \sum \left[\frac{(a^2+b^2)(b^4+b^4)}{4}\right]^7 \overset{AM-GM}{\geq} \sum \left[\frac{2ab\cdot 2a^2b^2}{4}\right]^7 \geq \sum \left(\frac{4a^3b^3}{4}\right)^7 \geq \sum a^{21}b^{21} \geq$$

$$\geq a^{21}b^{21} + b^{21}c^{21} + c^{21}a^{21} \geq a^{21}b^{21}c^{21}\left(\frac{1}{a^{21}} + \frac{1}{b^{21}} + \frac{1}{c^{21}}\right)$$

$$LHS = \sum \frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^4+b^4)(a^2+b^2)} \geq \left(\frac{1}{a^{21}} + \frac{1}{b^{21}} + \frac{1}{c^{21}}\right) \{\because abc = 1\}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

for $abc = 1, a, b, c > 0$*, we have*

$$\frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} + \frac{(b^{16}+c^{16})(b^{32}+c^{32})}{(b^2+c^2)(b^4+c^4)} + \frac{(c^{16}+a^{16})(c^{32}+a^{32})}{(c^2+a^2)(c^4+a^4)} \geq$$

$$\geq \frac{(a^2+b^2)(a^4+b^4)(a^{14}+b^{14})(a^{28}+b^{28})}{(a^2+b^2)(a^4+b^4)\times 4} + \cdots +$$

$$+\frac{(c^2+a^2)(c^4+a^4)(c^{14}+a^{14})(c^{28}+a^{28})}{(c^2+a^2)(c^4+a^4)\times 4}\geq$$

$$\geq\frac{(a^{21}+b^{21})^2}{4}+\frac{(b^{21}+c^{21})^2}{4}+\frac{(c^{21}+a^{21})^2}{4}\geq\frac{1}{4}(a^{21}+b^{21})(b^{21}+c^{21})+$$

$$+(b^{21}+c^{21})(c^{21}+a^{21})+(c^{21}+a^{21})(a^{21}+b^{21})=\frac{1}{4}\left[\begin{matrix}3\left((ab)^{21}+(bc)^{21}+(ca)^{21}\right)+\\+a^{42}+b^{42}+c^{42}\end{matrix}\right]$$

$$\geq\frac{1}{4}[4(a^{21}b^{21}+b^{21}c^{21}+c^{21}a^{21})]=a^{21}b^{21}+b^{21}c^{21}+c^{21}a^{21}=$$

$=\frac{1}{c^{21}}+\frac{1}{a^{21}}+\frac{1}{b^{21}}:(abc)^{21}=1$. ***Therefere is to be true.***

434. If $a,b,c>0$, $a+b+c=3$ then:

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$a(a-1)^2\geq 0\to a(a^2-2a+1)\geq 0\to 0\geq 2a^2-a(a^2+1)\to$$

$$\to 2\geq 2(a^2+1)-a(a^2+1)\to\frac{2}{a^2+1}\geq 2-a\to\sum\frac{2}{a^2+1}\geq 6-\sum a\to$$

$$\to\sum\frac{2}{a^2+1}\geq 6-3\to\sum\frac{1}{a^2+1}\geq\frac{3}{2}$$

435. If $a,b,c,d>0$ then:

$$\frac{a^7}{a^3+bcd}+\frac{b^7}{b^3+cda}+\frac{c^7}{c^3+dab}+\frac{d^7}{d^3+abc}\geq 2abcd$$

Proposed by Daniel Sitaru – Romania

Solution by Le Van-Ho Chi Minh-Vietnam

Applying Schwarz's inequality: $LHS=\sum\frac{a^8}{a^4+abcd}\geq\frac{(a^4+b^4+c^4+d^4)^2}{a^4+b^4+c^4+d^4+4abcd}$. ***Hence, it is enough to show that:*** $(a^4+b^4+c^4+d^4)^2\geq 2abcd(a^4+b^4+c^4+d^4)+8(abcd)^2$

Indeed, by AM-GM inequality: $\begin{cases} 2abcd \le \frac{a^4+b^4+c^4+d^4}{2} \\ 8(abcd)^2 \le 8\left(\frac{a^4+b^4+c^4+d^4}{4}\right)^2 = \frac{(a^4+b^4+c^4+d^4)^2}{2} \end{cases}$

Q.E.D. Equality holds when $a = b = c$.

436. If $x, y, z, t > 0$ then:

$$\frac{1}{\sqrt[3]{xyzt}}\left(\frac{x^2}{x+1}+\frac{y^2}{y+1}+\frac{z^2}{z+1}+\frac{t^2}{t+1}\right) \ge \frac{\sqrt[3]{x^2}}{x+1}+\frac{\sqrt[3]{y^2}}{y+1}+\frac{\sqrt[3]{z^2}}{z+1}+\frac{\sqrt[3]{t^2}}{t+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$LHS = \frac{1}{\sqrt[3]{xyzt}}\left(\frac{x^2}{x+1}+\frac{y^2}{y+1}+\frac{z^2}{z+1}+\frac{t^2}{t+1}\right)$$

$$LHS = \left(\frac{\sqrt[3]{x^2}}{x+1}\right)\left(\frac{x}{\sqrt[3]{yzt}}\right)+\left(\frac{\sqrt[3]{y^2}}{y+1}\right)\left(\frac{y}{\sqrt[3]{xzt}}\right)+\left(\frac{\sqrt[3]{z^2}}{z+1}\right)\left(\frac{z}{\sqrt[3]{xyt}}\right)+\left(\frac{\sqrt[3]{t^2}}{t+1}\right)\left(\frac{t}{\sqrt[3]{xyz}}\right)$$

Now, using Chebyshev's inequality {assume $x \ge y \ge z \ge t > 0$}

$$\left(\frac{LHS}{4}\right) \ge \sum_{cyclic}\frac{\left(\frac{\sqrt[3]{x^2}}{x+1}\right)}{4}\sum_{cyclic}\frac{\left(\frac{x}{\sqrt[3]{yzt}}\right)}{4}$$

AM – GM: $\frac{x}{\sqrt[3]{yzt}}+\frac{y}{\sqrt[3]{xzt}}+\frac{z}{\sqrt[3]{xyt}}+\frac{t}{\sqrt[3]{xyz}} \ge 4\sqrt[4]{\frac{xyz}{xyz}} \ge 4 \Rightarrow \left(\frac{LHS}{4}\right) \ge \sum_{cyclic}\frac{\left(\frac{\sqrt[3]{x^2}}{x+1}\right)}{4} \times \left(\frac{4}{4}\right) \Rightarrow$

$\Rightarrow LHS \ge \sum_{cyclic}\left(\frac{\sqrt[3]{x^2}}{x+1}\right)$ *(proved)*

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow \sum\frac{x^2}{x+1} \ge \sqrt[3]{xyzt}\left(\sum\frac{\sqrt[3]{x^2}}{x+1}\right) \Leftrightarrow \sum\frac{x^2}{x+1} \overset{(1)}{\ge} \sum\left(\left(\frac{x}{x+1}\right)\sqrt[3]{yzt}\right)$*. WLOG, we may assume* $x \ge y \ge z \ge t$; $\frac{x}{x+1} \ge \frac{y}{y+1} \Leftrightarrow xy + x \ge xy + y \Leftrightarrow x \ge y \rightarrow$ *true.*

$\therefore \frac{x}{x+1} \geq \frac{y}{y+1}$. *Similarly,*

$$\frac{y}{y+1} \geq \frac{z}{z+1} \,\&\, \frac{z}{z+1} \geq \frac{t}{t+1} \therefore \frac{x}{x+1} \geq \frac{y}{y+1} \geq \frac{z}{z+1} \geq \frac{t}{t+1} \therefore \sum \frac{x^2}{x+1} \underset{(a)}{\overset{Chebyshev}{\geq}} \frac{1}{4}\left(\sum x\right)\left(\sum \frac{x}{x+1}\right)$$

Also, $x \geq y \geq z \geq t \Rightarrow \sqrt[3]{yzt} \leq \sqrt[3]{ztx} \leq \sqrt[3]{txy} \leq \sqrt[3]{xyz}$

$$\therefore \sum\left(\left(\frac{x}{x+1}\right)\sqrt[3]{yzt}\right) \underset{(b)}{\overset{Chebyshev}{\leq}} \frac{1}{4}\left(\sum \frac{x}{x+1}\right)\left(\sqrt[3]{yzt}\right)$$

$$\overset{GM \leq AM}{\leq} \frac{1}{4}\left(\sum \frac{x}{x+1}\right)\left(\frac{y+z+t}{3} + \frac{z+t+x}{3} + \frac{t+x+y}{3} + \frac{x+y+z}{3}\right) =$$

$= \frac{1}{4}\left(\sum \frac{x}{x+1}\right)\left(\sum x\right)$; ***(a), (b)*** $\Rightarrow$ ***LHS of (1)*** $\geq \frac{1}{4}\left(\sum \frac{x}{x+1}\right)\left(\sum x\right) \geq$ ***RHS of (1)*** $\Rightarrow$ ***(1) is true***

(proved)

437. If $0 \leq a, b, c \leq 3$ then:

$$1 \leq \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + 2^{abc} \leq 8^9 + \frac{9}{10}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michail Stergioiu-Greece

$$1 \leq \sum_{cyc} \frac{a}{1+bc} + 2^{abc} < 8^9 + \frac{9}{10} \quad (1)$$

As $abc \geq 0 \to 2^{abc} \geq 1$ *and* $2^{abc} + \sum \frac{a}{1+bc} \geq 1$. *We can write (1):*

$$\sum_{cyc} \frac{a}{1+bc} < 2^{27} - 2^r + \frac{9}{10} \quad (2), \text{ where } r = abc$$

We also denote $p = \sum_{cyc} a \leq 9$ *and* $q = \sum_{cyc} ab \leq 27$

$$(2) \to -\sum_{cyc} \frac{a}{1+bc} \geq 2^r - 2^{27} - \frac{9}{10} \to p - \sum_{cyc} \frac{a}{1+bc} \geq 2^r - 2^{27} + p - \frac{9}{10} \to$$

$$\to r \cdot \sum \frac{1}{1+bc} \geq 2^r - 2^{27} + p - \frac{9}{10} \overset{BCS}{\to} r \cdot \frac{9}{q+3} \geq 2^r - 2^{27} + p - \frac{9}{10} \quad (3)$$

It suffices to show the stronger inequality: $r \cdot \frac{9}{30} \geq 2^r - 2^{27} + 9 - \frac{9}{10}$ *or*

$$3r + 10\left(2^{27} - 2^r\right) - 81 \geq 0 \quad (4)$$

of course, $3r^{\frac{1}{3}} \leq p \leq 9 \to r \leq 27$.

Consider the function $f(r) = 3r + 10\left(2^{27} - 2^r\right) - 81$

$f'(r) = 3 - 5\cdot 2^{r+1}\ln 2 < 0$ *(*) so,* $f(r)\downarrow$ *and* $f(r)\ge f(27)$ *for* $r\le 27$.

But $f(27) = 0$ *so we are done. (*) if* $r = 0$ *then* $3 < 5\cdot 2^1\cdot\ln 2$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$1 \overset{(A)}{\le} \frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}+2^{abc} \overset{(B)}{\le} 8^9+\frac{9}{10}$$

In order to prove (B), we shall first prove: $\frac{a}{1+bc}+\frac{2^{abc}}{3}\overset{(i)}{\le}\frac{8^9}{3}+\frac{3}{10} \because a\le 3\ \&\ bc\ge 0$

$\therefore (a-3)bc\le 0 \Rightarrow (abc-3bc)\ln 2\le 0\Rightarrow abc\ln 2\le 3bc\ln 2\Rightarrow \ln 2^{abc}\le \ln 2^{3bc}\Rightarrow$

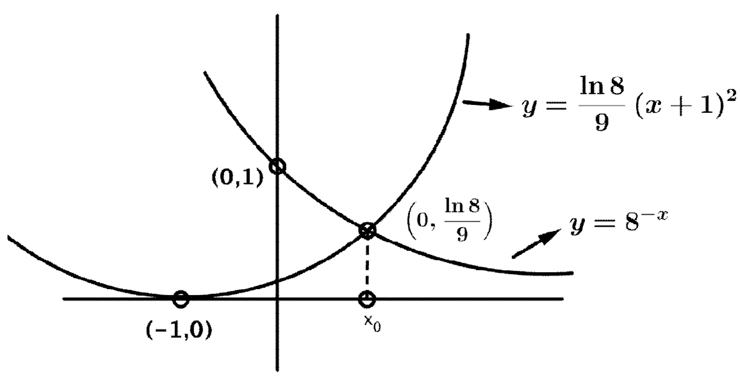
$\Rightarrow 2^{abc}\le 2^{3bc}$*. Also,* $a\le 3\Rightarrow \frac{a}{1+bc}\le\frac{3}{1+bc}$ $(\because 1+bc>0)$

$\therefore \frac{a}{1+bc}+\frac{2^{abc}}{3}\overset{(a)}{\le}\frac{3}{1+bc}+\frac{2^{3bc}}{3}=\frac{3}{1+bc}+\frac{8^{bc}}{3}$*. Let* $f(x)=\frac{3}{1+x}+\frac{8^x}{3}\ \forall x\in[0,9]$

$$f'(x)=\frac{(\ln 8)8^x}{3}-\frac{3}{(x+1)^2}\ \&\ f''(x)=\frac{(\ln^2 8)8^x}{3}+\frac{6}{(x+1)^3}>0$$

$\therefore f'(x)$ *is an increasing* f^n *in* $[0,9]$*. Now,* $f'(0)=\frac{\ln 8}{3}-3<0\ \&$

$$\&\ f'(x)=0\Leftrightarrow \frac{\ln 8}{9}(x+1)^2=8^{-x}$$



We see that $\frac{\ln 8}{9}(x+1)^2\ \&\ 8^{-x}$ *intersect at only one point* $x_0>0\ \&\ \forall x\in[0,x_0)$,

$8^{-x}>\frac{\ln 8}{9}(x+1)^2\ \&\ \forall x\in(x_0,9], \frac{\ln 8}{9}(x+1)^2>8^{-x}$*. So,* $\because \frac{\ln 8}{9}(x+1)^2\Big|_{x=1}>8^{-x}\big|_{x=1}$,

$\therefore 1>x_0\Rightarrow x_0\in(0,1)$ ***(1)***

$\because f'(x)=0$ *for some* $x_0\in(0,1)\because f'(x)$ *is an increasing* f' *in* $[0,9]\therefore\forall x\ge x_0$,

$f'(x)\ge f'(x_0)=0\ \therefore\forall x\in[0,x_0), f'(x)<0\ \&\ \forall x\in[x_0,9], f'(x)\ge 0\Rightarrow$

$\Rightarrow\forall x\in[0,x_0], f(x)$ *is a decreasing* $f^n\ \&\ \forall x\in[x_0,9], f(x)$ *is an increasing* f^n ***(2)***

$\because f(0) = \frac{10}{3} \& f(9) = \frac{3}{10} + \frac{8^9}{3} \therefore f(9) > f(0) \to$ ***(3)***

Combining (1), (2), (3) $\forall x \in [0,9], f_{\max} = f(9) = \frac{8^9}{3} + \frac{3}{10} \therefore x \in [0,9],$

$\frac{3}{1+x} + \frac{8^x}{3} \le \frac{8^9}{3} + \frac{3}{10}$. ***Putting*** $x = bc (\& bc \le 9), \frac{3}{1+bc} + \frac{8^{bc}}{3} \overset{(b)}{\le} \frac{8^9}{3} + \frac{3}{10}$

(a), (b) $\Rightarrow \frac{a}{1+bc} + \frac{2^{abc}}{3} \le \frac{8^9}{3} + \frac{3}{10} \Rightarrow$ ***(i) is true. Similarly,*** $\frac{b}{1+ca} + \frac{2^{abc}}{3} \overset{(ii)}{\le} \frac{8^9}{3} + \frac{3}{10} \&$

$$\frac{c}{1+ab} + \frac{2^{abc}}{3} \overset{(iii)}{\le} \frac{8^9}{3} + \frac{3}{10}$$

(i) + (iii) + (iii) $\Rightarrow$ ***(B) is true. Also,*** $\because a, b, c \ge 0, \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + 2^{abc} \ge$

$\ge 0 + 0 + 0 + 2^0 = 1 \Rightarrow$ ***(A) is true (proved)***

438. If $x, y, z > 0$ then:

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)} \ge \frac{(xy+1)(yz+1)(zx+1)}{(x^4+1)(y^4+1)(z^4+1)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $a, b > 0$. ***Let*** $z_1 = a + i, z_2 = b + i$. ***Now,*** $(a^2+1)(b^2+1) = |a+1|^2|b+i|^2 =$

$= |(a+i)(b+i)|^2 = |(ab-1) + i(a+b)|^2 = (ab-1)^2 + (a+b)^2 \ge$

$\ge (ab-1)^2 + \left(2\sqrt{ab}\right)^2 = (ab-1)^2 + 4ab = (ab+1) \Rightarrow$

$$(a^2+1)(b^2+1) \ge (ab+1)^2$$

$\therefore$ ***for*** $x, y, z > 0; (x^2+1)(y^2+1) \ge (xy+1)^2; (y^2+1)(z^2+1) \ge (yz+1)^2$

$(z^2+1)(x^2+1) \ge (zx+1)^2$. ***Multiplying above inequalities, we get***

$$[(x^2+1)(y^2+1)(z^2+1)]^2 \ge [(xy+1)(yz+1)(zx+1)]^2 \Rightarrow$$

$$\Rightarrow (x^2+1)(y^2+1)(z^2+1) \ge (xy+1)(yz+1)(zx+1) \Rightarrow$$

$$\Rightarrow (x^4+1)(y^4+1)(z^4+1) \ge (x^2y^2+1)(y^2z^2+1)(z^2x^2+1)$$

Multiplying above two inequalities, we get

$$(x^2+1)(y^2+1)(z^2+1)(x^4+1)(y^4+1)(z^4+1) \ge$$

$$\ge (xy+1)(yz+1)(zx+1)(x^2y^2+1)(y^2z^2+1)(z^2x^2+1) \Rightarrow$$

$$\Rightarrow \frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)} \ge \frac{(xy+1)(yz+1)(zx+1)}{(x^4+1)(y^4+1)(z^4+1)}$$

Solution 2 by Serban George Florin-Romania

$$(xy+1)^2 \overset{(C.B.S.)}{\le} (x^2+1)(y^2+1) \Rightarrow \prod (xy+1)^2 \le \prod (x^2+1)^2 \Rightarrow$$

$$\Rightarrow \prod(xy+1) \le \prod(x^2+1)\,;\,(x^2y^2+1)^2 \le (x^4+1)(y^4+1)\ \textbf{(C.B.S.)}\Rightarrow$$

$$\prod (x^2y^2+1)^2 \le \prod (x^4+1)^2 \Rightarrow \prod (x^2y^2+1) \le \prod (x^4+1) \Rightarrow$$

$$\Rightarrow \prod(xy+1)\cdot\prod(x^2y^2+1) \le \prod(x^2+1)\prod(x^4+1) \Rightarrow \frac{\prod(x^2+1)}{\prod(x^2y^2+1)} \ge \frac{\prod(xy+1)}{\prod(x^4+1)}\ \textit{True.}$$

439. If $a, b, c \in \mathbb{R}$ then:

$$(a-b)^2(b-c)^2(c-a)^2 \le 3(a^2+b^2+c^2)(a^4+b^4+c^4)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$3\sum a^6 + 2\sum a^4b^2 + 2\sum a^2b^4 + 2abc\left(\sum a^3\right) + 2\sum a^3b^3 + 6a^2b^2c^2 \overset{(1)}{\ge}$$

$\ge 2abc(\sum a^2b + \sum ab^2)$. ***Now,*** $\sum a^6 + 2\sum a^3b^3 = (\sum a^3)^2 \overset{(a)}{\ge} 0$. ***Also,***

$$(a^4b^2 + a^4c^2 + 2a^4bc) + (b^4c^2 + b^4c^2 + 2b^4ac) + (c^4a^2 + c^4b^2 + 2c^4ab) =$$

$$= (a^2b + a^2c)^2 + (b^2c + b^2a)^2 + (c^2a + c^2b)^2 \ge 0 \Rightarrow \sum a^4b^2 + \sum a^2b^4 +$$

$+2abc(\sum a^3) \overset{(b)}{\ge} 0$. ***Again,*** $\because a^2, b^2, c^2 \ge 0$, $\therefore$ ***applying Schur,*** $\sum(a^2)^3 + 3a^2b^2c^2 \ge$

$$\ge \sum a^4b^2 + \sum a^2b^4 = (a^4b^2 + b^4c^2 + c^4a^2) + (a^2b^4 + b^2c^4 + c^2a^4) \ge$$

$$\ge (a^2bb^2c + b^2c\cdot c^2a + c^2aa^2b) + (ab^2bc^2 + bc^2ca^2 + ca^2ab^2)$$

$$\left(\because \forall x, y, z \in \mathbb{R}, \sum x^2 \ge \sum xy\ as\ \sum x^2 - \sum xy = \frac{1}{2}\sum(x-y)^2 \ge 0\right)$$

$= abc(\sum a^2b + \sum ab^2) \Rightarrow 2\sum a^6 + 6a^2b^2c^2 \overset{(c)}{\ge} 2abc(\sum a^2b + \sum ab^2)$. ***Moreover,***

$\sum a^4b^2 + \sum a^2b^2 \overset{(d)}{\ge} 0$; ***(a)+(b) +(c) +(d)*** $\Rightarrow$ ***(1) is true (proved)***

440. If $x, y, z > 0$ then:

$$\frac{x}{3}\cdot\left(\frac{8}{3y+5z}\right)^7+\frac{y}{3}\left(\frac{8}{3z+5x}\right)^7+\frac{z}{3}\cdot\left(\frac{8}{3x+5y}\right)^7\ge\left(\frac{3}{x+y+z}\right)^6$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Inequality becomes: $\frac{x}{x+y+z}\left(\frac{8}{3y+5z}\right)^7+\frac{y}{x+y+z}\left(\frac{8}{3z+5x}\right)^7+\frac{z}{x+y+z}\left(\frac{8}{3x+5y}\right)\ge\frac{3^7}{(x+y+z)^7}$ *(1)*

But $f:(0,+\infty)\to\mathbb{R}; f(x)=x^7$ *is a convex function. From Jensen's inequality (general form)* $\Rightarrow p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\ge f(p_1x_1+p_2x_2+p_3x_3), p_1+p_2+p_3=1\Rightarrow$

$$\Rightarrow\frac{x}{x+y+z}\left(\frac{8}{3y+5z}\right)^7+\frac{y}{x+y+z}\left(\frac{8}{3z+5x}\right)^7+\frac{z}{x+y+z}\left(\frac{8}{3x+5y}\right)\ge$$

$$\ge\left(\frac{x}{x+y+z}\cdot\frac{8}{3y+5z}+\frac{y}{x+y+z}\cdot\frac{8}{3z+5x}+\frac{z}{x+y+z}\cdot\frac{8}{3x+5y}\right)^7\quad(2)$$

From (1) + (2) we must show: $\left(\frac{8x}{3y+5z}+\frac{8y}{3z+5x}+\frac{8z}{3x+5y}\right)^7\ge 3^7\Leftrightarrow$

$$\Leftrightarrow\frac{x}{3y+5z}+\frac{y}{3z+5x}+\frac{z}{3x+5y}\ge\frac{3}{8}\quad(3)$$

But from Cauchy's inequality we have: $\frac{x}{3y+5z}+\frac{y}{3z+5x}+\frac{z}{3x+5y}=\frac{x^2}{3xy+5xz}+\frac{y^2}{3yz+5xy}+$

$$+\frac{z^2}{3xz+5yz}\ge\frac{(x+y+z)^2}{8(xy+xz+yz)}\quad(4)$$

From (3)+(4) we must show: $\frac{(x+y+z)^2}{8(xy+xz+yz)}\ge\frac{3}{8}\Leftrightarrow(x+y+z)^2\ge 3(xy+xz+yz)\Leftrightarrow$

$\Leftrightarrow x^2+y^2+z^2\ge xy+xz+yz$ *which its true.*

441. If $a, b, c > 0, a+b+c=3$ then:

$$\sum(a+b-c)^3\cdot\sum(a+b-c)^5\ge 9abc$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

Using Power mean AM of m^th power ≥ m^th power of AM

$$\Rightarrow \frac{a_1^m+a_2^m+\cdots+a_n^m}{n} \ge \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m; \forall m \in \mathbb{R}\setminus(0,1)$$

$$\Rightarrow \frac{(a+b-c)^3+(b+c-a)^3+(c+a-b)^3}{3} \ge \left(\frac{a+b+c}{3}\right)^3 \Rightarrow$$

$$\Rightarrow \sum(a+b-c)^3 \ge 3\left(\frac{a+b+c}{3}\right)^3 \overset{AM-GM}{\ge} 3abc \quad \textbf{(1)}$$

Again, using power mean, $\frac{(a+b-c)^5+(b+c-a)^5+(c+a-b)^5}{3} \ge \left(\frac{a+b+c}{3}\right)^5 \Rightarrow$

$$\Rightarrow \sum(a+b-c)^5 \ge 3\left(\frac{a+b+c}{3}\right)^5; \sum(a+b-c)^5 \ge 3 \quad \textbf{(2)} \{\because a+b+c=3\}$$

Multiplying (1) & (2): $\sum(a+b-c)^3 \cdot \sum(a+b-c)^5 \ge 9abc$

442. If $1 \le a \le x, 1 \le b \le y, 1 \le c \le z$ then:

$$\frac{\sqrt{2}(a+x)(b+y)(c+z)}{(a+1)(b+1)(c+1)} \le \sqrt{(abc)^2+(xyz)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

RHS $\ge \frac{1}{\sqrt{2}}(abc+xyz) \overset{?}{\ge} \frac{\sqrt{2}(a+x)(b+y)(c+z)}{(a+1)(b+1)(c+1)} \Leftrightarrow (abc+xyz)(a+1)(b+1)(c+1) \ge$

$\underset{(1)}{\overset{?}{\ge}} 2(a+x)(b+y)(c+z) \because a,b,c \ge 1$, ***we can let*** $a = 1+m, b = 1+n, c = 1+p$

$(m,n,p \ge 0)$ ***&*** $\because x \ge a, y \ge b, z \ge c$, ***hence, we can let*** $x = a+u, y = b+v, z = c+w$

$$(u,v,w \ge 0) \therefore x = 1+m+u, y = 1+n+v, z = 1+p+w$$

$$\therefore \textbf{(1)} \Leftrightarrow \{(1+m)(1+n)(1+p)+(1+m+u)(1+n+v)(1+p+w)\}$$

$$(2+m)(2+n)(2+p) \ge 2(2+2m+u)(2+2n+v)(2+2p+w) \Leftrightarrow$$

$$\Leftrightarrow 2m^2n^2p^2 + m^2n^2pw + m^2np^2v + m^2npvw + mn^2p^2u + mn^2puw + mnp^2uv +$$

$$+mnpuvw + 6m^2n^2p + 2m^2n^2w + 6m^2np^2 + 3m^2npv + 3m^2npw + 2m^2nvw +$$

$$+2n^2p^2v + 2m^2pvw + 6mn^2p^2 + 3mn^2pu + 3mn^2pw + 2mn^2uw +$$

$$+3mnp^2u + 3mnp^2v + 3mnpuw + 3mnpvw +$$

$$+2mnuvw + 2mp^2uv + 2npuvw + 2n^2p^2u + 2n^2puw + 2np^2uv + 2npuvw +$$

$$+4m^2n^2 + 18m^2np + 2m^2nv + 6m^2nw + 4m^2p^2 + 6m^2pv + 2m^2pw + 4m^2vw +$$

$$+18mn^2p+2mn^2u+6mn^2w+18mnp^2+9mnpu+$$
$$9mnpv+9mnpw+2mnuv+$$
$$+6mnuw+4muvw+4n^2p^2+6n^2pu+2n^2pw+4n^2uw+$$
$$6np^2u+2np^2v+6npuv+$$
$$+6npuw+2npvw+4nuvw+4p^2uv+4puvw+$$
$$12m^2n+12m^2p+4m^2v+4m^2w+$$
$$+12mn^2+38mnp+6mnu+6mnv+10mnw+12mp^2+$$
$$6mpu+10mpv+6mpw+$$
$$+4muv+8mvw+12n^2p+4n^2u+4n^2w+12np^2+10npu+6npv+6npw+$$
$$+4nuv+8nuw+4nvw+4p^2u+4p^2v+8puv+4puw+4pvw+6uvw+8m^2+$$
$$+20mn+20mp+4mu+4mv+4mw+8n^2+20np+4nu+4nv+4nw+8p^2+$$
$$+4pu+4pv+4pw+4uv+4uw+4vw+8m+8n+8p\ge 0\rightarrow \textit{true}$$
$$\because m,n,p,u,v,w\ge 0\quad \textit{(proved)}$$

443. If $a,b,c,d>0, a+b+c+d=1$ then:

$$2^{16}abcd(1-a)(1-b)(1-c)(1-d)\le 81$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

Let $a=\frac{x}{x+y+z+t}, b=\frac{y}{x+y+z+t}, c=\frac{z}{x+y+z+t}, d=\frac{t}{x+y+z+t}$. *We must show:*

$$2^{16}xyzt(y+z+t)(x+z+t)(x+y+t)(x+y+z)\le 81(x+y+z+t)^8\quad \textit{(1)}$$

But $\sqrt[4]{xyzt}\le\frac{x+y+z+t}{4}\Leftrightarrow 2^8xyzt\le(x+y+z+t)^4$ *(2). From (1)+(2) we must show:*

$$2^8(y+z+t)(x+z+t)(x+y+t)(x+y+z)\le 81(x+y+z+t)^4\Leftrightarrow$$
$$\sqrt[4]{(y+z+t)(x+z+t)(x+y+t)(x+y+z)}\le\frac{3}{4}(x+y+z+t)\quad \textit{(3)}$$

But $\sqrt[4]{(y+z+t)(x+z+t)(x+y+z)(x+y+t)}\le\frac{3x+3y+3t+4z}{4}\Rightarrow$ *(3) its true.*

Solution 2 by Amit Dutta-Jamshedpur-India

We need to prove: $abcd(1-a)(1-b)(1-c)(1-d)\le\frac{81}{2^{16}}$. *Let*

$P = abcd(1-a)(1-b)(1-c)(1-d)$ **GM≤AM** $\Rightarrow (abcd)^{\frac{1}{4}} \le \left(\frac{a+b+c+d}{4}\right) \le \frac{1}{4}$

$$\{\because a+b+c+d=1\} \Rightarrow abcd \le \left(\frac{1}{4}\right)^4 \quad \textbf{(1)}$$

Again $GM \le AM$: $[(1-a)(1-b)(1-c)(1-d)]^{\frac{1}{4}} \le \left[\frac{4(a+b+c+d)}{4}\right] \le \left(\frac{4-1}{4}\right) \le \frac{3}{4}$

$$\Rightarrow (1-a)(1-b)(1-c)(1-d) \le \left(\frac{3}{4}\right)^3 \quad \textbf{(2)}$$

Multiplying (1) & (2), we have: $abcd(1-a)(1-b)(1-c)(1-d) \le \frac{3^4}{4^4 \cdot 4^4} \le \left(\frac{81}{2^{16}}\right)$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\because a,b,c,d > 0 \;\&\; \sum a = 1, \therefore 0 < a,b,c,d < 1 \Rightarrow (1-a) \textbf{ etc} > 0$$

$\sqrt[4]{abcd} \overset{G-A}{\le} \frac{\sum a}{4} = \frac{1}{4} \Rightarrow abcd \overset{(1)}{\le} \frac{1}{2^8}$. ***Also,*** $\sqrt[4]{(1-a)(1-b)(1-c)(1-d)} \le \frac{4-\sum a}{4} = \frac{3}{4}$

$$\Rightarrow (1-a)(1-b)(1-c)(1-d) \overset{(2)}{\le} \frac{81}{2^8}$$

$$\textbf{(1).(2)} \Rightarrow LHS \le \frac{2^{16} \cdot 81}{2^8 \cdot 2^8} = 81 \quad \textbf{(proved)}$$

444. Let $x,y,z \in (0;+\infty) \wedge xyz = 1$ and $\theta \ge 1$. Prove:

$$\frac{1}{\left(2\sqrt{x}+xy\right)^\theta} + \frac{1}{\left(2\sqrt{y}+yz\right)^\theta} + \frac{1}{\left(2\sqrt{z}+zx\right)^\theta} \ge 3^{1-\theta}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Marian Ursărescu-Romania

Because $2\sqrt{x} \le x+1 \Rightarrow$ ***we must show:*** $\frac{1}{(x+y+1)^\theta} + \frac{1}{(y+yz+1)^\theta} + \frac{1}{(z+zx+1)^\theta} \ge \frac{1}{3^{\theta-1}} \quad \textbf{(1)}$

Because $xyz = 1 \Rightarrow \exists a,b,c > 0$ ***such that*** $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a} \quad \textbf{(2)}$

From (1)+(2) we must show: $\frac{1}{\left(\frac{a}{b}+\frac{a}{b}\cdot\frac{b}{c}+1\right)^\theta} + \frac{1}{\left(\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+1\right)^\theta} + \frac{1}{\left(\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}+1\right)^\theta} \ge \frac{1}{3^{\theta-1}} \Leftrightarrow$

$$\frac{(bc)^\theta}{(ac+ab+bc)^\theta} + \frac{(ac)^\theta}{(ab+bc+ac)^\theta} + \frac{(ab)^\theta}{(bc+ac+ab)^\theta} \ge \frac{1}{3^{\theta-1}} \Leftrightarrow$$

$$\Leftrightarrow (bc)^\theta + (ac)^\theta + (ab)^\theta \ge \frac{(ab+bc+ac)^\theta}{3^{\theta-1}} \quad \textbf{(3)}$$

Let $ab = m, bc = n, ac = p, m, n, p > 0$ ***(4)***

From (3)+(4) we must show: $m^\theta + n^\theta + p^\theta \ge \frac{(m+n+p)^\theta}{3^{\theta-1}}$***, which it's true, because its Hölder's inequality (generalization).***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(2\sqrt{x} + xy)^\theta \overset{1=(xyz)^2}{=} \left(2\sqrt{x \cdot x^2y^2z^2} + xy\right)^\theta = \left(2xy\sqrt{xz \cdot z} + xy\right)^\theta \overset{G\le A}{\le}$$

$$\le (xy(xz + z) + xy)^\theta = (xy)^\theta(1 + z + zx)^\theta \Rightarrow \frac{1}{(2\sqrt{x} + xy)^\theta} \overset{(1)}{\ge} \frac{1}{(xy)^\theta} \cdot \frac{1}{(1 + z + zx)^\theta}$$

Again, $(2\sqrt{y} + yz)^\theta \overset{1=(xyz)^2}{=} \left(2\sqrt{y \cdot x^2y^2z^2} + yz\right)^\theta = \left(2yz\sqrt{xy \cdot x} + yz\right)^\theta \overset{G\le A}{\le}$

$$\le (yz(xy + z) + yz)^\theta = (yz)^\theta(1 + x + xy)^\theta = (yz)^\theta(xyz + x + xy)^\theta \ (\because 1 = xyz)$$

$$= (xyz)^\theta(1 + y + yz)^\theta = (xyz + y + yz)^\theta (\because 1 = xyz)$$

$$= y^\theta(1 + z + zx)^\theta \Rightarrow \frac{1}{(2\sqrt{y} + yz)^\theta} \overset{(2)}{\le} \frac{1}{y^\theta} \cdot \frac{1}{(1 + z + zx)^\theta}$$

Also, $(2\sqrt{z} + zx)^\theta \overset{1=(xyz)^2}{=} \left(2\sqrt{z \cdot x^2y^2z^2} + zx\right)^\theta = \left(2zx\sqrt{yz \cdot y} + zx\right)^\theta \le$

$$\overset{G\le A}{\le} (zx(yz + y) + zx)^\theta = (zx)^\theta(1 + y + yz)^\theta = (zx)^\theta(xyz + y + yz)^\theta \ (\because 1 = xyz)$$

$$= (zxy)^\theta(1 + z + zx)^\theta \Rightarrow \frac{1}{(2\sqrt{z} + zx)^\theta} \overset{(3)}{\ge} \frac{1}{(1 + z + zx)^\theta}$$

(1)+(2)+(3)⇒ LHS $\ge \frac{1}{(1+z+zx)^\theta}\left[\left(\frac{1}{xy}\right)^\theta + \left(\frac{1}{y}\right)^\theta + 1\right]$

$$= \frac{1 + z^\theta + (zx)^\theta}{(1 + z + zx)^\theta} \overset{Jensen}{\ge} 3\left(\frac{1 + z + zx}{3}\right)^\theta \left(\frac{1}{1 + z + zx}\right)^\theta = 3^{1-\theta}$$

($\because f(t) = t^\theta \ (t > 0)$ ***is convex as*** $f''(t) = \theta(\theta - 1)t^{\theta-2} \ge 0 \because \theta \ge 1$***) (proved)***

445. **If** $a, b, c \in \mathbb{N}, a + b + c = 8$ **then:**

$$\frac{81}{(a + 1)(b + 1)(c + 1)} > \frac{1}{\sqrt[4]{27}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sagar Kumar-Kolkata-India

$$a+b+c=8$$

$$\frac{81}{(a+1)(b+1)(c+1)}>\frac{1}{(27)^{\frac{1}{4}}}\Rightarrow(a+1)(b+1)(c+1)<3^4\cdot(3)^{\frac{3}{4}}<3^{\frac{19}{4}}<3^5$$

AM ≥ GM

$$\frac{(a+b+c)+3}{3}\ge\left((a+1)(b+1)(c+1)\right)^{\frac{1}{3}}\Rightarrow(a+1)(b+1)(c+1)\le\left(\frac{11}{3}\right)^3<(3)^{\frac{19}{4}}$$

Hence proved

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because a+b+c=8\therefore(a+1)+(b+1)+(c+1)=11$$

$$\therefore\sqrt[3]{(a+1)(b+1)(c+1)}\overset{G-A}{\le}\frac{(a+1)+(b+1)+(c+1)}{3}=\frac{11}{3}\Rightarrow$$

$$\Rightarrow(a+1)(b+1)(c+1)\le\frac{11^3}{27}\Rightarrow\frac{81}{(a+1)(b+1)(c+1)}\ge\frac{81\cdot27}{11^3}$$

$$\approx 1.643 \textit{ and} \because\frac{1}{\sqrt[4]{27}}\approx0.439\therefore\frac{81}{(a+1)(b+1)(c+1)}>\frac{1}{\sqrt[4]{27}}\quad\textit{(Done)}$$

Solution 3 by Abdallah El Farissi-Bechar-Algerie

$$\sqrt[4]{(1+a)(1+b)(1+c)1}\overset{AM-GM}{\le}\frac{1+a+1+b+1+c+1}{4}=3\textit{ then}\frac{81}{(1+a)(1+b)(1+c)}\ge1>\frac{1}{\sqrt[4]{27}}$$

446. **If $a,b,c\in\mathbb{R},a^2+b^2+c^2=3$ then:**

$$2(a^4+b^4+c^4)+12\ge3(a^3+b^3+c^3+a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$2(a^4+b^4+c^4)+12\ge3(a^3+b^3+c^3+a+b+c)\Leftrightarrow$$

$$\Leftrightarrow2(a^4+b^4+c^4)+4(a^2+b^2+c^2)\ge3(a^3+b^3+c^3+a+b+c)\Leftrightarrow$$

$$\Leftrightarrow(2a^4-3a^3+4a^2-3a)+(2b^4-3b^3+4b^2-3b)+(2c^4-3c^3+4c^2-3c)\ge0$$

$$\textit{or } 2\sum a^4-3\sum a^3+4\sum a^2-3\sum a\ge0\quad\textit{(1)}$$

$$\textit{How is } 2a^4-3a^3+4a^2-3a\ge2a^2-2\Leftrightarrow2a^4-3a^3+2a^2-3a+2\ge0\Leftrightarrow$$

$$\Leftrightarrow(a-1)^2(2a^2+a+2)\ge0\textit{ true }\forall a\in\mathbb{R}$$

$$2a^4-2a^3+4a^2-3a\ge 2a^2-2\quad (2)$$

$$2b^4-3b^3+4b^2-3b\ge 2b^2-2\quad (3)$$

$$2c^4-3c^3+4c^2-3c\ge 2c^2-2\quad (4)$$

(2)+(3)+(4)$\Rightarrow 2\sum a^4-3\sum a^3+4\sum a^2-3\sum a\ge 2\sum a^2-6=2\cdot 3-6=0\Rightarrow$ *(1)*

true

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$2(a^4+b^4+c^4)+12=2(a^4+b^4+c^4)+2(a^2+b^2+c^2)+b=$$

$$=(a^4+a^4+a^2+a^2)+(b^4+b^4+b^2+b^2)+(c^4+c^4+c^2+c^2)+6\ge$$

$$\ge 4(a^3+b^3+c^3)+6=3(a^3+b^3+c^3)+(a^3+1+1)+(b^3+1+1)+(c^3+1+1)$$

$$\ge 3(a^3+b^3+c^3)+3(a+b+c)=3(a^3+b^3+c^3+a+b+c)$$

Therefore it is to be true.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$3\left(\sum a^3+\sum a\right)=3\sum\{a(1+a^2)\}\overset{CBS}{\le}3\sqrt{\sum a^2}\sqrt{\sum(1+a^2)^2}\overset{\sum a^2=3}{=}$$

$$=3\sqrt{3}\sqrt{\sum(1+a^4+2a^2)}\overset{\sum a^2=3}{=}3\sqrt{3}\sqrt{3+\sum a^4+6}=3\sqrt{3}\sqrt{9+\sum a^4}$$

$$\overset{?}{\le}2\sum a^4+12\Leftrightarrow(2t+12)^2\overset{?}{\ge}27(9+t)\ \left(t=\sum a^4\right)\Leftrightarrow$$

$$\Leftrightarrow 4t^2+144+48t\overset{?}{\ge}243+27t\Leftrightarrow 4t^2+21t-99\overset{?}{\ge}0\Leftrightarrow(t-3)(4t+33)\underset{(1)}{\overset{?}{\ge}}0$$

Now, $t=\sum a^4\ge\frac{1}{3}(\sum a^2)^2=\frac{9}{3}=3\Rightarrow t\ge 3\Rightarrow$ *(1) is true (proved)*

447. **If $a,b,c\ge e$ then:**

$$(ln(ae))(ln(be))(ln(ce))+4\ge 4\,ln(abc)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\ln a=x,\ln b=y,\ln c=z\Rightarrow x,y,z\ge 1\Rightarrow$$

$$\Rightarrow(\ln a+\ln e)(\ln b+\ln e)(\ln c+\ln e)+4\ge 4(\ln a+\ln b+\ln c)$$

$$(x+1)(y+1)(z+1)+4\ge 4x+4y+4z$$

$$x+1=\alpha, y+1=\beta, z+1=\gamma \Rightarrow \alpha,\beta,\gamma \ge 2$$

$$\alpha\beta\gamma+4 \ge 4(\alpha-1)+4(\beta-1)+4(\gamma-1)$$

$$\alpha\beta\gamma+4-4\alpha-4\beta-4\gamma+12 \ge 0;\ \alpha\beta\gamma-4(\alpha+\beta+\gamma)+16 \ge 0$$

$$(\alpha-2)(\beta-2)(\gamma-2)=(\alpha-2)(\beta\gamma-2\beta-2\gamma+4)=\alpha\beta\gamma-2\sum\alpha\beta+4\sum\alpha-8 \Rightarrow$$

$$\Rightarrow \alpha\beta\gamma-4(\alpha+\beta+\gamma)+16=\prod_{\alpha,\beta,\gamma}(\alpha-2)+2\sum\alpha\beta-8\sum\alpha+24=\prod_{\alpha,\beta,\gamma}(\alpha-2)+$$

$$+2\sum\alpha\beta-4\sum\alpha-4\sum\alpha+8+8+8=\prod_{\alpha,\beta,\gamma}(\alpha-2)+2\alpha(\beta-2)+2\beta(\gamma-2)+$$

$$+2\gamma(\alpha-2)-4(\alpha-2)-4(\beta-2)-4(\gamma-2)=\prod_{\alpha,\beta,\gamma}(\alpha-2)+(\alpha-2)(2\gamma-4)+$$

$$+(\beta-2)(2\alpha-4)+(\gamma-2)(2\beta-4)=(\alpha-2)(\beta-2)(\gamma-2)+2(\alpha-2)(\gamma-2)+$$

$$+2(\beta-2)(\alpha-2)+2(\gamma-2)(\beta-2) \ge 0 \quad \textit{true}$$

$$\alpha-2 \ge 0, \beta-2 \ge 0, \gamma-2 \ge 0$$

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaidjan

$$(1+\ln a)(1+\ln b)(1+\ln c)+4 \ge 4(\ln a+\ln b+\ln c) \quad (1)$$

denote $\ln a = x, \ln b = y, \ln c = z, x \ge 1, y \ge 1, z \ge 1$

(1) $\Rightarrow (1+x)(1+y)(1+z)+4-4x-4y-4z \ge 0$

$$(y+z+yz-3)x+yz-3y-3z+5 \ge 0 \quad (2)$$

denote $f(x)=(y+z+yz-3)x+yz-3y-3z+5$***, where*** $y \ge 1, z \ge 1$ ***and*** $x \ge 1$.

This function is linear with respect to x.

$y+z+yz-3 \ge 0$ ***(since*** $y \ge 1, z \ge 1$***)***

if $y+z+yz-3=0$***, then*** $yz-3y-3z+5 \ge 0$ ***(true!)***

if $y+z+yz-3>0$***, then we must prove*** $f(1) \ge 0$.

$$f(1)=y+z+yz-3+yz-3y-3z+5=$$

$=2yz-2y-2z+2=2(y-1)(z-1) \ge 0$ ***(true!) Since*** $y \ge 1, z \ge 1$***. Q.E.D.***

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c \ge e$***, we have*** $ab+1>a+b, ab>1$ ***(*)***

$$4+(\ln ae)(\ln be)(\ln ce)=$$

$$=\ln a\ln b\ln c+\ln a\ln b+\ln b\ln c+\ln c\ln a+\ln a+\ln b+\ln c+1+4$$

$$4\ln(ax) = 4(\ln a + \ln b + \ln c)$$

If $a = b = c = e$

$$(\ln ae)(\ln be)(\ln ce) = 1+1+1+1+1+1+1+5 =$$

$$4(\ln abc) = 4\ln e^3 = 4 \times 3 = 12$$

If $a = b = e, c > e$

$$(\ln ae)(\ln be)(\ln ce) = \ln c + 1 + \ln e + \ln c + 1 + 1 + \ln e + 5 = 4\ln c + 8$$

$$4\ln(abc) = 4(\ln a + \ln b + \ln c) = 4\ln c + 8 \quad \textit{ok}$$

If $a = e, b, c > e$

$$(\ln ae)(\ln be)(\ln ce) = \ln b \ln e + \ln b + \ln b \ln c + \ln c + 1 + \ln b + \ln c + 8$$

$$= 2\ln bc + 2\ln b \ln c + 6$$

$$4\ln abc = 4\ln bc + 4$$

If $a, b, c > e \quad \ln a \ln b \ln c + \ln a \ln b + \ln b \ln c + \ln c \ln a + \ln a + \ln b + \ln c + 1 + 4$

$$> \ln a \ln b + \ln c + \ln a + \ln b + \ln b + \ln c + \ln c + \ln a + \ln a + \ln b + \ln c + 1$$

$$> \ln a + \ln b + \ln c + \ln a + \ln b + \ln b + \ln c + \ln c + \ln a + \ln a + \ln b + \ln e$$

$= 4\ln abc$. *Therefore It is to be true.*

Solution 4 by Soumava Chakraborty-Kolkata-India

$$(\ln(ae))(\ln(be))(\ln(ce)) + 4 \overset{(1)}{\geq} 4\ln(abc)$$

(1) $\Leftrightarrow (x+1)(y+1)(z+1) + 4 \overset{(2)}{\geq} 4(x+y+z)$, *where* $x = \ln a, y = \ln b, z = \ln c$

$$\because a, b, c \geq e \therefore x, y, z \geq 1$$

$\therefore$ *we can consider* $x = 1 + u, y = 1 + v, z = 1 + w \ (u, v, w \geq 0)$

$\therefore$ **(2)** $\Leftrightarrow (2+u)(2+v)(2+w) + 4 \geq 4(3 + u + v + w)$

$\Leftrightarrow 8 + 4\sum u + 2\sum uv + 4 \geq 12 + 4\sum u \Leftrightarrow 2\sum uv \geq 0 \to$ *true* $\because u, v, w \geq 0$ ***(proved)***

448. If $a, b, c > 0$ then:

$$2(a^2 + b^2 + c^2 + a^3 + b^3 + c^3) \leq \sqrt{2} \cdot \sum_{cyc(a,b,c)} \sqrt{a^6 + b^6} + \sqrt[3]{4} \cdot \sum_{cyc(a,b,c)} \sqrt[3]{a^6 + b^6}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Consider $2(a^6 + b^6) - (a^3 + b^3)^2 = a^6 + b^6 - 2a^3b^3 = (a^3 - b^3)^2 \geq 0 \Rightarrow$

$$\Rightarrow \sqrt{2}\sqrt{a^6+b^6} \ge a^3+b^3 \Rightarrow \sqrt{2}\sum_{cyc}\sqrt{a^6+b^6} \ge 2(a^3+b^3+c^3) \quad \textbf{(1)}$$

Also, $4(a^6+b^6)-(a^2+b^2)^3 = 3(a^6+b^6-a^4b^2-a^2b^4) =$

$$= 3(a^4)(a^2-b^2)+3b^4(b^2-a^2) = 3(a^2-b^2)^2(a^2+b^2) \ge 0 \Rightarrow$$

$$\Rightarrow 4^{\frac{1}{3}}(a^6+b^6)^{\frac{1}{3}} \ge a^2+b^2 \Rightarrow 4^{\frac{1}{3}}\sum_{cycl}(a^6+b^6)^{\frac{1}{3}} \ge 2(a^2+b^2+c^2) \quad \textbf{(2)}$$

Adding (1), (2), we get: $2(a^2+b^2+c^2+a^3+b^3+c^3) \le$

$$\le \sqrt{2}\sum_{cycl}\sqrt{a^6+b^6}+4^{\frac{1}{3}}\sum_{cycl}(a^6+b^6)^{\frac{1}{3}}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

for $a,b,c>0$*, we have:* $\sqrt{2}\left(\sqrt{a^6+b^6}+\sqrt{b^6+c^6}+\sqrt{c^6+a^6}\right)+$

$$+\sqrt[3]{4}\left(\sqrt[3]{a^6+b^6}+\sqrt[3]{b^6+c^6}+\sqrt[3]{c^6+a^6}\right) = \sqrt{2(a^6+b^6)}+\sqrt{2(b^6+c^6)}+$$

$$+\sqrt{2(c^6+a^6)}+\sqrt[3]{4(a^6+b^6)}+\sqrt[3]{4(b^6+c^6)}+\sqrt[3]{4(c^6+a^6)} \ge$$

$$\ge \sqrt{\frac{2(a^3+b^3)^2}{2}}+\sqrt{\frac{2(b^3+c^3)^2}{2}}+\sqrt{\frac{2(c^3+a^3)^2}{2}}+\sqrt[3]{\frac{4(a^2+b^2)^3}{4}}+\sqrt[3]{\frac{4(b^2+c^2)^3}{4}}+$$

$$+\sqrt[3]{\frac{4(c^3+a^2)^3}{4}} = a^3+b^3+b^3+c^3+c^3+a^3+a^2+b^2+b^2+c^2+c^2+a^2 =$$

$= 2(a^2+b^2+c^2+a^3+b^3+c^3)$. ***Therefore, it is to be true.***

449. If $a,b,c>0$ then:

$$\left(e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\right)^2 \le \left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sagar Kumar-Kolkata-India

Applying Cauchy: $\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right) \ge \left(e^{\frac{a^2+\frac{1}{b^2}}{2}}+e^{\frac{b^2+\frac{1}{c^2}}{2}}+e^{\frac{c^2+\frac{1}{a^2}}{2}}\right)^2$

$$AM \ge GM: \left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right) \ge \left(e^{\frac{2\frac{a}{b}}{2}}+e^{\frac{2\frac{b}{c}}{2}}+e^{\frac{2\frac{c}{a}}{2}}\right)^2 \ge \left(e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\right)^2$$

Solution 2 by Marian Ursărescu-Romania

From Cauchy inequality we have:

$$\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)=\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}+e^{\frac{1}{a^2}}\right)\ge$$

$$\ge\left(e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}+e^{\frac{1}{2}\left(b^2+\frac{1}{c^2}\right)}+e^{\frac{1}{2}\left(c^2+\frac{1}{a^2}\right)}\right)^2\Rightarrow \textit{we must show:}$$

$$e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}+e^{\frac{1}{2}\left(b^2+\frac{1}{c^2}\right)}+e^{\frac{1}{2}\left(c^2+\frac{1}{a^2}\right)}\ge e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\quad (1)$$

But $e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}\ge e^{\frac{a}{b}}\Leftrightarrow\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)\ge\frac{a}{b}\Leftrightarrow a^2b^2+1\ge 2ab\Leftrightarrow(ab-1)^2\ge 0$

and similarly, $(bc-1)^2\ge 0;\ (ac-1)^2\ge 0\Rightarrow$ *(1) its true*

450. Let $\Delta\ ABC\wedge abc=1$ and $x\in(0;1)$. Prove:

$$\frac{1}{(a^2+2ab+3)^x}+\frac{1}{(b^2+2bc+3)^x}+\frac{1}{(c^2+2ca+3)^x}\le\frac{3}{6^x}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$A-G\Rightarrow(a^2+2ab+3)^x\ge(2a+2ab+2abc)^x\ (\because 1=abc)$$

$$=(2a)^x(1+b+bc)^x\Rightarrow\frac{1}{(a^2+2ab+3)^x}\overset{(1)}{\le}\frac{1}{2^x a^x(1+b+bc)^x}$$

Also, $(b+2bc+3)^x\overset{A-G}{\ge}(2b+2bc+2abc)^x=(2b)^x(1+c+ca)^x=$

$$=(2b)^x(abc+c+ca)^x=(2bc)^x(1+a+ab)^x=(2bc)^x(abc+a+ab)^x$$

$$=(2abc)^x(1+b+bc)^x\overset{abc=1}{=}2^x(1+b+bc)^x\Rightarrow$$

$\Rightarrow\frac{1}{(b^2+2bc+3)^x}\overset{(2)}{\le}\frac{1}{2^x(1+b+bc)^x}$. *Also,* $(c^2+2ca+3)^x\overset{A-G}{\ge}(2c+2ca+2abc)^x$

$$=(2c)^x(1+a+ab)^x\overset{1=abc}{=}(2c)^x(abc+a+ab)^x=(2ca)^x(1+b+bc)^x\Rightarrow$$

$$\Rightarrow\frac{1}{(c^2+2ca+3)^x}\overset{(3)}{\le}\frac{1}{2^x(ca)^x(1+b+bc)^x}$$

(1)+(2)+(3)⇒LHS

$$\le \frac{1}{2^x}\left(\frac{1}{a^x}+1+\frac{1}{(ca)^x}\right)\frac{1}{(1+b+bc)^x}=\frac{1+b^x+(bc)^x}{2^x(1+b+bc)^x}\overset{?}{\le}\frac{3}{6^x}\Leftrightarrow$$

$$\Leftrightarrow 1+b^x+(bc)^x\le 3\left(\frac{1+b+bc}{3}\right)^x \to \textbf{true by Jensen} \because f(t)=t^m(t>0)$$

$(0\le m\le 1)$ ***is concave as*** $f''(t)=m(m-1)t^{m-2}\le 0$ ***as*** $0\le m\le 1$ ***(Proved)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$abc=1 \text{ and } x\in(0,1)$$

$$\sum_{cyc}\frac{1}{(a^2+2ab+3)^x}\overset{AM\ge GM}{\le}\sum_{cyc}\frac{1}{(2a+2ab+2)^x}=\frac{1}{2^x}\sum_{cyc}\frac{1}{(a+ab+1)^x}\le$$

$$\le\frac{3}{2^x}\left(\frac{1}{3}\sum_{cyc}\frac{1}{a+ab+1}\right)^x \textbf{[since } x\in(0,1)\textbf{]}=\frac{3}{6^x}\left(\sum_{cyc}\frac{1}{a+ab+1}\right)^x$$

$$=\frac{3}{6^x}\left(\frac{1}{a+ab+abc}+\frac{1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x$$

$$=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+abc}+\frac{1}{c+ca+1}\right)^x=$$

$$=\frac{3}{6^x}\left(\frac{ac(bc+1)}{1+c+ac}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{c+ac+1}{c+ca+1}\right)^x=\frac{3}{6^x}\quad \textbf{(proved)}$$

451. If $x,y,z>0, x^4+y^4+z^4=x^2y^2z^2$ then:

$$\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2+\left(\frac{xy^2+xz^2}{y^4+z^4}\right)^2+\left(\frac{yz^2+zx^2}{z^4+x^4}\right)^2\le 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

Let $P=\sum_{cyc}(x,y,z)\left[\frac{z(x^2+y^2)}{(x^4+y^4)}\right]^2$***. From power mean inequality***

$$\frac{x^4+y^4}{2}\ge\left(\frac{x^2+y^2}{2}\right)^2\Rightarrow\frac{x^2+y^2}{x^4+y^4}\le\frac{2}{(x^2+y^2)}\quad \textbf{(1)}$$

$$\Rightarrow P\le\sum_{cyc(x,y,z)}\left(\frac{2z}{x^2+y^2}\right)^2. \textbf{ Using (1)}$$

$$\Rightarrow P\overset{AM-GM}{\le}\sum_{cyc(x,y,z)}\left(\frac{2z}{2xy}\right)^2\le\sum_{cyc}\frac{z^2}{x^2y^2}$$

$$P \le \sum_{cyc(x,y,z)} \frac{z^4}{x^2y^2z^2} \le \sum_{cyc} \frac{z^4}{x^4+y^4+z^4} \{\because x^2y^2z^2 = x^4+y^4+z^4\}$$

$$\Rightarrow P \le \left(\frac{x^4+y^4+z^4}{x^4+y^4+z^4}\right) \le 1 \Rightarrow P \le 1$$ ***(proved)***

Solution 2 by Artan Ajredini-Presheva-Serbie

By Cauchy-Schwarz inequality we have:

$$x^4+y^4 \ge \frac{(x^2+y^2)^2}{2} \quad (1)$$

$$y^4+z^4 \ge \frac{(y^2+z^2)^2}{2} \quad (2)$$

$$z^4+x^4 \ge \frac{(z^2+x^2)^2}{2} \quad (3)$$

From (1), (2) and (3) we have: $LHS \le \frac{4z^2}{(x^2+y^2)^2} + \frac{4x^2}{(y^2+z^2)^2} + \frac{4y^2}{(z^2+x^2)^2}$ ***(4)***

By applying AM-GM inequality we have: $(x^2+y^2)^2 \ge 4x^2y^2$ ***(5)***

From (5) to (4) we have: $LHS \le \frac{z^2}{x^2y^2} + \frac{x^2}{y^2z^2} + \frac{y^2}{x^2z^2} = \frac{z^4x^2y^2z^2+x^4x^2y^2z^2+y^4x^2y^2z^2}{x^4y^4z^4} =$

$$= \frac{(z^4+x^4+y^4)x^2y^2z^2}{x^4y^4z^4} = \frac{x^2y^2z^2\cdot x^2y^2z^2}{x^4y^4z^4} = 1.$$ ***Q.E.D.***

Solution 3 by Lahiru Samarakoon-Sri Lanka

$x,y,z > c, x^4+y^4+z^4 = x^2y^2z^2; \sum\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2 \le 1.$ ***Holds inequality***

$x^4+y^4 \ge \frac{1}{2}(x^2+y^2)$***. So,*** $LHS \le \sum z^2 \frac{(x^2+y^2)^2}{(x^2+y^2)^4} \times 4 = \sum \frac{4z^2}{(x^2+y^2)^2}$

AM-GM

$$\le \sum \frac{4z^2}{(2xy)^2} = \sum \frac{z^4}{x^2y^2z^2} = \frac{z^4+y^4+z^4}{x^2y^2z^2} = 1$$ ***(proved)***

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z > 0$ ***we get that***

1. $(x^4+y^4)^2 \ge (x^2+y^2)^3(xy)^2$

2. $(y^4+z^4)^2 \ge (y^2+z^2)^2(yz)^2$

3. $(z^4+x^4)^2 \ge (y^2+z^2)^2(yz)^2$

Hence, $\frac{z^2(x^2+y^2)^2}{(x^4+y^4)^2} \le \frac{z^2}{(xy)^2}; \frac{x^2(y^2+z^2)^2}{(y^4+z^4)^2} \le \frac{x^2}{(yz)^2}; \frac{y^2(z^2+x^2)^2}{(z^4+x^4)^2} \le \frac{y^2}{(xz)^2}$

That is $\left(\frac{z(x^2+y^2)}{x^4+y^4}\right)^2 + \left(\frac{x(y^2+z^2)}{y^4+z^4}\right)^2 + \left(\frac{y(z^2+x^2)}{z^4+x^4}\right)^2 \le \frac{x^2}{(yz)^2} + \frac{y^2}{(zx)^2} + \frac{z^2}{(xy)^2}$

Hence $\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2 + \left(\frac{xy^2+xz^2}{y^4+z^4}\right)^2 + \left(\frac{yz^2+yx^2}{z^4+x^4}\right)^2 \le 1$. ***Because*** $x^4 + y^4 + z^4 = (xyz)^2$

Therefore it is to be true.

Solution 5 by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{z^2(x^4+y^4+2x^2y^2)}{(x^4+y^4)^2} = \sum \frac{z^2}{x^4+y^4} + 2x^2y^2z^2 \sum \frac{1}{(x^4+y^4)^2}$$

$$\overset{A-G}{\le} \sum \frac{z^2}{2x^2y^2} + 2x^2y^2z^2 \sum \frac{1}{4x^4y^4} = \sum \frac{z^4}{2x^2y^2z^2} + \sum \frac{z^4}{2x^2y^2z^2} = \frac{2\sum x^4}{2x^2y^2z^2} = 1 \textit{ (Done)}$$

Solution 6 by Marian Ursărescu-Romania

We must show: $z^2\left(\frac{x^2+y^2}{x^4+y^4}\right)^2 + x^2\left(\frac{y^2+z^2}{y^4+z^4}\right)^2 + y^2\left(\frac{z^2+x^2}{z^4+x^4}\right)^2 \le 1$ ***(1)***

But $x^4 + y^4 \ge xy(x^2+y^2)$ ***(2) because*** $\Leftrightarrow x^4 - x^3y + y^4 - xy^3 \ge 0 \Leftrightarrow$

$\Leftrightarrow (x-y)^2(x^2+xy+y^2) \ge 0$ ***true (2)*** $\Rightarrow \frac{x^2+y^2}{x^4+y^2} \le \frac{1}{xy}$ ***(2)***

From (1)+(2) we must show: $\frac{z^2}{x^2y^2} + \frac{x^2}{y^2z^2} + \frac{y^2}{x^2z^2} \le 1$ ***(3)***

But $x^4 + y^4 + z^4 = x^2y^2z^2 \Leftrightarrow \frac{x^2}{y^2z^2} + \frac{y^2}{x^2z^2} + \frac{z^2}{x^2y^2} = 1 \Rightarrow$ ***(3) its true.***

452. If $x, y, z > 1, xyz = 2\sqrt{2}$ then:

$$x^y + y^z + z^x + y^x + z^y + x^z > 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

From Bernoulli's inequality $x^y > 1 + y(x-1) \overset{AM-GM}{\ge} 2\sqrt{y(x-1)} > 2\sqrt{y}$

Similarly, $y^z > 2\sqrt{z}; z^x > 2\sqrt{x}; y^x > 2\sqrt{x}; z^y > 2\sqrt{y}; x^z > 2\sqrt{z}$

Therefore $LHS > 4(\sqrt{x} + \sqrt{y} + \sqrt{z}) \overset{AM-GM}{\ge} 12(xyz)^{\frac{1}{6}} = 12\sqrt[4]{2} > 9$

Solution 2 by Cristian Viorel Hrenciuc-Romania

$$\text{Let } xyz = 2\sqrt{2} = a; \frac{x^y+y^z+z^x}{3} \ge \sqrt[3]{x^y \cdot y^z \cdot z^x} = a^{\frac{a}{3}};\ x^y + y^z + z^x \ge 3a^{\frac{a}{3}}$$

$$\text{Likewise, } x^z + z^y + y^x \ge 3\sqrt[3]{x^z \cdot z^y \cdot y^x} = 3a^{\frac{a}{3}}$$

$$\text{Thus, } x^y + y^z + z^x + y^x + z^y + x^z \ge 6a^{\frac{a}{3}} = 6(2\sqrt{2})$$

453. If $x, y, z > 0, \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} = 1$ then:

$$\sum_{cyc(x,y,z)} \frac{y^3 + z^3 + 3}{x^3} \ge 3xyz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum\left(\frac{y^3+z^3}{x^3} + \frac{3}{x^3}\right) \ge 3xyz \Leftrightarrow \sum\frac{y^3+z^3}{x^3} + 3 \ge 3xyz \quad (1)$$

$$\text{Let } x = \sqrt[3]{\frac{a+b+c}{a}}, y = \sqrt[3]{\frac{a+b+c}{b}}, z = \sqrt[3]{\frac{a+b+c}{c}} \quad (2)$$

$$\text{From (1)+(2) we must show: } \sum\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}} + 3 \ge \frac{3(a+b+c)}{\sqrt[3]{abc}} \Leftrightarrow \sum\frac{a(b+c)}{bc} + 3 \ge \frac{3(a+b+c)}{\sqrt[3]{abc}} \quad (3)$$

$$\text{But } \sqrt[3]{abc} \ge \frac{3abc}{ab+ac+bc} \Leftrightarrow \frac{1}{\sqrt[3]{abc}} \le \frac{ab+ac+bc}{3abc} \quad (4)$$

$$\text{From (3)+(4) we must show: } \sum\frac{a(b+c)}{bc} + 3 \ge \frac{(ab+ac+bc)(a+b+c)}{abc} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sum a^2(b+c)}{abc} + 3 \ge \frac{(ab+bc+ac)(a+b+c)}{abc} \Leftrightarrow$$

$$\sum a^2(b+c) + 3abc \ge (ab+bc+ac)(a+b+c) \quad (5)$$

$$\text{But (5)} \Leftrightarrow a^2(b+c) + b^2(a+c) + c^2(a+b) + 3abc \ge a^2b + ab^2 + abc + abc +$$

$$+b^2c + bc^2 + a^2c + abc + ac^2 \Leftrightarrow 0 \ge 0 \Leftrightarrow \text{(5) its true.}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{For } x, y, z > 0 \text{ and } \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} = 1 \text{ we have } x^3 + y^3 + z^3 \ge 3xyz \quad \text{(fact)}$$

$$\Rightarrow 1 \ge \frac{3xyz}{x^3 + y^3 + z^3} \Rightarrow \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \ge \frac{3(xyz)}{x^3 + y^3 + z^3} \Rightarrow$$

$$\Rightarrow \frac{x^3+y^3+z^3}{x^3}+\frac{x^3+y^3+z^3}{y^3}+\frac{x^3+y^3+z^3}{z^3}\geq 3xyz \Rightarrow$$

$$\Rightarrow \frac{x^3+y^3}{z^3}+\frac{y^3+z^3}{x^3}+\frac{z^3+x^3}{y^3}+3\geq 3xyz \Rightarrow$$

$$\Rightarrow \frac{x^3+y^3}{z^3}+\frac{y^3+z^3}{x^3}+\frac{z^3+x^3}{y^3}+\frac{3}{x^3}+\frac{3}{y^3}+\frac{3}{z^3}\geq 3xyz \Rightarrow$$

$$\Rightarrow \frac{x^3+y^3+3}{z^3}+\frac{y^3+z^3+3}{x^3}+\frac{z^3+x^3+3}{y^3}\geq 3xyz$$

Therefore, it is true.

454. Let $a,b,c\in[0;+\infty)\wedge a+b+c=3$. Prove:

$$a^2+b^4+c^4\leq 81+abc$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Tran Hong-Vietnam

Because: $a+b+c=3$ ***we have:*** $a^4+b^4+c^4\leq 81+abc \Leftrightarrow$

$$\Leftrightarrow a^4+b^4+c^4\leq (a+b+c)^4+abc \Leftrightarrow (a+b+c)^4+abc-(a^4+b^4+c^4)\geq 0$$

$$\Leftrightarrow \begin{bmatrix} a^4+b^4+c^4+4(a^3b+a^3c+ab^3+ac^3+bc^3+cb^3)+ \\ +6(a^2b^2+b^2c^2+c^2a^2)+12abc(a+b+c) \end{bmatrix}+abc-(a^4+b^4+c^4)\geq 0$$

$$\Leftrightarrow 4(a^3b+a^3c+ab^3+bc^3+cb^3)+6(a^2b^2+b^2c^2+c^2a^2)+12abc(a+b+c)+abc\geq 0$$

Which is true because $a,b,c\geq 0$***. Equality*** $\Leftrightarrow (a,b,c)=(3,0,0)$ ***or*** $(a,b,c)=(0,3,0)$ ***or*** $(a,b,c)=(0,0,3)$

Solution 2 by Michael Stergiou-Greece

$$\sum_{cyc} a^4\leq 81+abc \quad \textit{(1)}$$

Let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)=(p,q,r).\ p=3, r\leq \left(\frac{q}{3}\right)^{\frac{3}{2}}$ ***[AM-GM]***

$$\sum a^4=\left(\sum_{cyc} a^2\right)^2-2\sum_{cyc} a^2b^2=(9-2q)^2-2(q^2-6r)=2q^2-36q+12r+81$$

So (1) $\rightarrow 2q^2-36q+11r\leq 0$ ***or the stronger inequality*** $2q^2-36q+11\left(\frac{q}{3}\right)^{\frac{3}{2}}\leq 0\rightarrow$

$\rightarrow \frac{1}{9}q\left(18q+11\sqrt{3}\sqrt{q}-324\right)\le 0$ *which is obviously true as* $q\le\frac{p^2}{3}=3$. *Done!*

(Equality for $a=0, b=0, c=3$ *and permutations)*

Solution 3 by Daniel Sitaru-Romania

$$a,b,c\in[0,3],$$

$[0,3]X[0,3]X[0,3]$ *– convexe domain with vertex:* $(0,0,0),(3,0,0),$

$$(0,3,0),(0,0,3),(3,3,0),(3,0,3),(0,3,3),(3,3,3)$$

$$f:[0,3]X[0,3]X[0,3]\rightarrow\mathbb{R}, f(a,b,c)=a^4+b^4+c^4-abc$$

$$\begin{cases} f'_a=4a^3-bc, f''_{aa}=12a^2\ge 0\\ f'_b=4b^3-ac, f''_{bb}=12b^2\ge 0\\ f'_a=4c^3-ba, f''_{cc}=12c^2\ge 0\end{cases}$$

f *– convexe in each variable – max is attained in a vertex*

$$\max\{f(a,b,c)|a+b+c=3\}=\max\{f(3,0,0),f(0,3,0),f(0,0,3)\}=81$$

$$f(a,b,c)\le 81, a^4+b^4+c^4-abc\le 81$$

455. If $a,b,c>0$ then:

$$\sum_{cyc(a,b,c)}\left(\frac{1}{a^2b^2}-\frac{1}{ab}\right)+2\sum_{cyc(a,b,c)}\frac{bc^2(ab+1)}{a(b^2c^2+1)}\ge 6$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum\frac{bc^2(ab+1)}{a(b^2c^2+1)}=2\sum\left(\frac{b^2c^2}{b^2c^2+1}\right)\left(\frac{ab+1}{ab}\right)=2\sum\left\{\frac{(b^2c^2+1)-1}{b^2c^2+1}\right\}\left(1+\frac{1}{ab}\right)$$

$$=2\sum\left(1-\frac{1}{b^2c^2+1}\right)\left(1+\frac{1}{ab}\right)=2\sum\left\{1+\frac{1}{ab}-\frac{1}{b^2c^2+1}-\frac{1}{ab(b^2c^2+1)}\right\}$$

$$\underset{(1)}{\overset{A-G}{\ge}}2\sum\left(1+\frac{1}{ab}-\frac{1}{2bc}-\frac{1}{2ab^2c}\right)=6+2\sum\frac{1}{ab}-\sum\frac{1}{ab}-\frac{1}{abc}\sum\frac{1}{a}$$

$$\text{(1)}\Rightarrow LHS\ge\sum\frac{1}{a^2b^2}-\sum\frac{1}{ab}+6+\sum\frac{1}{ab}-\frac{1}{abc}\sum\frac{1}{a}\overset{?}{\ge}6$$

$$\Leftrightarrow\sum\frac{1}{a^2b^2}\overset{?}{\ge}\frac{1}{abc}\sum\frac{1}{a}\Leftrightarrow\sum x^2y^2\overset{?}{\ge}xyz\left(\sum x\right)\left(x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)$$

$$\Leftrightarrow \sum u^2 \overset{?}{\geq} \sum uv \quad (xy = u, yz = v, zx = w) \rightarrow$$ ***true (Proved)***

456. If $x, y, z > 0, \sqrt{xy} + \sqrt{yz} + \sqrt{zx} = 3\sqrt{xyz}$ then:

$$\frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)} + \frac{(y^2+1)(z^2+1)}{(y^3+1)(z^3+1)} + \frac{(z^2+1)(x^2+1)}{(z^3+1)(x^3+1)} \leq 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $a \geq 0$ *then* $2(a^3+1) \geq (a+1)(a^2+1) \Leftrightarrow (a-1)^2(a+1) \geq 0,$

Which is true, then $\sum_{cyc} \frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)} \sum_{cyc} \left(\frac{2}{x+1}\right)\left(\frac{2}{y+1}\right) \overset{AM\geq GM}{\lesseqqgtr} \sum_{cyc} \frac{1}{\sqrt{xy}} \leq \frac{1}{3}\left(\sum_{cyc} \frac{1}{\sqrt{x}}\right)^2 = 3$

(proved) since, $\sum_{cyc} \sqrt{xy} = 3\sqrt{xyz}$

Solution 2 by Tran Hong-Vietnam

$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = 3\sqrt{xyz} \Leftrightarrow \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} = 3.$ *Let* $u = \frac{1}{\sqrt{x}}; v = \frac{1}{\sqrt{y}}; w = \frac{1}{\sqrt{z}} \Rightarrow$

$\Rightarrow u + v + w = 3.$ *We have:* $a(1+a^2)^2 \leq (1+a^3)^2$ ***(*)***

$$\Leftrightarrow a(1+a^2)^2 - (1+a^3)^2 \leq 0$$

$\Leftrightarrow a^5 + a - a^6 - 1 \leq 0 \Leftrightarrow (a-1)^2(a^4+a^3+a^2+1) \geq 0$ ***(true for all*** $a > 0$***)***

Using (*) we have: LHS $\leq uv + vw + wu$. *We need to prove:*

$uv + vw + wu \leq 3 = \frac{(u+v+w)^2}{3} \Leftrightarrow (u-v)^2 + (v-w)^2 + (w-u)^2 \geq 0.$ ***True.***

Equality $\Leftrightarrow u = v = w = 1 \Leftrightarrow x = y = z = 1.$

457. If $x, y, z \geq 0$ then:

$$\left(\sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2}\right)^2 \geq 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z \geq 0$, *we have* $\left(\sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2}\right)^2 \geq$

$$\geq\left(\sqrt{(x+y)^2+(y+z)^2}+\sqrt{z^2+x^2}\right)^2\geq\left(\sqrt{(x+y+z)^2+(x+y+z)^2}\right)^2$$

$$=\left(\sqrt{2(x+y+z)^2}\right)^2\geq\left(\sqrt{2(x^2+y^2+z^2)}\right)^2 : x,y,z\geq 0$$

$$=\sqrt{4(x^2+y^2+z^2)^2}=2\sqrt{(x^2+y^2+z^2)^2}$$

$$\geq 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}\ \textbf{ok}: (a+b+c)^2\geq 3(ab+bc+ca), \forall a,b,c$$

Therefore, it is true.

Solution 2 by Safal Das Biswas-Chinsurah-India

Note that if any two of x,y,z is 0 then the inequality is trivial, however we can check the equality holds if $x=y=z=0$. Now, suppose any one of x,y,z is 0, then WLOG suppose $x=0$ and $yz>0$ then the equation

$$\left(\sum_{cyc}\sqrt{x^2+y^2}\right)^2=\left(y+z+\sqrt{y^2+z^2}\right)^2\overset{RMS-AM}{\geq}\left(y+z+\frac{y+z}{\sqrt{2}}\right)^2$$

$$\left(y+z+\frac{y+z}{\sqrt{2}}\right)^2=(y+z)^2\left(1+\frac{1}{\sqrt{2}}\right)^2=\left(1+\sqrt{2}+\frac{1}{2}\right)(y+z)^2\overset{AM-GM}{\geq}\left(\frac{3}{2}+\sqrt{2}\right)4yz$$

as $3+2\sqrt{2}=\left(\frac{3}{2}+\sqrt{2}\right)2>\sqrt{3}\Rightarrow\left(\frac{3}{2}+\sqrt{2}\right)4yz>2\sqrt{3}yz=2\sqrt{3(y^2z^2+x^2y^2+x^2z^2)}$

(as $x=0$)

Lemma 1: If x,y,z are all positive reals then $(x+y+z)^2>\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$

Proof: As $x^2+y^2\overset{AM-GM}{\geq}2xy\Rightarrow\sum_{cyc\ x,y,z}x^2+y^2\geq\sum_{cyc\ x,y,z}2xy\Leftrightarrow$

$$\Leftrightarrow x^2+y^2+z^2\geq xy+yz+zx$$

$$(x+y+z)^4=\left(x^2+y^2+z^2+2xy+2yz+2zx\right)^2\geq\left(3(xy+yz+zx)\right)^2=9(xy+yz+zx)^2$$

$$9(xy+yz+zx)^2>3(x^2y^2+y^2z^2+z^2x^2+2x^2yz+2y^2zx+2z^2xy)>$$

$$>3(x^2y^2+y^2z^2+z^2x^2)$$

Thus $(x+y+z)^4>3\left(x^2y^2+y^2z^2+z^2x^2\right)\Rightarrow(x+y+z)^2>\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$

Back to the problem: Note

$$\sqrt{x^2+y^2}\overset{RMS-AM}{\geq}\frac{x+y}{\sqrt{2}}\Rightarrow\left(\sum_{cyc\ (x,y,z)}\sqrt{x^2+y^2}\right)^2\geq\left(\sum_{cyc\ (x,y,z)}\frac{x+y}{\sqrt{2}}\right)^2$$

Now, $\left(\sum_{cyc\,(x,y,z)}\frac{x+y}{\sqrt{2}}\right)^2 = 2(x+y+z)^2 > 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$ ***(by Lemma 1)***

thus $\left(\sum_{cyc\,(x,y,z)}\sqrt{x^2+y^2}\right)^2 \ge \left(\sum_{cyc\,(x,y,z)}\frac{x+y}{\sqrt{2}}\right)^2 > 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$ ***if*** $x,y,z>0$

Thus, $\left(\sum_{cyc\,(x,y,z)}\sqrt{x^2+y^2}\right)^2 \ge 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$ ***equality at*** $x=y=z=0$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS = 2\sum x^2 + 2\sum\sqrt{x^2+y^2}\sqrt{y^2+z^2} =$$

$$= 2\sum x^2 + 2\sum\sqrt{y^4+\sum x^2y^2} \ge 2\sum x^2 + 2\left(\sum\sqrt{\sum x^2y^2}\right)$$

$$= 2\sum x^2 + 6\sqrt{\sum x^2y^2} \ge 2\sum x^2 + 2\sqrt{3\sum x^2y^2}$$

$$\left(\because \sum x^2 \ge 0 \ \&6 > 2\sqrt{3} \ along\ with\ \sqrt{\sum x^2y^2} \ge 0\right)$$ ***(proved)***

Solution 4 by Tran Hong-Vietnam

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2 \ge 3\sum\left(\sqrt{x^2+y^2}\right)\left(\sqrt{y^2+z^2}\right) \ge$$

$$\ge \frac{3}{2}\sum(x+y)(y+z) = \frac{3}{2}\left[\sum x^2 + 3\sum xy\right] \ge \frac{3}{2}\left[\sum xy + 3\sum xy\right] = 6\sum xy$$

We need to prove: $6\sum xy \ge 2\sqrt{3\sum x^2y^2} \Leftrightarrow 3\left(\sum xy\right)^2 \ge \sum x^2y^2 \Leftrightarrow$

$$\Leftrightarrow 3(xy+yz+zx)^2 \ge x^2y^2+y^2z^2+z^2x^2 \Leftrightarrow 2\sum x^2y^2 + 6xyz(x+y+z) \ge 0$$

True because $x,y,z \ge 0$. ***Equality*** $\Leftrightarrow x=y=z=0$.

458. If $a,b,c>0, a^3+b^3+c^3=3$ then:

$$\left(\frac{a^2+1}{a+1}\right)^3 + \left(\frac{b^2+1}{b+1}\right)^3 + \left(\frac{c^2+1}{c+1}\right)^3 \ge 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

Let $x=a^3, y=b^3, c=z^3 \Rightarrow \begin{cases} a=\sqrt[3]{x}, b=\sqrt[3]{y}, c=\sqrt[3]{z} \\ x+y+z=3 \end{cases} (x,y,z>0)$

$$LHS=\sum\left(\frac{x^{\frac{2}{3}}+1}{x^{\frac{1}{3}}+1}\right)^3. \text{ Let } f(t)=\left(\frac{t^{\frac{2}{3}}+1}{t^{\frac{1}{3}}+1}\right)^3, 0<t<3$$

$$\Rightarrow f''(t)=\frac{2\left(\sqrt[3]{t}-1\right)^2\left(t^{\frac{2}{3}}+1\right)\left(t^{\frac{2}{3}}+4\sqrt[3]{t}+1\right)}{3\left(\sqrt[3]{t}+1\right)^5\cdot t^{\frac{5}{3}}}\geq 0, \forall t\in(0,3)$$

Using Jensen's inequality: LHS$= f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3f(1)=3.$

$\Rightarrow$ ***Proved. Equality*** $\Leftrightarrow x=y=z=1\Leftrightarrow a=b=c=1.$

Solution 2 by Boris Colakovic-Belgrade-Serbie

$$f(x)=\frac{(x^2+1)^3}{(x+1)^3}\Rightarrow f(1)=1$$

$$f'(x)=\frac{6x(x^2+1)^2(x+1)^3-3(x^2+1)^3(x+1)^2}{(x+1)^6}\Rightarrow f'(1)=\frac{3}{2}$$

$$g(x)=kx^3+m\Rightarrow g(1)=k+m;\ g'(x)=3kx^2\Rightarrow g'(1)=3k$$

$$f(1)=g(1)\Rightarrow k+m=1\Rightarrow m=\frac{1}{2};\ f'(1)=g'(1)\Rightarrow 3k=\frac{3}{2}\Rightarrow k=\frac{1}{2}$$

$$g(x)=\frac{1}{2}x^3+\frac{1}{2}$$

$$f(x)\geq g(x)\Leftrightarrow\frac{(x^2+1)^3}{(x+1)^3}\geq\frac{1}{2}(x^3+1)\Leftrightarrow x^6-3x^5+3x^4-2x^3+3x^2-3x+1\geq 0$$

$\Leftrightarrow (x-1)^4(x^2+x+1)\geq 0$ ***true sign "=" holds for*** $x=1$

Now, we have LHS $\geq\frac{1}{2}\sum a^3+\frac{3}{2}=\frac{1}{2}\cdot 3+\frac{3}{2}=3$ ***sign "=" holds for*** $a=b=c=1.$

Solution 3 by Avishek Mitra-India

By AM ≥ GM: $\frac{\left(\frac{a^2+1}{a+1}\right)^3+\left(\frac{b^2+1}{b+1}\right)^3+\left(\frac{c^2+1}{c+1}\right)^3}{3}\geq\frac{(a^2+1)(b^2+1)(c^2+1)}{(a+1)(b+1)(c+1)}$ ***and*** $\frac{a^3+b^3+c^3}{3}\geq abc$

[by AM ≥ GM]$\Rightarrow abc\leq\frac{3}{3}\Rightarrow abc\leq 1$

as $abc\leq 1$ ***and*** $a^3+b^3+c^3=3\Leftrightarrow$ ***equality holds only when*** $a=b=c=1$

Hence $\Rightarrow\left(\frac{a^2+1}{a+1}\right)^3+\left(\frac{b^2+1}{b+1}\right)^3+\left(\frac{c^2+1}{c+1}\right)^3\geq 3\frac{(1^2+1)}{(1+1)}\cdot\frac{(1^2+1)}{(1+1)}\cdot\left(\frac{1^2+1}{1+1}\right)\geq 3$ ***(proved)***

459. Let $x, y, z > 0 \wedge x + y + z \leq 1$. Prove:

$$(x + y + z)^3 + \frac{1}{xyz} \geq 28$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$ *and* $x + y + z \leq 1$*, we have* $xyz \leq \frac{1}{27}$ *and* $xyz \leq 1 \Rightarrow \left(xyz - \frac{1}{27}\right) \leq 0$

and $(xyz - 1) \leq 0 \Rightarrow \left(xyz - \frac{1}{27}\right)(xyz - 1) \geq 0 \Rightarrow (27xyz - 1)(xyz - 1) \geq 0 \Rightarrow$

$$\Rightarrow 27xyz^2 - 28xyz + 1 \geq 0 \Rightarrow 27xyz + \frac{1}{xyz} \geq 28 \Rightarrow (x + y + z)^3 + \frac{1}{xyz} \geq 28$$

Therefore, it is to be true.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall x, y, z > 0 \mid \sum x \leq 1, \left(\sum x\right)^3 + \frac{1}{xyz} \geq 28$$

$\sqrt[3]{xyz} \overset{G \leq A}{\leq} \frac{\sum x}{3} \Rightarrow xyz \leq \frac{(\sum x)^3}{27} \Rightarrow \frac{1}{xyz} \geq \frac{27}{t}$*, where* $t = (\sum x)^3$

$$\Rightarrow \left(\sum x\right)^3 + \frac{1}{xyz} \geq t + \frac{27}{t} \overset{?}{\geq} 28 \Leftrightarrow t^2 - 28t + 27 \overset{?}{\geq} 0 \Leftrightarrow (t - 1)(t - 27) \underset{(1)}{\overset{?}{\geq}} 0$$

Now, $\because \sum x \leq 1 \therefore t = (\sum x)^3 \leq 1 \Rightarrow t - 1 \overset{(a)}{\leq} 0$

Also, $t - 27 \leq 1 - 27 \overset{(b)}{<} 0 \therefore$ *(a).(b)*$\Rightarrow (t - 1)(t - 27) \geq 0 \Rightarrow$ *(1) is true (proved)*

Solution 3 by Sudhir Jha-Kolkata-India

Since $x, y, z > 0$*; AM ≥ GM*$\Rightarrow \frac{x+y+z}{3} \geq (xyz)^{\frac{1}{3}} \Rightarrow xyz \leq \left(\frac{x+y+z}{3}\right)^3$

$\Rightarrow xyz \leq \left(\frac{1}{3}\right)^3 \because x + y + z \leq 1 \Rightarrow \frac{1}{xyz} \geq 27$ *(1)*

(Equality holds when $x = y = z = \frac{1}{3}$*)* $\because x + y + z \leq 1$

then $(x + y + z)^3 \leq 1$ *(2)*

adding (1)+(2) $\Rightarrow (x + y + z)^3 + \frac{1}{xyz} \geq 28$ *[The equality holds when* $x = y = z = \frac{1}{3}$*]*

460. If $a,b,c\in\mathbb{N}, a,b,c\geq 4$ then:

$$a^{\frac{1}{a+1}}\cdot b^{\frac{1}{b+1}}\cdot c^{\frac{1}{c+1}}>(a+1)^{\frac{1}{a+2}}\cdot(b+1)^{\frac{1}{b+2}}\cdot(c+1)^{\frac{1}{c+2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{For } a\geq 4, \left(1+\frac{1}{a}\right)^a<3\leq a-1$$

$$\Rightarrow\left(1+\frac{1}{a}\right)^a\left(1+\frac{1}{a}\right)<(a-1)\left(1+\frac{1}{a}\right)\Rightarrow\left(1+\frac{1}{a}\right)^{a+1}<a-\frac{1}{a}<a$$

$$\Rightarrow(1+a)^{a+1}<a^{a+2}\Rightarrow a^{\frac{1}{(a+1)}}>(1+a)^{\frac{1}{(a+2)}}$$

Similarly, for b and c. Multiplying the inequalities, we get:

$$a^{\frac{1}{(a+1)}}b^{\frac{1}{(b+1)}}c^{\frac{1}{(c+1)}}>(1+a)^{\frac{1}{(a+2)}}(1_b)^{\frac{1}{(b+2)}}\times(1+c)^{\frac{1}{(c+2)}}$$

Solution 2 by Michail Sterghiou-Greece

$$\prod_{a,b,c}{}_{cyc}\, a^{\frac{1}{a+1}}>\prod_{a,b,c}{}_{cyc}\,(a+1)^{\frac{1}{a+2}}\quad (1)$$

For $x\geq 4, f(x)=x^{\frac{1}{x+1}}$ *is decreasing as* $f'(x)=-\frac{x^{-\frac{x}{x+1}}[x(\ln x-1)-1]}{(x+1)^2}<0$ *as* $x\geq 4\rightarrow$

$$\rightarrow\ln x>1.38\rightarrow\ln x-1>0{,}38\rightarrow x(\ln x-1)>1.52>1$$

So, as $a+1>a\rightarrow f(a)>f(a+1)\rightarrow a^{\frac{1}{a+1}}>(a+1)^{\frac{1}{a+2}}$.

Cyclic application and multiplication gives (1)

461. If $x,y,z,t\in[-5,5], x+y+z+t=0$ then:

$$\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\leq 20$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

Using Cauchy's Schwarz inequality:

$$(1^2+1^2+1^2+1^2)(25-x^2+25-y^2+25-z^2+25-t^2)\geq$$

$$\geq\left(\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right)^2$$

$$4\left(100-(x^2+y^2+z^2+t^2)\right)\geq\left[\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right]^2 \quad \textbf{(1)}$$

Using Cauchy's Schwarz inequality:

$$(1^2+1^2+1^2+1^2)(x^2+y^2+z^2+t^2)\geq(x+y+z+t)^2 \because (x+y+z+t)=0$$

$$\Rightarrow 4\left(\sum x^2\right)\geq 0\Rightarrow\sum x^2\geq 0\Rightarrow x^2+y^2+z^2+t^2\geq 0\Rightarrow -(x^2+y^2+z^2+t^2)\leq 0$$

From (i):

$$\left(\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right)\leq\{4(100+0)\}^{\frac{1}{2}}\leq(400)^{\frac{1}{2}}\leq 20$$

$$\therefore \sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\leq 20 \textit{ (proved)}$$

462. If $a,b,c>0$ then:

$$\sum\frac{1}{a+1}-3\sum\frac{1}{a+2}+3\sum\frac{1}{a+3}-\sum\frac{1}{a+4}<\frac{\sqrt{6}}{8}\left(\frac{\sqrt{a}}{a^2}+\frac{\sqrt{b}}{b^2}+\frac{\sqrt{c}}{c^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{6}\sqrt{a}}{8a^2}\overset{(1)}{>}\frac{1}{a+1}-3\frac{1}{a+2}+3\frac{1}{a+3}-\frac{1}{a+4}\Leftrightarrow$$

$$\Leftrightarrow\frac{\sqrt{6a}}{8a^2}>3\left(\frac{1}{a+3}-\frac{1}{a+2}\right)+\left(\frac{1}{a+1}-\frac{1}{a+4}\right)=\frac{3(a+2-a-3)}{(a+2)(a+3)}+\frac{a+4-a-1}{(a+1)(a+4)}$$

$$=3\left\{\frac{1}{(a+1)(a+4)}-\frac{1}{(a+2)(a+3)}\right\}=\frac{3(a^2+5a+6-a^2-5a-4)}{(a+1)(a+4)(a+2)(a+3)}=$$

$$=\frac{6}{(a+1)(a+2)(a+3)(a+4)}\Leftrightarrow\frac{6a}{64a^4}>\frac{36}{(a+1)^2(a+2)^2(a+3)^2(a+4)^2}\Leftrightarrow$$

$$\Leftrightarrow (a+1)^2(a+2)^2(a+3)^2(a+4)^2\overset{(2)}{>}384a^3. \textit{ Now,}$$

$$(a+1)^2(a+2)^2(a+3)^2(a+4)^2\overset{A-G}{\geq}(a+1)^2(4a\cdot 2)(4a\cdot 3)(4a\cdot 4)=1536a^3(a+1)^2$$
$$=384a^3\{4(a+1)^2\}>384a^3\ (\because 2a+1>0\therefore 2(a+1)>1\Rightarrow 4(a+1)^2>1)$$

$\Rightarrow$ ***(2) is true*** $\Rightarrow$ ***(1) is true*** $\forall a>0$. ***Similarly,*** $\forall b>0,\frac{\sqrt{6b}}{8b^2}\overset{(3)}{>}\frac{1}{b+1}-\frac{3}{b+2}+\frac{3}{b+3}-\frac{1}{b+4}$ **&**

$$\forall c>0,\frac{\sqrt{6c}}{8c^2}\overset{(4)}{>}\frac{1}{c+1}-\frac{3}{c+2}+\frac{3}{c+3}-\frac{1}{c+4}$$

(1)+(3)+(4)$\Rightarrow \sum \frac{\sqrt{6a}}{8a^2} > \sum \frac{1}{a+1} - 3\sum \frac{1}{a+2} + 3\sum \frac{1}{a+3} - \sum \frac{1}{a+4}$ ***(proved)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \left(\frac{1}{a+1} - \frac{1}{a+4}\right) - 3\sum \left(\frac{1}{a+2} - \frac{1}{a+3}\right) =$$

$$= 3\sum \frac{1}{(a+1)(a+4)} - 3\sum \frac{1}{(a+2)(a+3)} =$$

$$= 3\sum \left(\frac{1}{(a+1)(a+4)} - \frac{1}{(a+2)(a+3)}\right) = \sum \frac{6}{(a+1)(a+2)(a+3)(a+4)}$$

$$\sum \frac{6}{(a+1)(a+2)(a+3)(a+4)} < \frac{\sqrt{6}}{8}\sum \frac{\sqrt{a}}{a^2}$$

$$\frac{6}{(a+1)(a+2)(a+3)(a+4)} < \frac{\sqrt{6}}{8} \cdot \frac{\sqrt{a}}{a^2} \quad \textit{(ASSURE)}$$

$$a > 0 \Rightarrow a + 4 > 1 \quad \textit{(TRUE)}$$

$$4\sqrt{6}a(a+4) > 4\sqrt{6}a$$

$$4\sqrt{6}a < 4\sqrt{6}a(a+4) = 2\sqrt{2a} \cdot 2\sqrt{3a}(a+4) \overset{M_g \le M_a}{\le} (a+2)(a+3)(a+4)$$

$$4\sqrt{6}a < (a+2)(a+3)(a+4) | \cdot 2\sqrt{a}$$

$$8\sqrt{6}a\sqrt{a} < 2\sqrt{a}(a+2)(a+3)(a+4) \overset{M_g \le M_a}{\le} (a+1)(a+2)(a+3)(a+4)$$

$$8\sqrt{6}a\sqrt{a} < (a+1)(a+2)(a+3)(a+4) | \cdot \sqrt{6a}$$

$$8 \cdot 6a^2 < \sqrt{6} \cdot \sqrt{a}(a+1)(a+2)(a+3)(a+4)$$

$$\frac{6}{(a+2)(a+3)(a+4)} < \frac{\sqrt{6}}{8} \cdot \frac{\sqrt{a}}{a^2}$$

463. If $x, y, z \in (0,1)$, $x^6 + y^6 + z^6 = \frac{1}{9}$ then:

$$\left(\frac{2}{1-x^2}\right)^6 + \left(\frac{2}{1-y^2}\right)^6 + \left(\frac{2}{1-z^2}\right)^6 \ge 3^7$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Van Nho-Nghe An-Vietnam

Hope: $\left(\frac{2}{1-x^2}\right)^6 \ge 3^9x^6 \leftrightarrow (1-x^2)^6(2x^2)^3 \le \left(\frac{2}{3}\right)^9 \to$ ***(*)***

Use AM-GM: LHS (*) $= (1-x^2)^6(2x^2)^3 \le \left(\frac{6(1-x^2)+3(2x^2)}{9}\right)^9 = \left(\frac{2}{3}\right)^9 \to$ *(*) is true*

Similarly: $\left(\frac{2}{1-y^2}\right)^6 \ge 3^9y^6$ *and* $\left(\frac{2}{1-z^2}\right)^6 \ge 3^9z^6$. *So: LHS* $\ge 3^9\sum x^6 = 3^9\frac{1}{9} = 3^7$ *(done)*

Solution 2 by Boris Colakovic-Belgrade-Serbie

$$f(x) = \left(\frac{2}{1-x^2}\right)^6; g(x) = kx^6 + m;\ f\left(\frac{1}{\sqrt{3}}\right) = 3^6; g\left(\frac{1}{\sqrt{3}}\right) = \frac{k}{3^3} + m$$

$$f\left(\frac{1}{\sqrt{3}}\right) = g\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \frac{k}{3^3} + m = 3^6 \quad (1)$$

$$f'(x) = 6\left(\frac{2}{1-x^2}\right)^5 \cdot \frac{4x}{(1-x^2)^2}; g'(x) = 6kx^5;\ f'\left(\frac{1}{\sqrt{3}}\right) = 6\cdot\frac{3^7}{\sqrt{3}}; g'\left(\frac{1}{\sqrt{3}}\right) = \frac{6k}{3^2\sqrt{3}}$$

$$f'\left(\frac{1}{\sqrt{3}}\right) = g'\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \frac{6\cdot 3^7}{\sqrt{3}} = \frac{6k}{3^2\sqrt{3}} \Rightarrow k = 3^9 \quad (2)$$

From (1) and (2) $\Rightarrow m = 0;\ g(x) = 3^9x^6$. *Prove that*

$$f(x) \ge g(x) \forall x \in (0,1) \Leftrightarrow \left(\frac{2}{1-x^2}\right)^6 \ge 3^9x^6 \Leftrightarrow \frac{2}{1-x^2} \ge 3\sqrt{3}x \Leftrightarrow$$

$$\Leftrightarrow 3\sqrt{3}x^3 - 3\sqrt{3}x + 2 \ge 0 \Leftrightarrow \left(x - \frac{1}{\sqrt{3}}\right)^2\left(x + \frac{2}{\sqrt{3}}\right) \ge 0 \text{ true } \forall x \in (0,1)$$

Similarly, $\left(\frac{1}{1-y^2}\right)^6 \ge 3^9y^6; \left(\frac{2}{1-z^2}\right)^6 \ge 3^9z^6$; *LHS* $\ge 3^9(x^6+y^6+z^6) = 3^9\cdot\frac{1}{3^2} = 3^7$

sign "=" holds for $x = y = z = \frac{1}{\sqrt{3}}$

464. **If $x, y, z > 0$ then:**

$$\frac{(x+y)^4+1}{(x+y)^6+1} + \frac{(y+z)^4+1}{(y+z)^6+1} + \frac{(z+x)^4+1}{(z+x)^6+1} \le \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We use this inequality: $\frac{a^4+1}{a^6+1} \le \frac{1}{a} \overset{\forall a>0}{\Leftrightarrow} a^5 + a \le a^6 + 1 \Leftrightarrow a^6 - a^5 - a + 1 \ge 0 \Leftrightarrow$

$$a^5(a-1) - (a-1) \ge 0 \Leftrightarrow (a-1)(a^5-1) \ge 0 \Leftrightarrow$$

$$(a-1)^2(a^4+a^3+a^2+a+1) \ge 0, \forall a \ge 0 \text{ true}$$

(with equality for $a = 1) \Rightarrow \frac{(x+y)^4+1}{(x+y)^6+1} \le \frac{1}{x+y} \Rightarrow \sum\frac{(x+y)^4+1}{(x+y)^6+1} \le \sum\frac{1}{x+y}$ *(1)*

But $\frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right), \forall x, y > 0$ ***(2) because***

$$\Leftrightarrow \frac{1}{x+y} \le \frac{x+y}{4xy} \Leftrightarrow (x+y)^2 \ge 4xy \Leftrightarrow (x-y)^2 \ge 0 \text{ with equality } x = y.$$

From (1)+(2)$\Rightarrow \sum \frac{(x+y)^4+1}{(x+y)^6+1} \le \frac{1}{2}\sum \frac{1}{x}$***, with equality for*** $x = y = z = \frac{1}{2}$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

For $a \ge 0$ ***we know,*** $2(a^6+1) \ge (a^4+1)(a^2+1) \Leftrightarrow (a^2-1)^2(a^2+1) \ge 0$

$$\therefore \sum_{cyc} \frac{(x+y)^4+1}{(x+y)^6+1} \le \sum_{cyc} \frac{2}{(x+y)^2+1} \overset{AM\ge GM}{\le} \sum_{cyc} \frac{1}{x+y} \le \frac{1}{4}\sum_{cyc}\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{1}{2}\sum_{cyc}\frac{1}{x}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b > 0$***, we have:*** $(a+b)((a+b)^6+1) = (a+b)((a+b)^6+1)$

$$\ge (a+b)\left((a+b)^5+(a+b)\right) = (a+b)^2((a+b)^4+1)$$

$$\ge 4ab((a+b)^4+1) \Rightarrow \frac{(a+b)^4+1}{(a+b)^6+1} \le \frac{a+b}{4ab} = \frac{1}{4a}+\frac{1}{4b}$$

Hence $\frac{(x+y)^4+1}{(x+y)^6+1} + \frac{(y+z)^4+1}{(y+z)^6+1} + \frac{(z+x)^4+1}{(z+x)^6+1} \le \frac{1}{4x}+\frac{1}{4y}+\frac{1}{4y}+\frac{1}{4z}+\frac{1}{4z}+\frac{1}{4x} \le \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Therefore, it is true.

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{4}\cdot\sum\left(\frac{1}{x}+\frac{1}{y}\right) \ge \sum\frac{1}{x+y} = \sum\frac{2}{2(x+y)\cdot 1} \ge \sum\frac{2}{(x+y)^2+1} =$$

$$= 2\sum\frac{(x+y)^4+1}{((x+y)^2+1)((x+y)^4+1)} \overset{Chebyshev}{\ge} \sum\frac{(x+y)^4+1}{(x+y)^6+1}$$

465. If $a, b, c > 0, a+b+c = 3$ **then:**

$$a^6+b^6+c^6+\frac{1}{32}((3-a)^6+(3-b)^6+(3-c)^6) \ge 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Tam Path-Vietnam

Using Cauchy-Schwarz's Inequality we have: $a^6+b^6+c^6 \ge \frac{(a^3+b^3+c^3)^2}{3}$

$$(3-a)^6+(3-b)^2+(3-c)^6 \ge \frac{[(3-a)^3+(3-b)^3+(3-c)^3]^2}{3}$$

Using Holder's Inequality, we have: $a^3+b^3+c^3 \ge \frac{(a+b+c)^3}{9}=3$

$(3-a)^3+(3-b)^3+(3-c)^3 \ge \frac{(3-a-b-c)^3}{9}=24$***. So,*** $a^6+b^6+c^6 \ge \frac{3^2}{3}=3$

$$(3-a)^6+(3-b)^6+(3-c)^6 \ge \frac{24^2}{3}=192$$

$$\Rightarrow a^6+b^6+c^6+\frac{1}{32}((3-a)^6+(3-b)^6+(3-c)^6) \ge 3+\frac{192}{39}=9$$

Solution 2 by Tran Hong-Vietnam

Let $f(t)=t^6+\frac{1}{32}(t-3)^6, 0<t<3 \Rightarrow f''(t)=30\left\{t^4+\frac{(t-3)^4}{32}\right\}>0\ (0<t<3)$

Using Jensen's inequality, we have: $LHS=f(a)+f(b)+f(c) \ge 3f\left(\frac{a+b+c}{3}\right)=3f(1)$

$=3\left(1^6+\frac{1}{32}\cdot 2^6\right)=3\cdot 3=9$***. Proved. Equality*** $\Leftrightarrow a=b=c=1.$

Solution 3 by Sudhir Jha-Kolkata-India

Considering three numbers a, b, c ***and applying*** m^{th} ***power theorem by taking*** $m=6$***,***

we get $\frac{a^6+b^6+c^6}{3} \ge \left(\frac{a+b+c}{3}\right)^6 \Rightarrow a^6+b^6+c^6 \ge 3$ ***(i)*** $\because (a+b+c=3)$

(Equality holds when $a=b=c=1$***)***

Considering three numbers $b+c, c+a$ ***&*** $a+b$ ***and applying mth power theorem by***

taking $m=6$***, we get:*** $\frac{(b+c)^6+(c+a)^6+(a+b)^6}{3} \ge \left(\frac{b+c+c+a+a+b}{3}\right)^6$

$$\Rightarrow (3-a)^6+(3-b)^6+(3-c)^6 \ge 3\times 2^6\ [\because a+b+c=3]$$

$$\Rightarrow \frac{1}{32}[(3-a)^6+(3-b)^6+(3-c)^6] \ge 6 \quad \textbf{(ii)}$$

(equality holds when $a=b=c=1$***)***

Adding (i) & (ii), we get: $a^6+b^6+c^6+\frac{1}{32}[(3-a)^6+(3-b)^6+(3-c)^6] \ge 9$

(proved) (the equality holds when $a=b=c=1$***)***

Solution 4 by Soumava Chakraborty-Kolkata-India

Let $f(x)=x^6+\frac{1}{32}(3-x)^6$

$$f'(x)=6x^5-\frac{3(3-x)^5}{16}=0\Leftrightarrow 32x^5=(3-x)^5\Leftrightarrow 2x=3-x\Leftrightarrow x=1$$

$$\Rightarrow f'(1)=0 \text{ and } f''(1)=30(1)^4+\frac{15(3-1)^4}{16}>0$$

$\therefore f(x)$ ***attains a minima at*** $x=1$ ***and*** $\because f(x)$ ***never attains a maxima,***

$\therefore f(x)_{\min}=f(1)=3\Rightarrow x^6+\frac{1}{32}(3-x)^6\ge 3\forall x\in\mathbb{R}$***, equality at*** $x=1\to$ ***(1)***

By (1), $\sum a^6+\frac{1}{32}\sum(3-a)^6=\sum\left(a^6+\frac{1}{32}(3-a)^6\right)\ge\sum(3)=9,$

equality at $a=b=c=1$

466. If $a,b>0$ then:

$$\left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$\left[\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)-\sqrt{ab}\right]^2+ab=$$

$$=\left(\frac{2a}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2+2ab+ab-2\sqrt{ab}\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)+ab\le$$

$$\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2\Rightarrow 2\sqrt{ab}\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)\ge 4ab \text{ true}$$

Because of AM-GM: $2\sqrt{ab}\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)\overset{AM-GM}{\ge}2\sqrt{ab}\cdot 2\sqrt{ab}=4ab$

Solution 2 by Catinca Alexandru-Romania

$$a,b>0;\left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2$$

$x=\frac{a+b}{2};y=\sqrt{ab};z=\frac{2ab}{a+b};x\ge y\ge z>0$ ***since*** $AM\ge GM\ge HM$

$$(z-y+x)^2+y^2\le z^2+x^2\Leftrightarrow(z-y+x)^2-z^2\le x^2-y^2\Leftrightarrow$$

$$\Leftrightarrow(2z-y+x)(x-y)\le(x+y)(x-y)|:(x-y) \text{ if } x\ne y$$

$$\Leftrightarrow 2z-y+x\le x+y\Leftrightarrow 2z\le 2y\Leftrightarrow z\le y \text{ (T)}$$

Now if $x=y\Rightarrow$ ***the equality holds in the AM≥GM*** $\Leftrightarrow a=b$

So, $(2z-y+x)\cdot 0\le (x+y)\cdot 0 \Leftrightarrow 0\le 0$ *(T)*

Solution 3 by Remus Florin Stanca-Romania

$$\left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2 \quad \textit{(1) (we have to prove)}$$

Let $\frac{2ab}{a+b}=x^2$ and $\frac{a+b}{2}=y^2$ such that $x,y>0$

$xy=\sqrt{ab}$ so, the inequality can be written as:

$$(1)\Leftrightarrow (x^2-xy+y^2)^2+x^2y^2\le x^4+y^4\Leftrightarrow$$

$$x^4+x^2y^2+y^4-2x^3y+2x^2y^2-2xy^3+x^2y^2\le x^4+y^4\Leftrightarrow$$

$$4x^2y^2-2x^3y-2xy^3\le 0\Leftrightarrow x^3y+xy^3-2x^2y^2\ge 0\Leftrightarrow$$

$$\left(x\sqrt{xy}-y\sqrt{xy}\right)^2\ge 0 \ \textit{(true)} \Rightarrow \left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2 \quad \textit{(Q.E.D.)}$$

Solution 4 by Marian Ursărescu-Romania

$$\text{Inequality}\Leftrightarrow\left(\frac{2a}{a+b}\right)^2+ab+\left(\frac{a+b}{2}\right)^2-\frac{2\cdot 2ab}{a+b}\sqrt{ab}-2\sqrt{ab}\frac{a+b}{2}$$

$$+2\cdot\frac{2ab}{a+b}\cdot\frac{a+b}{2}+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2\Leftrightarrow$$

$$2ab+2ab\le\frac{4ab\sqrt{ab}}{a+b}+\sqrt{ab}(a+b)4ab\le\frac{4ab\sqrt{ab}}{a+b}+\sqrt{ab}(a+b)\Leftrightarrow$$

$$4\sqrt{ab}\le\frac{4ab}{a+b}+a+b \quad (1)$$

But $\frac{4ab}{a+b}+a+b\ge 2\sqrt{\frac{4ab}{a+b}(a+b)}\Rightarrow\frac{4ab}{a+b}+a+b\ge 4\sqrt{ab}\Rightarrow$ *(1) it's true.*

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\forall a,b>0,\left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\overset{(1)}{\le}\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2$$

Let $H=\frac{2ab}{a+b}, G=\sqrt{ab}, A=\frac{a+b}{2}\therefore (1)\Leftrightarrow (H-G+A)^2+G^2\le H^2+A^2$

$$\Leftrightarrow (H-G+A)^2+G^2\le (H+A)^2-2AH$$

$$\Leftrightarrow (H+A+G)(H+A-G)-(H-G+A)^2\ge 2AH$$

$$\Leftrightarrow (H+A-G)(H+A+G-H+G-A)\ge 2AH\Leftrightarrow G(H+A-G)\ge AH$$

$$\Leftrightarrow HG+AG-G^2-AH\ge 0\Leftrightarrow G(A-G)-H(A-G)\ge 0$$

$$\Leftrightarrow (A-G)(G-H)\ge 0\rightarrow \text{true}\because A\ge G\ge H \ \textit{(Proved)}$$

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467. **Let $a, b, c > 0$. Prove:**

$$\frac{\left((a+b)(b+c)(c+a)\right)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \ge 16\sqrt{2}abc$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because (x+y)^2 \ge 4xy \therefore (a^2+b^2+2ab)^2 \ge 4(a^2+b^2)2ab \Rightarrow (a+b)^4 \ge 8ab(a^2+b^2)$$

$$\Rightarrow (a+b)^2 \ge 2\sqrt{2}\sqrt{ab}\sqrt{a^2+b^2} \Rightarrow \frac{(a+b)^2}{\sqrt{a^2+b^2}} \overset{(1)}{\ge} 2\sqrt{2}\sqrt{ab}$$

Similarly, $\frac{(b+c)^2}{\sqrt{b^2+c^2}} \overset{(2)}{\ge} 2\sqrt{2}\sqrt{bc}, \frac{(c+a)^2}{\sqrt{c^2+a^2}} \overset{(2)}{\ge} 2\sqrt{2}\sqrt{ca}$

(1), (2), (3) $\Rightarrow \frac{\left((a+b)(b+c)(c+a)\right)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \ge 16\sqrt{2}abc$ ***(Proved)***

Solution 2 by Tran Hong-Vietnam

Inequality $\Leftrightarrow \{(a+b)(b+c)(c+a)\}^4 \ge 2\cdot 4^4(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a^2)$

$$\Leftrightarrow (a+b)^4(b+c)^4(c+a)^2 \ge 512(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a^2) \text{ (*)}$$

$$(a+b)^4 \ge 8ab(a^2+b^2) \text{ (1)}$$

$\Leftrightarrow (a-b)^4 \ge 0$ ***(true). Same:***

$$(b+c)^4 \ge 8bc(b^2+c^2) \text{ (2);}$$

$$(c+a)^4 \ge 8ca \ge (c^2+b^2) \text{ (3);}$$

From (1), (2), (3) we have: $LHS_{(*)} \ge 8^3(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a)^2 = RHS_{(*)}$

468. **If $x, y, z > 0$, then:**

$$(x+y+z)\left(\frac{2x+y+z}{x^2+xy+yz+zx} + \frac{2y+z+x}{y^2+yz+zx+xy} + \frac{2z+x+y}{z^2+zx+xy+yz}\right) \ge 9$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Amit Dutta-Jamshedpur-India

For $a, b, c > 0$***; AM*** $\ge$ ***HM:*** $\left(\frac{a+b+c}{3}\right) \ge \frac{3}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$

$$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9 \Rightarrow (a+b+c)\left(\frac{2}{a}+\frac{2}{b}+\frac{2}{c}\right)\ge 18$$

$$\Rightarrow (a+b+c)\left[\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{c}+\frac{1}{a}\right)\right]\ge 18$$

$$\Rightarrow (a+b+c)\left[\left(\frac{a+b}{ab}\right)+\left(\frac{b+c}{bc}\right)+\left(\frac{c+a}{ac}\right)\right]\ge 18$$

Put $a=x+y, b=y+z, c=x+z$

$$\Rightarrow 2(x+y+z)\left[\frac{2x+y+z}{x^2+xy+yz+zx}+\frac{2y+z+x}{y^2+yz+zx+xy}+\frac{2z+x+y}{z^2+zx+xy+yz}\right]\ge 18$$

$$\Rightarrow (x+y+z)\left[\sum_{cyc}\left(\frac{2x+y+z}{x^2+xy+yz+zx}\right)\right]\ge 9 \text{ (proved)}$$

Solution 2 by Michael Sterghiou-Greece

$$(x+y+z)\left(\sum_{cyc}\frac{2x+y+z}{x^2+xy+yz+zx}\right)\ge 9 \quad (1)$$

Let $(a,b,c)=(x+y,y+z,z+x)$ ***(1)*** $\rightarrow \frac{\sum_{cyc}a}{2}\left(\sum_{cyc}\frac{a+b}{ab}\right)\ge 9\rightarrow$

$\rightarrow \frac{\sum_{cyc}a}{2}\cdot\frac{\sum_{cyc}(a+b)c}{abc}\ge 9\rightarrow \sum_{cyc}a\cdot\sum_{cyc}ab\ge 9$ ***which holds***

$$\left(\sum_{cyc}a\ge 3(abc)^{\frac{1}{3}},\sum_{cyc}ab\ge 3(abc)^{\frac{2}{3}}\right)$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z>0$***, we get that:*** $(x+y+z)\left[\frac{2x+y+z}{x^2+xy+yz+zx}+\frac{2y+z+x}{y^2+xy+yz+zx}+\frac{2z+xy+z}{z^2+xy+yz+zx}\right]$

$$=(x+y+z)\left[\frac{(x+y)+(x+z)}{(x+y)(x+z)}+\frac{(x+y)+(z+y)}{(y+x)(y+z)}+\frac{(z+x)+(z+y)}{(z+x)(z+y)}\right]$$

$$=(x+y+z)\left[\frac{1}{x+y}+\frac{1}{x+z}+\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{x+z}+\frac{1}{z+y}\right]$$

$$=6+\frac{z}{x+y}+\frac{y}{x+z}+\frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x}+\frac{x}{z+y}=6+2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)$$

$\ge 6+3 \because \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\ge\frac{3}{2}=9$***. Therefore, it's true.***

Solution 4 by Soumava Chakraborty-Kolkata-India

$$(x+y+z)\left(\frac{2x+y+z}{x^2+xy+yz+zx}+\frac{2y+z+x}{y^2+yz+zx+xy}+\frac{2z+x+y}{z^2+zx+xy+yz}\right)\overset{(1)}{\ge} 9$$

(1) $\Leftrightarrow \sum x\left\{\frac{x+y+z+x}{(x+y)(z+x)}+\frac{x+y+y+z}{(x+y)(y+z)}+\frac{z+x+y+z}{(z+x)(y+z)}\right\}\geq 9$

$\Leftrightarrow 2\sum x\left(\sum\frac{1}{x+y}\right)\overset{(2)}{\geq} 9$. *Let* $x+y=a, y+z=b, z+x=c$

Then, $a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ *are sides of a triangle with semi-perimeter, circumradius & in-radius* $= s, R, r$ *respectively (say)*

(2) becomes $2S\left(\sum\frac{1}{a}\right)\geq 9$

$$\left(\because 2\sum x=\sum a=2S\Rightarrow \sum x=S\Rightarrow z=s-a, x=s-b, y=s-c\right)$$

$$\Leftrightarrow 2S\left(\frac{\sum ab}{4Rrs}\right)\geq 9\Leftrightarrow S^2+4Rr+r^2\geq 18Rr\Leftrightarrow S^2-(16Rr-5r^2)+2r(R-2r)\geq 0$$

$\rightarrow$ *true* $\because S^2\overset{Gerretsen}{\geq} 16Rr-5r^2$ *&* $R\overset{Euler}{\geq} 2r$ *(proved)*

469. Let $x,y,z>0$ and $xyz=1$. Prove:

$$8\sum x\sqrt{\sum x}\leq 3\sqrt{3}\prod(x+y)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$8\left(\sum x\right)\sqrt{\sum x}\leq 3\sqrt{3}\prod(x+y)\Leftrightarrow 27\left(\prod(x+y)\right)^2\overset{(1)}{\geq} 64\left(\sum x\right)^3$$

$$\because \prod(x+y)\geq\frac{8}{9}\left(\sum x\right)\left(\sum xy\right)\therefore 27\left(\prod(x+y)\right)^2\geq\frac{27\cdot 64}{81}\left(\sum x\right)^2\left(\sum xy\right)^2$$

$\geq\frac{64}{3}(\sum x)^2(3xyz(\sum x))=64(\sum x)^3(\because \prod x=1)\Rightarrow$ *(1) is true (Proved)*

470. If $x,y,z>0$ then:

$$\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2\cdot\sum\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}\right)^2\cdot\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2\geq\left(\sum\left(\frac{x}{y}+\frac{y}{x}\right)\right)^3$$

Proposed by Daniel Sitaru-Romania

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Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Since $$\left(\frac{\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)^2}{3}\right)^3$$

$$=\left(\frac{\left(\frac{x^8}{y^8}+\frac{y^8}{z^8}+\frac{z^8}{x^8}+\frac{x^8}{z^8}+\frac{z^8}{y^8}+\frac{y^8}{x^8}+2\left(\begin{array}{c}\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4+\\+\frac{x^8}{(yz)^4}+\frac{y^8}{(xz)^4}+\frac{z^8}{(xy)^4}+\frac{(xy)^4}{z^8}+\frac{(yz)^4}{x^8}+\frac{(zx)^4}{y^8}+1+1+1\end{array}\right)\right)}{3}\right)^3$$

$$\geq\left(\frac{\begin{array}{c}\frac{x^8}{y^8}+\frac{y^8}{z^8}+\frac{z^8}{x^8}+\frac{x^8}{z^8}+\frac{z^8}{y^8}+\frac{y^8}{x^8}+2\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)+\frac{x^8}{(yz)^4}+\frac{y^8}{(zx)^4}+\frac{z^8}{(xy)^4}+\\+\frac{(xy)^4}{z^8}+\frac{(yz)^4}{x^8}+\frac{(zx)^4}{y^8}+2\left(\frac{x^4}{(yz)^2}+\frac{y^4}{(zx)^2}+\frac{z^4}{(xy)^2}+\frac{(xy)^2}{z^4}+\frac{(yz)^2}{x^4}+\frac{(zx)^2}{y^4}\right)\end{array}}{3}\right)^3$$

$$\geq\left(\frac{\begin{array}{c}4\left(\frac{x}{2}+\frac{z}{4}+\frac{y}{x}\right)+\frac{x^8}{y^8}+\frac{y^8}{z^8}+\frac{z^8}{x^8}+\frac{x^8}{z^8}+\frac{z^8}{y^8}+\frac{y^8}{x^8}+\frac{x^8}{(yz)^4}+\frac{y^8}{(zx)^4}+\frac{z^8}{(xy)^4}+\frac{(xy)^4}{z^8}+\frac{(yz)^4}{x^8}+\frac{(zx)^8}{y^8}+\\+2\left(\frac{x^4}{(yz)^2}+\frac{y^4}{(zx)^2}+\frac{z^4}{(xy)^2}+\frac{x^4}{z^4}+\frac{z^4}{y^4}+\frac{y^4}{x^4}\right)\end{array}}{3}\right)^3$$

$$\geq\left(\frac{4\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)+6\left(\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right)+\frac{x^8}{y^8}+\frac{y^8}{z^8}+\frac{z^8}{x^8}+\frac{x^8}{(yz)^4}+\frac{y^8}{(zx)^4}+\frac{z^8}{(xy)^4}}{3}\right)^3$$

$$\geq\left(\frac{6\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+6\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)}{3}\right)^3=\left(\left(\frac{x}{y}+\frac{y}{z}\right)^2+\left(\frac{y}{z}+\frac{z}{x}\right)^2+\left(\frac{z}{x}+\frac{x}{y}\right)^2\right)^3$$

and since $$\left(\left(\frac{x}{y}\right)^8+\left(\frac{y}{x}\right)^8\right)^2+\left(\left(\frac{y}{z}\right)^8+\left(\frac{z}{y}\right)^8\right)^2+\left(\left(\frac{z}{x}\right)^8+\left(\frac{x}{z}\right)^8\right)^2\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)^2+\left(\left(\frac{y}{z}\right)^4+\left(\frac{z}{y}\right)^4\right)^2+$$
$$+\left(\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4\right)^2\left(\left(\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2\right)^2+\cdots\right)$$

$$\geq \frac{\begin{pmatrix}\left(\frac{x}{y}\right)^8+\left(\frac{y}{z}\right)^8+\left(\frac{z}{x}\right)^8+\left(\frac{x}{z}\right)^8+\left(\frac{z}{y}\right)^8+\left(\frac{y}{x}\right)^8\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)\cdot\\ \cdot\left(\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{y}\right)^2+\left(\frac{y}{x}\right)^2\right)\end{pmatrix}^2}{3^3}$$

$$\geq \frac{\left(\left(\left(\frac{x}{y}\right)^6+\left(\frac{y}{z}\right)^6+\left(\frac{z}{x}\right)^6+\left(\frac{x}{z}\right)^6+\left(\frac{z}{4}\right)^6+\left(\frac{y}{x}\right)^6\right)\left(\left(\frac{x}{4}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)\left(\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{y}\right)^2+\left(\frac{y}{x}\right)^2\right)\right)^2}{3^3}$$

$$\geq \frac{\left(\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{x}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)^3\right)^2}{27}=\left(\frac{\left(\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\left(\frac{x}{z}\right)^4+\left(\frac{z}{y}\right)^4+\left(\frac{y}{x}\right)^4\right)^2}{3}\right)^3$$

Therefore, it's true

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2=\sum\left(\frac{x^{16}}{y^{16}}+\frac{y^{16}}{x^{16}}+2\right)\overset{(a)}{=}\sum\frac{x^{16}}{y^{16}}+\sum\frac{y^{16}}{x^{16}}+6$$

Now, $\sum\frac{x^{16}}{y^{16}}=\frac{x^{16}}{y^{16}}+\frac{y^{16}}{z^{16}}+\frac{z^{16}}{x^{16}}\geq\frac{x^8}{z^8}+\frac{y^8}{x^8}+\frac{z^8}{y^8}\,(\because a^2+b^2+c^2\geq ab+bc+ca)$

$$\geq\frac{y^4}{z^4}+\frac{z^4}{x^4}+\frac{x^4}{y^4}\left(\because\sum a^2\geq\sum ab\right)\overset{(1)}{\geq}\frac{y^2}{x^2}+\frac{z^2}{y^2}+\frac{x^2}{z^2}\left(\because\sum a^2\geq\sum ab\right)=\sum\frac{y^2}{x^2}$$

Similarly, $\sum\frac{y^{16}}{x^{16}}\overset{(2)}{\geq}\sum\frac{x^2}{y^2}\therefore$ ***(1)+(2)***$\Rightarrow\sum\frac{x^{16}}{y^{16}}+\sum\frac{y^{16}}{x^{16}}+6$

$$\overset{(b)}{\geq}\sum\frac{x^2}{y^2}+\sum\frac{y^2}{x^2}+6=\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\right)=\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

(a), (b)$\Rightarrow\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2\overset{(i)}{\geq}\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$. ***Again,*** $\sum\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}\right)^2=\sum\frac{x^8}{y^8}+\sum\frac{y^8}{x^8}+6$

$\overset{(ii)}{\geq}\sum\frac{x^2}{y^2}+\sum\frac{y^2}{x^2}+6$ ***(proceeding in previous fashion)*** $=\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$

Also, $\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2=\sum\frac{x^4}{y^4}+\sum\frac{y^4}{x^4}+6\geq\sum\frac{y^2}{x^2}+\sum\frac{x^2}{y^2}+6$

$$\Rightarrow\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2\overset{(iii)}{\geq}\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

(i).(ii).(iii)$\Rightarrow$ ***given inequality is true (proved)***

471. If $x,y,z>0, xyz=9$ then:

$$\sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2+\sqrt{x}\cdot\left(\frac{y+z}{\sqrt{y}+\sqrt{z}}\right)^2+\sqrt{y}\cdot\left(\frac{z+x}{\sqrt{z}+\sqrt{x}}\right)^2\ge 9$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dimitris Kastriotis-Greece

$$(x+y)\overset{C-S}{\ge}\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)^2\to\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge\frac{1}{4}\left(\sqrt{x}+\sqrt{y}\right)^2\ge\sqrt{xy}$$

$$\to\sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge\sqrt{xyz}=\sqrt{9}=3\to\sum\sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge 9$$

Solution 2 by El Harati-Youness-Morocco

$$\sum_{cyc}\sqrt{x}\left(\frac{y+z}{\sqrt{y}+\sqrt{z}}\right)^2\ge\sum_{cyc}\frac{\sqrt{x}}{2}\cdot\frac{(z+y)^2}{z+y}=\sum_{cyc}\frac{\sqrt{x}}{2}(z+y)\ge\sum\sqrt{xyz}=3\times 3=9$$

Solution 3 by Lahiru Samarakoon-Sri Lanka

$$x+y\ge\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)^2\ \therefore\ \textit{LHS}\ge\sum\frac{\sqrt{z}}{4}\left[\frac{\left(\sqrt{x}+\sqrt{y}\right)^2}{\left(\sqrt{x}+\sqrt{y}\right)}\right]^2\ge\sum\frac{\sqrt{z}}{4}\left(2\sqrt[4]{xy}\right)^2=\sum\sqrt{xyz}$$

$$=3\sqrt{xyz}\ \ \left(\because\ \sqrt{xyz}=9\right)=9\ \ \textit{(proved)}$$

Solution 4 by Remus Florin Stanca-Romania

$$\frac{x+y}{\sqrt{x}+\sqrt{y}}\ge\frac{\sqrt{x}+\sqrt{y}}{2}\ge\sqrt[4]{xy}\Rightarrow\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge\sqrt{xy}\Rightarrow\sqrt{z}\cdot\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge\sqrt{xyz}=3$$

$$\Rightarrow\sum\sqrt{z}\cdot\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2\ge 9\Rightarrow\ \textit{Q.E.D.}$$

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\textit{LHS}=\sum\sqrt{z}\frac{(x+y)^2}{x+y+2\sqrt{xy}}\overset{A-G}{\ge}\sum\sqrt{z}\frac{(x+y)^2}{2(x+y)}=\sum\sqrt{z}\left(\frac{x+y}{2}\right)\overset{A-G}{\ge}\sum\sqrt{z}\sqrt{xy}=3\sqrt{xyz}=3\sqrt{9}=9$$

Solution 6 by Marian Ursărescu-Romania

From Cauchy's inequality we have: $2(x+y)\ge\left(\sqrt{x}+\sqrt{y}\right)^2\Rightarrow$

$\frac{x+y}{\left(\sqrt{x}+\sqrt{y}\right)^2}\ge\frac{1}{2}$ ***(1). From (1) inequality becomes:*** $\frac{\sqrt{z}(x+y)}{2}+\frac{\sqrt{x}(y+z)}{2}+\frac{\sqrt{y}(z+x)}{2}\ge 9$ ***(2)***

But $x+y\geq 2\sqrt{xy}$ (3). Form (2)+(3) we must show:

$$\left.\begin{array}{c}\sqrt{xyz}+\sqrt{xyz}+\sqrt{xyz}\geq 9\\ But\ xyz=9\end{array}\right\}\Rightarrow \text{it's true.}$$

472. If $a,b,c,d>0, a+b+c+d=4$ then:

$$\frac{(a+1)(b+1)(c+1)(d+1)}{abcd}\geq 16$$

Proposed by Daniel Sitaru – Romania

Proposed by Marian Ursărescu – Romania

We must show: $(a+1)(b+1)(c+1)(d+1)\geq 16abcd$ (1)

Let $a=\frac{4x}{x+y+z+t}, b=\frac{4y}{x+y+z+t}, c=\frac{4z}{x+y+z+t}, d=\frac{4t}{x+y+z+t}$

$$(1)\Leftrightarrow \prod\left(\frac{4x}{x+y+z+t}+1\right)\geq 16\cdot 4^4\frac{xyzt}{(x+y+z+t)^4}\Leftrightarrow$$

$$\prod(5x+y+z+t)\geq 4^6xyzt \quad (2)$$

$$\text{But}\ \left.\begin{array}{l}5x+y+z+t\geq 8\sqrt[8]{x^5yzt}\\ x+5y+z+t\geq 8\sqrt[8]{xy^5zt}\\ x+y+5z+t\geq 8\sqrt[8]{xyz^5t}\\ x+y+z+5t\geq 8\sqrt[8]{xyzt^5}\end{array}\right\}\Rightarrow \prod(5x+y+z+t)\geq 8^4xyzt\Rightarrow (2)\ \text{it's true.}$$

473. Let $a,b,c\in(0;+\infty)\wedge a+b+c=3$. Prove:

$$\sqrt{a^8+\frac{1}{a^2}+\frac{1}{a}}+\sqrt{b^8+\frac{1}{b^2}+\frac{1}{b}}+\sqrt{c^8+\frac{1}{c^2}+\frac{1}{c}}\geq 3\sqrt{3}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

By Minkowski inequality: $\sqrt{a^8+\frac{1}{a^2}+\frac{1}{a}}+\sqrt{b^8+\frac{1}{b^2}+\frac{1}{b}}+\sqrt{c^8+\frac{1}{c^2}+\frac{1}{c}}\geq$

$$\geq\sqrt{(a^4+b^4+c^4)^2+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2+\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^2}\quad (1)$$

$$a^4+b^4+c^4\geq\frac{(a+b+c)^4}{27}=\frac{3^4}{27}=3$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{81}{(a+b+c)^2}=9$$

$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\geq\frac{9}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\geq\frac{9}{\sqrt{3(a+b+c)}}=\frac{9}{\sqrt{3\cdot 3}}=3$$

$$\Rightarrow LHS\geq\sqrt{3^2+9+3^2}=3\sqrt{3}\leftrightarrow a=b=c=1.$$

474. If $a,b,c\geq 0$ then:

$$e^{2\sqrt{3}(a+b+c)}\geq\left((a^2+a+1)(b^2+b+1)(c^2+c+1)\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\text{Let } F(x)=e^{-\frac{2x}{\sqrt{3}}}(x^2+x+1)$$

$$F'(x)=e^{-\frac{2x}{\sqrt{3}}}(2x+1)+(x^2+x+1)e^{-\frac{2x}{\sqrt{3}}}\left(-\frac{2}{\sqrt{3}}\right)$$

$$=e^{-\frac{2x}{\sqrt{3}}}\{(2x+1)-2\sqrt{3}\,(x^2+x+1)\}=-e^{-\frac{2x}{\sqrt{3}}}\{2x^2+(2-2\sqrt{3})x+(2-\sqrt{3})\}$$

$$=-\frac{2}{\sqrt{3}}e^{-\frac{2x}{\sqrt{3}}}\left\{x^2+(1-\sqrt{3})x+\left(\frac{2-\sqrt{3}}{2}\right)\right\}=-\frac{2}{\sqrt{3}}e^{-\frac{2x}{\sqrt{3}}}\left\{x-\left(\frac{\sqrt{3}-1}{2}\right)\right\}^2\leq 0$$

$$\Rightarrow F'(x)\leq 0\Rightarrow F(x)\text{ is a decreasing function; }x\geq 0\Rightarrow F(x)\leq F(0)$$

$$\Rightarrow e^{-\frac{2x}{\sqrt{3}}}(x^2+x+1)\leq 1\Rightarrow(x^2+x+1)\leq e^{\frac{2x}{\sqrt{3}}}$$

Putting $x=a,b,c$ and multiplying, then, we get:

$$(a^2+a+1)(b^2+b+1)(c^2+c+1)\leq e^{\frac{2}{\sqrt{3}}(a+b+c)}$$

$$\text{or }\{(a^2+a+1)(b^2+b+1)(c^2+c+1)\}^3\leq e^{2\sqrt{3}(a+b+c)}\text{ proved}$$

Solution 2 by Boris Colakovic-Belgrade-Serbie

$$e^{2\sqrt{3}(a+b+c)}\geq\left((a^2+a+1)(b^2+b+1)(c^2+c+1)\right)^3\Leftrightarrow$$

$$\Leftrightarrow e^{\frac{2\sqrt{3}}{3}(a+b+c)}\geq(a^2+a+1)(b^2+b+1)(c^2+c+1)\Leftrightarrow$$

$$\Leftrightarrow\frac{2\sqrt{3}}{3}(a+b+c)\geq\ln(a^2+a+1)+\ln(b^2+b+1)+\ln(c^2+c+1)\quad\textit{(1)}$$

$$y = \frac{2\sqrt{3}}{3}x - \ln(x^2 + x + 1)$$

$$y(0) = 0; y'_x = \frac{2\sqrt{3}}{3} - \frac{2x+1}{x^2+x+1} = \frac{2\sqrt{3}x^2 + (2\sqrt{3}-6)x + 2\sqrt{3} - 3}{3(x^2+x+1)} =$$

$$= \frac{\left(x - \frac{3-\sqrt{3}}{2\sqrt{3}}\right)^2}{3(x^2+x+1)} \geq 0; \forall x \in (0, \infty)$$

$f(x)$ ***is increasing function*** $\Rightarrow f(x) \geq f(0) \Rightarrow \frac{2\sqrt{3}}{3}x - \ln(x^2 + x + 1) \geq 0$

$$\left.\begin{array}{l} \frac{2\sqrt{3}}{3}a \geq \ln(a^2 + a + 1) \\ \frac{2\sqrt{3}}{3}b \geq \ln(b^2 + b + 1) \\ \frac{2\sqrt{3}}{3}c \geq \ln(c^2 + c + 1) \end{array}\right\}_{\oplus} \Rightarrow \textit{(1)}$$

Solution 3 by Lazaros Zachariades-Thessaloniki-Greece

$f(x) = \frac{2\sqrt{3}}{3}x - \ln(x^2 + x + 1), x \geq 0, f(0) = 0; f'(x) = \frac{2\sqrt{3}}{3} - \frac{2x+1}{x^2+x+1} \geq \frac{2\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} = 0$

$f'(x) \geq 0$, "=" $x = \frac{\sqrt{3}-1}{2}$

x	0	$\frac{\sqrt{3}-1}{2}$	$+\infty$
f'	+ + + + + + + +	0	+ + + + + + + +
f	↗	↗	↗

So, $f(x) \geq f(0); \forall x \geq 0;$ $f(x) \geq 0;$ $\frac{2\sqrt{3}x}{3} - \ln(x^2 + x + 1) \geq 0$. ***Thus,***

$$\frac{2\sqrt{3a}}{3} - \ln(a^2 + a + 1) \geq 0$$

$$\frac{2\sqrt{3}}{3}b - \ln(b^2 + b + 1) \geq 0$$

$$\frac{2\sqrt{3}}{3}c - \ln(c^3 + c + 1) \geq 0$$

$$- - - - - - - - - - - - - - - (+)$$

$$\frac{2\sqrt{3}}{3}\cdot\sum a \geq \sum \ln(a^2+a+1) \Leftrightarrow$$

$$2\sqrt{3}\sum a \geq 3\ln\left(\prod(a^2+a+1)\right) \Leftrightarrow$$

$$e^{2\sqrt{3}\sum a} \geq \left(\prod(a^2+a+1)\right)^3$$

$$g(x)=\frac{2x+1}{x^2+x+1}, x\geq 0;\ g'(x)=\frac{-2x^2-2x+1}{(x^2+x+1)^2};\ g'(x)=0 \Leftrightarrow x=\frac{-1+\sqrt{3}}{2}$$

x	0	$\frac{\sqrt{3}-1}{2}$	$+\infty$
g'	+ + + + + + + +	0	− − − − − − − −
g	↗		↘

$$g(x)\geq g\left(\frac{\sqrt{3}-1}{2}\right)=\frac{2\sqrt{3}}{3}$$

475. If $a,b,c>0$ then:

$$(a+b+c)\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)\geq\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $(a+b+c)(a^{11}c^{10}+b^{11}a^{10}+c^{11}b^{10})\geq(a^6c^5+b^6a^5+c^6b^5)^2$ ***(1)***

From Cauchy's inequality we have:

$$\left(\left(a^{\frac{1}{2}}\right)^2+\left(b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}\right)^2\right)\left(\left(a^{\frac{11}{2}}\right)^2(c^5)^2+\left(b^{\frac{11}{2}}\right)^2(a^5)^2+\left(c^{\frac{11}{2}}\right)^2(b^5)^2\right)\geq$$

$$\geq(a^6b^5+b^6a^5+c^6c^5)^2 \Rightarrow \textbf{\textit{(1) it's true.}}$$

Solution 2 by Catinca Alexandru-Romania

$$(a+b+c)\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)=\left((\sqrt{a})^2+(\sqrt{b})^2+(\sqrt{c})^2\right)\left(\left(\frac{\sqrt{a}}{b^5}\right)^2+\left(\frac{\sqrt{b}}{c^5}\right)^2+\left(\frac{\sqrt{c}}{a^5}\right)^2\right)\geq$$

$$\overset{CBS}{\geq}\left(\frac{\sqrt{a}\cdot\sqrt{a}}{b^5}+\frac{\sqrt{b}\cdot\sqrt{b}}{c^5}+\frac{\sqrt{c}\cdot\sqrt{c}}{a^5}\right)^2=\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)^2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$(a+b+c)\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)=\frac{a^2}{b^{10}}+\frac{b^2}{c^{10}}+\frac{c^2}{a^{10}}+\left(\frac{a}{b^9}+\frac{ab}{c^{10}}\right)+\left(\frac{ac}{b^{10}}+\frac{c}{a^9}\right)+\left(\frac{b}{c^9}+\frac{bc}{a^{10}}\right)$$

$$\geq\frac{a^2}{b^{10}}+\frac{b^2}{c^{10}}+\frac{c^2}{a^{10}}+\frac{2a}{b^4c^5}+\frac{2c}{b^5a^4}+\frac{2b}{c^4a^{10}}=\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)^2$$

. Equality when $a=b=c$.

476. If $a,b\geq 1$ then:

$$\left|\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right|\leq(a-b)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b\geq 1$***, we give*** $a=x^{15}\leftrightarrow a^{\frac{1}{15}}=x$; $b=y^{15}\leftrightarrow b^{\frac{1}{15}}=y$***. Consider as follows***

(1) $x^{30}+y^{30}+x^{13}y^{17}+x^{17}y^{13}\geq x^{27}y^3+x^3y^{27}+x^{13}y^{17}x^{17}y^{13}$

$$\geq 2(x^{22}y^8+x^8y^{22})\geq x^{22}y^8+x^8y^{22}+2x^{15}y^{15}$$

Hence, $a^2+b^2+(a^{13}b^{17})^{\frac{1}{15}}+(a^{17}b^{13})^{\frac{1}{15}}\geq(a^{22}b^8)^{\frac{1}{15}}+(a^8b^{22})^{\frac{1}{15}}+2ab$

$$(a^{22}b^8)^{\frac{1}{15}}+(a^8b^{22})^{\frac{1}{15}}-(a^{13}b^{17})^{\frac{1}{15}}-(a^{17}b^{13})^{\frac{1}{15}}\leq a^2+b^2-2ab$$

$$\left((a^{10}b^5)^{\frac{1}{15}}-(a^5b^{10})^{\frac{1}{15}}\right)\left((a^{12}b^3)^{\frac{1}{15}}-(a^3b^{12})^{\frac{1}{15}}\right)\leq(a-b)^2$$

$$\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\leq(a-b)^2$$

(2) $x^{30}+y^{30}+x^{22}y^8+x^8y^{22}\geq x^{12}y^{18}+x^{18}y^{12}+x^{22}y^8+x^8y^{22}$

$$\geq 2(x^{17}y^{13}+x^{13}y^{17})\geq 2x^{15}y^{15}+x^{17}y^{13}+x^{13}y^{17}$$

Hence $a^2+b^2+(a^{22}b^8)^{\frac{1}{15}}+(a^8b^{22})^{\frac{1}{15}}\geq 2ab+(a^{17}b^{13})^{\frac{1}{15}}+(a^{13}b^{17})^{\frac{1}{15}}$

$$(a^{28}b^8)^{\frac{1}{15}}+(a^8b^{22})^{\frac{1}{15}}-(a^{17}b^{13})^{\frac{1}{15}}-(a^{13}b^{17})^{\frac{1}{15}}\geq 2ab-a^2-b^2$$

$$\left((a^{10}b^5)^{\frac{1}{15}}-(a^5b^{10})^{\frac{1}{15}}\right)\left((a^{12}b^3)^{\frac{1}{15}}-(a^3b^{12})^{\frac{1}{15}}\right)\geq-(a-b)^2$$

$$\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\geq-(a-b)^2$$

From (1) and (2) we get: $\left|\left(\sqrt[3]{a^2b}-\sqrt[5]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right|\le(a-b)^2$

Therefore, it's true.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c,d\ge 1$***;*** $2\sqrt{abcd}\le ad+bc\Rightarrow -ad-bc\le -2\sqrt{abcd}$

$$\Rightarrow ac+bd-ad-bc\le ac+bd-2\sqrt{abcd}\Rightarrow (a-b)(c-d)\le\left(\sqrt{ac}-\sqrt{bd}\right)^2$$

and since $a^{\frac{22}{15}}b^{\frac{8}{15}}+a^{\frac{8}{15}}b^{\frac{22}{15}}\le a^{\frac{30}{15}}+b^{\frac{30}{15}}$

$$\Rightarrow a^{\frac{22}{15}}b^{\frac{8}{15}}-2ab+a^{\frac{8}{15}}b^{\frac{22}{15}}\le a^2-2ab+b^2$$

$$\Rightarrow (a^2b)^{\frac{1}{3}}\left((a^4b)^{\frac{1}{5}}-2\right)\sqrt{(a^2b)^{\frac{1}{3}}(a^4b)^{\frac{1}{5}}(ab^2)^{\frac{1}{3}}(ab^4)^{\frac{1}{5}}}+(ab^2)^{\frac{1}{3}}(ab^4)^{\frac{1}{5}}\le(a-b)^2$$

$$\Rightarrow\left(\sqrt{(a^2b)^{\frac{1}{3}}(a^4b)^{\frac{1}{15}}}-\sqrt{(ab^2)^{\frac{1}{3}}(ab^4)^{\frac{1}{5}}}\right)^2\le(a-b)^2$$

$$\Rightarrow\left|\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right|$$

$$\le\left|\left(\sqrt{(a^2b)^{\frac{1}{3}}(a^4b)^{\frac{1}{5}}}\right)-\left(\sqrt{(ab^2)^{\frac{1}{3}}(ab^4)^{\frac{1}{5}}}\right)^2\right|\le(a-b)^2.$$ ***Therefore, it's true.***

Solution 3 by Michael Sterghiou-Greece

$$\left|\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right|\le(a-b)^2\quad (1)$$

LHS of (1) is always ≥ 0 ***so, we can get rid of the absolute value.***

(1) is homogeneous so, WLOG, assume $ab=1$***. Then (1) becomes***

$$\left(a^{\frac{1}{3}}-\frac{1}{a^{\frac{1}{3}}}\right)\left(a^{\frac{1}{5}}-\frac{1}{a^{\frac{1}{5}}}\right)-\left(a-\frac{1}{a}\right)^2\le 0\quad (2)$$

Let $a^{\frac{1}{15}}=x\ge 1$ ***(2)***$\rightarrow$

$$\left(x^5-\frac{1}{x^5}\right)\left(x^3-\frac{1}{x^3}\right)-\left(a^{15}-\frac{1}{a^{15}}\right)^2\le 0\rightarrow x^{30}+\frac{1}{x^{30}}+x^2+\frac{1}{x^2}-x^8-\frac{1}{x^8}+2\ge 0$$

But $x^{30}\ge x^8, x^2\ge\frac{1}{x^8}$ ***so, we are done!***

Solution 4 by Long Nhat-Vietnam

WLOG, suppose that $a\ge b\ge 0$***. Let*** $b=ka\Rightarrow k\le 1$***. The inequality can be written:***

$$\left|\left(\sqrt[3]{k}-\sqrt[3]{k^2}\right)\left(\sqrt[5]{k}-\sqrt[5]{k^4}\right)\right|\le (k-1)^2 \Leftrightarrow \left(\sqrt[3]{k}-\sqrt[3]{k^2}\right)\left(\sqrt[5]{k}-\sqrt[5]{k^4}\right)\le (k-1)^2$$

Let** $k=x^{15}$. **Hence, we need to prove: $(x^5-x^{10})(x^3-x^{12})\le (x^{15}-1)^2$

$$\Leftrightarrow (x-1)^2(x+1)^2(x^{26}+2x^{24}+3x^{22}+4x^{20}+4x^{18}+4x^{16}+4x^{14}+x^{13}+$$
$$+4x^{12}+4x^{10}+4x^8+4x^6+3x^4+2x^2+1)\ge 0$$ ***(True)***

477. **If** $a,b,c>0, a+b+c\le 1$ **then:**

$$\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c\ge\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Remus Florin Stanca – Romania

$$\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)(a+b+c)\ge\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$$

$$\Rightarrow \frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\ge\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}\Rightarrow \frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c\ge$$

$$\ge\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}+1-a-b-c$$ ***(1). We note*** $x=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ ***and*** $y=a+b+c$

$$y\le 1\Rightarrow 1-y\ge 0\Rightarrow (x-y)^2(1-y)\ge 0\Rightarrow (x^2+y^2-2xy)(1-y)\ge 0\Rightarrow$$

$$\Rightarrow x^2(1-y)+y^2(1-y)+2xy(y-1)\ge 0\Rightarrow$$

$$\Rightarrow x^2-x^2y+y^2-y^3+2xy^2-2xy\ge 0 > x^2\ge x^2y+2xy+y^3-2xy^2-y^2 >$$

$$x^2+y-y^2\ge x^2y+y+y^3+2xy-2xy^2-2y^2$$

$$\Rightarrow \frac{x^2}{y}+1-y\ge x^2+1+y^2+2x-2xy-2y=(x+1-y)^2\Rightarrow$$

$$\Rightarrow\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}+1-a-b-c\ge\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$ ***(2)***

$$\overset{(1)(2)}{\Rightarrow}\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c\ge\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$ ***(Q.E.D.)***

478. If $a, b > 0$ then:

$$\frac{\left((ab)^6+\left(\frac{a+b}{2}\right)^{12}\right)\left(ab+\left(\frac{a+b}{2}\right)^2\right)}{\left((ab)^3\sqrt{ab}+\left(\frac{a+b}{2}\right)^7\right)^2}\le\frac{(a^5+b^5)^2}{4(ab)^5}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Vietnam

$$\because a^5+b^5\ge ab(a^3+b^3)=ab(a+b)[(a+b)^2-3ab];$$

We must show that:

$$(ab)^3\left[(ab)^6+\left(\frac{a+b}{2}\right)^{12}\right]\left[ab+\left(\frac{a+b}{2}\right)^2\right]\le(a+b)^2[(a+b)^2-3ab]^2\times$$

$$\times\left[(ab)^3\sqrt{ab}+\left(\frac{a+b}{2}\right)^7\right]^2$$ ***(*). Let*** $u=\sqrt{ab};v=\frac{a+b}{2}\ (v\ge u>0)$

$$(*)\Leftrightarrow u^6[u^{12}+v^{12}][u^2+v^2]\le v^2(4v^2-3u^2)^2(u^7+v^7)^2$$

$$(\text{Let } u=tv, 0<u\le v\Rightarrow 0<t\le 1)\Leftrightarrow t^6(1+t^{12})(1+t^2)\le(4-3t^2)^2(1+t^7)^2$$

$$\Leftrightarrow[t^3(1+t^{12})][t^3(1+t^2)]\le[(4-3t^2)(1+t^7)][(4-3t^2)(1+t^7)]\quad (**)$$

$$t^3(1+t^{12})\le(4-3t^2)(1+t^7)\quad (1)$$

(1) true because: $\begin{cases}0<t\le1\Rightarrow t^3\le1\le4-3t^2\\ 0\le t\le1\Rightarrow t^5\le1\Rightarrow1+t^{12}\le1+t^7\end{cases}$

$$t^3(1+t^2)\le(4-3t^2)(1+t^7)\quad (2)$$

$\because t\le1\Rightarrow4-3t^2\ge1$. ***We must show that:*** $t^3(1+t^2)\le1+t^7$

$$\Leftrightarrow t^3+t^5\le1+t^7\Leftrightarrow(t-1)(t^6-t^5-t^2-t-1)\ge0$$

It is true because: $\because t\le1\Rightarrow t-1\le0, t^6\le t^2, t^5\le t\Rightarrow t^5<t+1$

From (1) and (2) $\Rightarrow$ ***(*) true. Proved.***

Solution 2 by Soumava Chakraborty-Kolkata-India

Let $\frac{a+b}{2}=x, \sqrt{ab}=y$. ***Now,*** $a^5+b^5\overset{Chebyshev}{\ge}\frac{1}{2}(a^2+b^2)(a^3+b^3)$

$$\overset{Chebyshev}{\ge}\frac{1}{4}(a+b)^2(a^3+b^3)\ge\frac{1}{4}(a+b)^2\cdot ab(a+b)=\frac{(2x)^3y^2}{4}=2x^3y^2$$

$$\Rightarrow \frac{(a^5+b^5)^2}{4(ab)^5} \ge \frac{(2x^3y^2)^2}{4y^{10}} = \frac{x^6}{y^6} \Rightarrow RHS \overset{(1)}{\ge} \frac{x^6}{y^6}.$$ *Also, LHS* $= \frac{(y^{12}+x^{12})(y^2+x^2)}{(y^7+x^7)^2} \underset{(2)}{\overset{?}{\le}} \frac{x^6}{y^6}$

$$\Leftrightarrow x^6(x^{14}+y^{14}+2x^7y^7) \overset{?}{\ge} y^6(x^{14}+y^{14}+x^{12}y^2+x^2y^{12})$$

$$\Leftrightarrow t^{20}-t^{14}+2t^{13}-t^{12}+t^6-t^2-1 \overset{?}{\ge} 0 \left(t=\frac{x}{y}\right)$$

$$\Leftrightarrow (t-1)(t^{19}+t^{18}+t^{17}+t^{16}+t^{15}+t^{14}+2t^{12}+t^{11}$$

$$+t^{10}+t^9+t^8+t^7+t^6+2t^5+2t^4+2t^3+2t^2+t+1)$$

$\overset{?}{\ge} 0 \rightarrow$ *true* $\because t = \frac{x}{y} = \frac{\frac{a+b}{2}}{\sqrt{ab}} \overset{A-G}{\ge} 1 \Rightarrow$ ***(2) is true (1), (2)*** $\Rightarrow LHS \le RHS.$ ***(proved)***

479. If $a, b, c \ge 0$ then:

$$3abc \le \sqrt{a^2+b^2+c^2} \cdot \sqrt[3]{a^3+b^3+c^3} \cdot \sqrt[5]{a^5+b^5+c^5}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$3abc \le \sqrt{a^2+b^2+c^2} \cdot \sqrt[3]{a^3+b^3+c^3} \cdot \sqrt[5]{a^5+b^5+c^5} \quad (1)$$

Let $\sum_{cyc} a = p$. *We have:* $\sum_{cyc} a^2 \ge \frac{p^2}{3} \rightarrow \left(\sum_{cyc} a^2\right)^{\frac{1}{2}} \ge \frac{p\sqrt{3}}{3}, \sum_{cyc} a^3 \ge 3\left(\frac{p}{3}\right)^3 \rightarrow$

$\rightarrow \left(\sum_{cyc} a^3\right)^{\frac{1}{3}} \ge \frac{p}{3}\sqrt[3]{3}$ *and likewise* $\left(\sum_{cyc} a^5\right)^{\frac{1}{5}} \ge \frac{p}{3}\sqrt[5]{3}$ *(Jensen)*

Then (1)→ RHS of (1) $\ge \frac{p^3}{3^3} \cdot \underbrace{\left(\sqrt{3}+\sqrt[3]{3}+\sqrt[5]{3}\right)}_{\theta} \overset{AM-GM}{\ge} \frac{\left(3\sqrt[3]{abc}\right)^3}{3^3} \cdot \theta = \theta \cdot abc.$

But $\theta \simeq 3 \cdot 11$ *so RHS of (1)* $\ge 3abc$. *Done. Equality when* $a=b=c=0$.

Solution 2 by Avishek Mitra-West Bengal-India

By $AM \ge GM \Rightarrow \frac{a^2+b^2+c^2}{3} \ge (abc)^{\frac{2}{3}} \Rightarrow \sqrt{a^2+b^2+c^2} \ge 3^{\frac{1}{2}}(abc)^{\frac{1}{3}}$

$$\Rightarrow \frac{a^3+b^3+c^3}{3} \ge abc \Rightarrow \sqrt[3]{a^3+b^3+c^3} \ge 3^{\frac{1}{3}}(abc)^{\frac{1}{3}}$$

$\Rightarrow \frac{a^5+b^5+c^5}{3} \ge (abc)^{\frac{5}{3}} \Rightarrow \sqrt[5]{a^5+b^5+c^5} \ge 3^{\frac{1}{5}}(abc)^{\frac{1}{3}}$. *Hence, we get:*

$$\Rightarrow p = \left(\sqrt{a^2+b^2+c^2}\right) \cdot \left(\sqrt[3]{a^3+b^3+c^3}\right) \cdot \left(\sqrt[5]{a^5+b^5+c^5}\right) \ge 3^{\frac{1}{30}} \cdot 3abc$$

as $3^{\frac{1}{30}} > 1 \Rightarrow 3^{\frac{1}{30}} \cdot 3abc > 3abc$. Hence $p > 3abc$ but at $a = b = c = 1$

$\Leftrightarrow \left(\sqrt{a^2 + b^2 + c^2}\right) \cdot \left(\sqrt[3]{a^3 + b^3 + c^3}\right) \cdot \left(\sqrt[5]{a^5 + b^5 + c^5}\right) = 3abc$ hence $p \geq 3abc$

Solution 3 by Tran Hong-Vietnam

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \Rightarrow \sqrt{a^2 + b^2 + c^2} \geq \frac{a+b+c}{\sqrt{3}}$$

$$a^3 + b^3 + c^3 \geq \frac{(a+b+c)^3}{3^2} \Rightarrow \sqrt[3]{a^3 + b^3 + c^3} \geq \frac{a+b+c}{\sqrt[3]{3^2}}$$

$$a^5 + b^5 + c^5 \geq \frac{(a+b+c)^5}{3^4} \Rightarrow \sqrt[5]{a^5 + b^5 + c^5} \geq \frac{a+b+c}{\sqrt[5]{3^4}}$$

$\Rightarrow LHS \geq \frac{(a+b+c)^3}{\sqrt{3}\cdot\sqrt[3]{3^2}\cdot\sqrt[5]{3^4}} = \frac{(a+b+c)^3}{3^{\frac{1}{2}+\frac{2}{3}+\frac{4}{5}}} = \frac{(a+b+c)^3}{3^{\frac{59}{30}}}$. We must show that: $\frac{(a+b+c)^3}{3^{\frac{59}{30}}} \geq 3abc$

$$3abc \leq 3 \cdot \frac{(a+b+c)^3}{27} = \frac{(a+b+c)^3}{3^2}$$

$$\Rightarrow \frac{(a+b+c)^3}{3^{\frac{59}{30}}} \geq \frac{(a+b+c)^3}{3^2} \Leftrightarrow (a+b+c)^3\left[3^2 - 3^{\frac{59}{30}}\right] \geq 0$$

(true: $a, b, c \geq 0, 3^2 - 3^{\frac{59}{30}} > 0\left(\because 2 > \frac{59}{30}\right)$). Proved. Equality $\Leftrightarrow a = b = c = 0$.

Solution 4 by Abdul Aziz-Semarang-Indonesia

A Fact: $3^{\frac{89}{30}} < 3^3 \Leftrightarrow abc \cdot 3^{\frac{89}{30}} \leq 3^3 abc \overset{AM-GM}{\leq} (a+b+c)^3 \Leftrightarrow 3abc \leq \frac{(a+b+c)^3}{59}$

$$= \frac{(a+b+c)}{3^{\frac{1}{2}}} \cdot \frac{(a+b+c)}{3^{\frac{2}{3}}} \cdot \frac{(a+b+c)}{3^{\frac{4}{5}}}$$

$$\overset{Holder}{\leq} \sqrt{a^2 + b^2 + c^2} \cdot \sqrt[3]{a^3 + b^3 + c^3} \cdot \sqrt[5]{a^5 + b^5 + c^5}$$

Equality holds when $a = b = c = 0$.

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c \geq 0$, we get that: $(a^2 + b^2 + c^2)^{15} \geq 3^{15}(abc)^{10}$

$$(a^3 + b^3 + c^3)^{10} \geq 3^{10}(abc)^{10};\ \left(a^5 + b^5 + c^5\right)^6 \geq 3^6(abc)^{10}$$

Hence, we have $(a^2 + b^2 + c^2)^{15}(a^3 + b^3 + c^3)^{10}\left(a^5 + b^5 + c^5\right)^6 \geq 3^{31}(abc)^{30}$

$$\geq 3^{30}(abc)^{30}: a, b, c \geq 0$$

Hence, $\left((a^2+b^2+c^2)^{15}(a^3+b^3+c^3)^{10}\left(a^5+b^5+c^5\right)^6\right)^{\frac{1}{30}} \ge 3(abc)$

That is, $(a^2+b^2+c^2)^{\frac{1}{2}}(a^3+b^3+c^3)^{\frac{1}{3}}(a^5+b^5+c^5)^{\frac{1}{5}} \ge 3abc$

Therefore, $\sqrt{a^2+b^2+c^2}\cdot\sqrt[3]{a^3+b^3+c^3}\cdot\sqrt[5]{a^5+b^5+c^5} \ge 3abc$*. Ok.*

480. **If** $x, y, z > 0, \frac{x+y}{2x+y}+\frac{y+z}{2y+z}+\frac{z+x}{2z+x}=2$ **then:**

$$\frac{3x^2+xy+2y^2}{2x^2+y^2}+\frac{3y^2+yz+2z^2}{2y^2+z^2}+\frac{3z^2+zx+2x^2}{2z^2+x^2}\le 6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x+y}{2x+y}+\frac{y+z}{2y+z}+\frac{z+x}{2z+x}\overset{(1)}{=}2;\ \sum\frac{3x^2+xy+2y^2}{2x^2+y^2}\overset{(a)}{\le}6$$

(1)$\Leftrightarrow (x+y)(2y+z)(2z+x)+(y+z)(2z+x)(2x+y)+(z+x)(2x+y)(2y+z)=$

$=2(2x+y)(2y+z)(2z+x)\Leftrightarrow \sum xy^2 \overset{(2)}{=} 3xyz$ *(upon simplification)*

But $\sum xy^2 \overset{A-G}{\ge} 3xyz$*, with equality when* $xy^2=yz^2=zx^2 \Rightarrow$ *when*

$xy=z^2, yz=x^2$ *&* $zx=y^2 \Rightarrow$ *when* $\sum xy=\sum x^2 \Rightarrow \sum\left[\frac{1}{2}(x-y)^2\right]=0$

$\Rightarrow$ *when* $x=y=z$ $\therefore$ *(2)* $\Rightarrow x=y=z \therefore$ *LHS of (6)* $=\sum\frac{3x^2+x^2+2x^2}{2x^2+x^2}=\sum(2)$

$=6 \therefore$ *LHS* $=6\Rightarrow$ *LHS* ≤ 6 *(proved)*

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\therefore \sum_{cyc}\frac{3x^2+xy+2y^2}{2x^2+y^2}\le 6$$

$\Leftrightarrow \sum_{cyc}\frac{3x^2+xy+2y^2}{2x^2+y^2}\le 3\cdot\sum_{cyc}\frac{x+y}{2x+y}$ *(1)*

$\because$ *Must show that:* $\frac{3x^2+xy+2y^2}{2x^2+y^2}\le 3\cdot\frac{x+y}{2x+y}$ *(2)*

$\Leftrightarrow (3x^2+xy+2y^2)(2x+y)\le 3(x+y)(2x^2+y^2)$

$\Leftrightarrow 6x^3+5yx^2+5xy^2+2y^3\le 3(2x^3+xy^2+2yx^2+y^3)$

$\Leftrightarrow 2xy^2 \le yx^2 + y^3 \Leftrightarrow y(y^2 - 2xy + x^2) \ge 0 \Leftrightarrow y(x-y)^2 \ge 0$ *(true because $y > 0$).*

Similarly: $\frac{3y^2+yz+2z^2}{2y^2+z^2} \ge 3 \cdot \frac{y+z}{2y+z}$ *(3);* $\frac{3z^2+xz+2x^2}{2z^2+x^2} \ge 3 \cdot \frac{z+x}{2z+x}$ *(4)*

From (2)+(3)+(4)⇒ (1) true.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0, \frac{x+y}{2x+y} + \frac{y+z}{2y+z} + \frac{z+x}{2z+x} = 2$*. We have as follows*

$$\frac{2x^2+xy+2y^2}{2x^2+y^2} = \frac{2x^2+y^2}{2x^2+y^2} + \frac{x^2+xy+y^2}{2x^2+y^2} = 1 + \frac{x^2+xy+y^2}{2x^2+y^2} \le 1 + \frac{x+2y}{2x+y} = \frac{3(x+y)}{2x+y}$$

$$\frac{3y^2+yz+2z^2}{2y^2+z^2} = \frac{2y^2+z^2}{2y^2+z^2} + \frac{y^2+yz+z^2}{2y^2+z^2} = 1 + \frac{y^2+yz+z^2}{2y^2+x^2} \le 1 + \frac{y+2z}{2y+z} = \frac{3(y+z)}{2y+z}.$$

$$\frac{3z^2+zx+2x^2}{2z^2+x^2} = \frac{2z^2+x^2}{2z^2+x^2} + \frac{z^2+zx+x^2}{2z^2+x^2} = 1 + \frac{z^2+zx+x^2}{2z^2+x^2} \le$$

$$1 + \frac{z+2x}{2z+x} = \frac{3(z+x)}{2z+x}$$

Hence $\frac{3x^2+xy+2y^2}{2x^2+y^2} + \frac{3y^2+yz+2z^2}{2y^2+z^2} + \frac{3z^2+zx+2x^2}{2z^2+x^2} \le \frac{3(x+y)}{2x+y} + \frac{3(y+z)}{2y+z} + \frac{3(z+x)}{2z+x} = 3(2) = 6$

Therefore, it's true.

481. **If $a, b, c > 0$ then:**

$$a^a \cdot b^b \cdot c^c \cdot (4a+4b+4c)^{a+b+c} \ge 3^{a+b+c} \cdot (a+b)^{a+b} \cdot (b+c)^{b+c} \cdot (c+a)^{c+a}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Inequality $\Leftrightarrow a^a \cdot b^b \cdot c^c \cdot (a+b+c)^{a+b+c} \ge \left(\frac{3}{4}\right)^{a+b+c} (a+b)^{a+b}(b+c)^{b+c}(c+a)^{c+a}$ *(*)*

Let $f(x) = x\log x \ (x > 0) \Rightarrow f''(x) = \frac{1}{x} > 0 \ (\forall x > 0)$

Using Popoviciu's inequality, with $f(x) = x\log x \ (x > 0)$ *we have:* Δ

$$\Leftrightarrow \sum a\log a + 3 \cdot \frac{a+b+c}{3}\log\left(\frac{a+b+c}{3}\right) \ge 2\sum\left(\frac{a+b}{2} \cdot \log\frac{a+b}{2}\right)$$

$$\Leftrightarrow \sum a\log a + \log\left(\frac{a+b+c}{3}\right)^3 \ge \sum \log\left(\frac{a+b}{2}\right)^{a+b}$$

$$\Leftrightarrow \log\left[a^a \cdot b^b \cdot c^c \cdot \left(\frac{a+b+c}{3}\right)^{a+b+c}\right] \ge \log\left[\left(\frac{a+b}{2}\right)^{a+b}\left(\frac{b+c}{2}\right)^{b+c}\left(\frac{c+a}{2}\right)^{c+a}\right]$$

$$\Leftrightarrow a^a\cdot b^b\cdot c^c\cdot(a+b+c)^{a+b+c}\cdot\frac{1}{3^{a+b+c}}\ge(a+b)^{a+b}(b+c)^{b+c}(c+a)^{c+a}\cdot\frac{1}{4^{a+b+c}}$$

$$\Leftrightarrow a^a\cdot b^b\cdot c^c\cdot(a+b+c)^{a+b+c}\ge\left(\frac{3}{4}\right)^{a+b+c}\cdot(a+b)^{a+b}\cdot(b+c)^{b+c}\cdot(c+a)^{c+a}$$

$\Rightarrow$ ***(*) true. Proved. Equality*** $\Leftrightarrow a=b=c$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=x\ln x$ ***for all*** $x>0$. ***Now,*** $f'(x)=1+\ln x$, $f''(x)=\frac{1}{x}>0$

Hence f ***is convex for all*** $x>0$, ***By Tiberiu Popoviciu's Inequality, we have:***

$$\frac{f(a)+f(b)+f(c)}{3}+f\left(\frac{a+b+c}{3}\right)\ge\frac{2}{3}\sum_{cyc}f\left(\frac{a+b}{2}\right)\quad\text{where } a,b,c>0$$

$$\frac{1}{3}\sum_{cyc}a\ln a+\frac{a+b+c}{3}\ln\left(\frac{a+b+c}{3}\right)\ge\frac{2}{3}\sum_{cyc}\left(\frac{a+b}{2}\right)\ln\left(\frac{a+b}{2}\right)$$

$$\Rightarrow\ln\left(a^ab^bc^c\right)+(a+b+c)\ln\left(\frac{a+b+c}{3}\right)\ge\sum_{cyc}(a+b)\ln\left(\frac{a+b}{2}\right)$$

$$a^a\cdot b^b\cdot c^c\cdot(4a+4b+4c)^{a+b+c}\ge3^{a+b+c}\cdot(a+b)^{a+b}\cdot(b+c)^{b+c}\cdot(c+a)^{c+a}$$

Solution 3 by Michael Sterghiou-Greece

$$a^ab^bc^c(4a+4b+4c)^{a+b+c}\ge3^{a+b+c}(a+b)^{a+b}(b+c)^{b+c}(c+a)^{c+a}\quad\textbf{\textit{(1)}}$$

(1)$\to a^ab^bc^c\cdot4^{a+b+c}\cdot(a+b+c)^{a+b+c}\ge3^{a+b+c}\cdot\prod_{cyc}(a+b)^a(c+a)^a$

Denoting $a+b+c=p$ ***we have*** $\frac{a^ab^bc^c\cdot p^a\cdot p^b\cdot p^c}{\prod_{cyc}(a+b)^a(c+a)^a}\ge\left(\frac{3}{4}\right)^p$ ***(2)***

Yet $(a+b)(c+a)\overset{AM-GM}{\le}\left(\frac{a+p}{2}\right)^2$***, so it suffices to show from (2)***

$\prod_{cyc}\left[\frac{4ap}{(a+p)^2}\right]^a\ge\left(\frac{3}{4}\right)^p$ ***or*** $\prod_{cyc}\left[\frac{ap}{(a+p)^2}\right]^a\ge\left(\frac{3}{16}\right)^p$***. Taking logarithms***

$\sum_{cyc}a\ln\frac{ap}{(a+p)^2}\ge p\cdot\ln\frac{3}{16}$ ***(3) Consider the function*** $f(x)=x\ln x$

with $f''(x)=\frac{1}{x}>0\to f$ ***convex. By Jensen on LHS of (3):***

LHS of (3) $\ge3\cdot\frac{p}{3}\cdot\ln\frac{\frac{p}{3}\cdot p}{\left(\frac{p}{3}+p\right)^2}=p\cdot\ln\frac{\frac{1}{3}}{\left(\frac{4}{3}\right)^2}=p\cdot\ln\frac{3}{16}$***. Done!***

482. If $x,y,z\ge0,x+y+z=2$ then:

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$$\frac{2}{5} \le \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \le \frac{18}{13}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Tran Hong-Vietnam

$$\frac{2}{5} \overset{(1)}{\le} \sum \frac{x}{1+x^2} \overset{(2)}{\le} \frac{18}{13}$$

$$\textit{(1)} \Leftrightarrow \sum \frac{x}{1+x^2} \ge \sum \frac{x}{5} \ (\because \sum x = 2) \Leftrightarrow \sum x\left(\frac{5-1-x^2}{5}\right) \ge 0 \Leftrightarrow \sum \frac{x(4-x^2)}{5} \ge 0$$

(true because: $0 \le x, y, z \le 2$***).***

$$\textit{(2)} \Leftrightarrow 13\sum x(1+y^2)(1+z^2) \le 18(1+x^2)(1+y^2)(1+z^2)$$

$$\Leftrightarrow 8 + 13[xy(x+y) + xz(z+x) + yz(y+z) + xyz(xy+xz+yz)]$$

$$\le 18(x^2+y^2+z^2+x^2y^2+y^2z^2+z^2x^2+x^2y^2z^2)$$

$$\Leftrightarrow 8 + 13[2(xy+xz+yz) + xyz(xy+xz+yz-3)]$$

$$\le 18[4 - 2(xy+xz+yz) + \sum x^2y^2 + (xyz)^2] \quad (*)$$

Let $s = x+y+z = 2; q = xy+yz+zx, r = xyz$

$$(*) \Leftrightarrow 8 + 13[2q + r(q-3)] \le 18[4 - 2q + q^2 - 4r + r^2]$$

$$\Leftrightarrow 8 + 26q + 13qr - 39r \le 72 - 36q + 18q^2 - 72r + 18r^2$$

$$\Leftrightarrow 18q^2 + 18r^2 + 64 \ge 33r + 62q + 13qr;$$

$$\Leftrightarrow 18\left(q - \frac{4}{3}\right)^2 + 18\left(r - \frac{8}{27}\right)^2 + \frac{2464}{81} \ge 13qr + 14q + \frac{67}{3}r$$

Must show: $\frac{2464}{81} \ge 13qr + 14q + \frac{67}{3}r$***. It is true because:*** $q \le \frac{p^2}{3} = \frac{4}{3}$ ***and*** $r \le \frac{p^3}{27} = \frac{8}{27}$

$$\Rightarrow 13qr + 14q + \frac{67}{3}r \le 13 \cdot \frac{8}{27} \cdot \frac{4}{3} + 14 \cdot \frac{4}{3} + \frac{67}{3} \cdot \frac{8}{27} = \frac{2464}{81} \Rightarrow \textit{(2) true. Proved.}$$

Solution 2 by Michael Sterghiou-Greece

$$\frac{2}{5} \le \sum_{cyc} \frac{x}{1+x^2} \le \frac{18}{13} \quad (1)$$

$$\sum_{cyc} \frac{x}{1+x^2} = \sum_{cyc} \frac{x^2}{x+x^3} \overset{BCS}{\ge} \frac{\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right) + \left(\sum_{cyc} x^3\right)} = \frac{4}{2 + \sum_{cyc} x^3} \overset{?}{\ge} \frac{2}{5} \to \sum_{cyc} x^3 \le 8$$

This is true because $x \le 2 \to x^2 \le 4 \to x^2 - 4 \le 0 \to x(x^2-4) \le 0 \to$

$$\to x^3 - 4x \le 0 \to \sum_{cyc} x^3 \le 4\sum_{cyc} x = 8$$

Consider the function $f(t)=\frac{t}{1+t^2}$ ***on*** $[0,2]$ $f'(t)=\frac{1-t^2}{(1+t^2)^2}$ ***with***

root $t=1$ ***in*** $[0,2]$ $f''(t)=\frac{2t(t^3-3)}{(t^2+1)^3}$ ***with root*** $\sqrt{3}$***.*** $t=1$ ***is a max for*** $f(t)$ ***and also*** $f(t)$

is concave in $[0,\sqrt{3}]$ ***as*** $f''(t)\le 0$ ***in this interval. Assume*** $\max\{x,y,z\}=x\le\sqrt{3}$***.***

Then by Jensen we have:

$$\sum_{cyc}\frac{x}{1+x^2}\le 3\cdot\frac{\frac{1}{3}\sum_{cyc}x}{1+\left(\frac{5x}{3}\right)^2}=\frac{2}{1+\frac{4}{9}}=\frac{18}{13}$$ ***and we are done. Assume*** $x>\sqrt{3}$ ***then***

$y+z\le 2-\sqrt{3}<\sqrt{3}$ ***and by Jensen*** $\frac{y}{1+y^2}+\frac{z}{1+z^2}\le 2\cdot\frac{\frac{y+z}{2}}{1+\left(\frac{y+z}{2}\right)^2}\le 2\cdot\frac{\frac{2-\sqrt{3}}{2}}{1+\left(\frac{2-\sqrt{3}}{2}\right)^2}<0{,}3(2)$

because $f(t)$ ***is*** $\uparrow$ ***in*** $[0,1]$ ***and*** $2-\sqrt{3}<1$***. Also,*** $f(t)$ ***is*** $\downarrow$ ***in*** $[1,2]$ $(f'(t)\le 0)$ ***so***

$f(x)<f(1)$ ***as*** $x>\sqrt{3}>1$ ***or*** $\frac{x}{1+x^2}\le\frac{1}{2}$***. Combining this with (2) we have***

$\sum_{cyc}\frac{x}{1+x^2}<0{,}3+0{,}5=0{,}8<\frac{18}{13}$***. We are done.***

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum x=2,\ \text{then}\ \frac{2}{5}\overset{(1)}{\le}\sum\frac{x}{1+x^2}\overset{(2)}{\le}\frac{18}{13}$$

$$(1)\Leftrightarrow\sum\frac{x}{1+x^2}\ge\frac{\sum x}{5}\Leftrightarrow\sum\left(\frac{x}{1+x^2}-\frac{x}{5}\right)\ge 0\Leftrightarrow\sum\frac{x(4-x^2)}{5(1+x^2)}\ge 0\Leftrightarrow\sum\frac{x(2+x)(2-x)}{5(1+x^2)}\overset{(1a)}{\ge}0$$

$$\because x=2-y-z\le 2(\because y\le 0\ \&-z\le 0)\therefore 2-x\ge 0\ \because x\ge 0\therefore x(2-x)\ge 0$$

$$\Rightarrow x(2-x)(2+x)\ge 0\Rightarrow\frac{x(2-x)(2+x)}{5(1+x^2)}\overset{(i)}{\ge}0.\ \text{Similarly,}\ \frac{y(2-y)(2+y)}{5(1+y^2)}\overset{(ii)}{\ge}0\ \&\ \frac{z(2-z)(2+z)}{5(1+z^2)}\overset{(iii)}{\ge}0$$

(i)+(ii)+(iii)⇒ (1a)⇒ (1) is true.

$$\text{Now, } 1+x^2=x^2+\frac{4}{9}+\frac{5}{9}=\left(x-\frac{2}{3}\right)^2+\frac{4x}{3}+\frac{5}{9}\ge\frac{4x}{3}+\frac{5}{9}=\frac{12x+5}{9}\Rightarrow\frac{1}{1+x^2}\le\frac{9}{12x+5}$$

$$\Rightarrow\frac{x}{1+x^2}\overset{(a)}{\le}\frac{9x}{12x+5}\ (\because x\ge 0).\ \text{Similarly,}\ \frac{y}{1+y^2}\overset{(b)}{\le}\frac{9y}{12y+5}\ \&\ \frac{z}{1+z^2}\overset{(c)}{\le}\frac{9z}{12+5}$$

$$(a)+(b)+(c)\Rightarrow\sum\frac{x}{1+x^2}\overset{(2a)}{\le}9\sum\frac{x}{12+5}$$

$$\text{Let } f(t)=\frac{t}{12t+5}.\ \text{Then, } f''(t)=-\frac{120}{(12t+5)^3}<0$$

$$\Rightarrow f(t)\ \text{is concave}\therefore 9\sum\frac{x}{12x+5}\underset{(2b)}{\overset{Jensen}{\le}}3\cdot 9\left[\frac{\left(\frac{\sum x}{3}\right)}{12\left(\frac{\sum x}{3}\right)+5}\right]=\frac{9\left(\frac{2}{3}\right)\cdot 3}{12\left(\frac{2}{3}\right)+5}=\frac{18}{13}$$

(2a), (2b) $\Rightarrow \sum \frac{x}{1+x^2} \le \frac{18}{13} \Rightarrow$ ***(2) is true (Proved)***

483. If $a, b, c > 0$ then:

$$\left(1+\frac{b}{a}+\frac{c}{a}+\frac{a}{a+b+c}\right)\left(1+\frac{c}{b}+\frac{a}{b}+\frac{b}{a+b+c}\right)\left(1+\frac{a}{c}+\frac{b}{c}+\frac{c}{a+b+c}\right) \ge \frac{1000}{27}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Athens-Greece

$$\prod_{cyc}\left(1+\frac{b}{a}+\frac{c}{a}+\frac{a}{a+b+c}\right) \ge \frac{1000}{27} \quad (1)$$

Let $(p, q, r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)$

(1) is homogeneous, so we can assume $a+b+c = p = 1$ ***(1)***$\rightarrow$

$$\rightarrow \prod_{cyc}\left(1+\frac{1-a}{a}+a\right) \ge \frac{1000}{27} \text{ or } \prod_{cyc}\frac{a^2+1}{a} \ge \frac{1000}{27} \text{ or } \prod_{cyc}(a^2+1) \ge \frac{1000}{27}\cdot r$$

or $2\left(\sum_{cyc} a^2c^2\right) + \left(\sum_{cyc} a^2\right) + a^2b^2c^2 + 1 \ge \frac{1000}{27} r$ ***which reduces to***

$$27r^2 + 27q^2 - 1054r - 54q + 54 \ge 0 \quad (2)$$

$$\left(\because \sum_{cyc} a^2c^2 = q^2 - 2pr = q^2 - 2r \ \ and \sum_{cyc} a^2 = p^2 - 2q = 1 - 2q\right)$$

But $q^2 \ge \left(3r^{\frac{2}{3}}\right)^2$ ***(AM-GM) and*** $q \le \frac{p^3+9r}{4p} = \frac{1+9r}{4}$ ***(Schur) so, (2) yields the stronger***

inequality $27r^2 + 243r^{\frac{4}{3}} - 1054r - \frac{54}{4}(1+9r) + 54 \ge 0$ ***or***

$$\frac{1}{2}\left(3r^{\frac{1}{3}} - 1\right)\left(18r^{\frac{5}{3}} + 6r^{\frac{4}{3}} - 729r^{\frac{2}{3}} + 164r - 243r^{\frac{1}{3}-81}\right) \ge 0$$

$$3r^{\frac{1}{3}} - 1 \le 0 \text{ and } \underbrace{18r^{\frac{5}{3}} - 18r^{\frac{2}{3}}}_{<0} + \underbrace{6r^{\frac{4}{3}} - 6r^{\frac{2}{3}}}_{<-} + \underbrace{164r - 81}_{<0} - 705r^{\frac{2}{3}} - 243r^{\frac{1}{3}} < 0 \text{ as } r \le \frac{1}{27}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$1+\frac{b}{a}+\frac{c}{a}+\frac{a}{b+a+c} = 1+\frac{b}{a}+\frac{c}{a}+\frac{1}{1+\frac{b}{a}+\frac{c}{a}}$. ***Suppose,*** $x = \frac{b}{a}, y = \frac{c}{a}$

$$1+\frac{b}{a}+\frac{c}{a}+\frac{a}{a+b+c} = 1+x+y+\frac{1}{1+x+y}$$

$1+x+y+\frac{1}{1+x+y} = \frac{(x+y+1)^2+1}{(x+y+1)}$. ***Suppose:*** $a \le b \le c$ ***then:*** $\frac{b}{a} \ge 1$ ***and*** $\frac{c}{a} \ge 1$

So: $\gamma = x + y + 1 \geq 3$. *Let us prove:* $\frac{(1+x+y)^2+1}{1+x+y} \overset{?}{\geq} \frac{10}{3}$ *(*)*

$$3(1 + x + y)^2 + 3 \overset{?}{\geq} 10(1 + x + y);\ 3\gamma + 3 \overset{?}{\geq} 10\gamma;\ 3\gamma^2 - 30\gamma + 3 \overset{?}{\geq} 0$$

$$3\gamma^2 - 10\gamma + 3 = 0 \left\langle \begin{array}{l} \gamma = \frac{1}{3} \\ \gamma = 3 \end{array} \right.$$

γ	$-\infty$ $\quad \frac{1}{3} \quad$ 3 $\quad +\infty$
$3\gamma^2 - 10\gamma + 3$	+ + + + + 0 − − − − − 0 + + + + + +

So: The inequality (*) is true and since: $1 + \frac{b}{a} + \frac{c}{a} + \frac{b}{b+a+c} \geq \frac{10}{3}$ *and then:*

$$\left(1 + \frac{b}{a} + \frac{c}{a} + \frac{a}{a+b+c}\right)\left(1 + \frac{c}{a} + \frac{a}{b} + \frac{b}{a+b+c}\right)\left(1 + \frac{a}{c} + \frac{b}{c} + \frac{c}{a+b+c}\right) \geq \frac{10}{3} \cdot \frac{10}{3} \cdot \frac{10}{3}$$

$$L_1 \geq \frac{1000}{27}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\left(1 + \frac{b}{a} + \frac{c}{a} + \frac{a}{a+b+c}\right)\left(1 + \frac{c}{b} + \frac{a}{b} + \frac{b}{a+b+c}\right)\left(1 + \frac{a}{c} + \frac{b}{c} + \frac{c}{a+b+c}\right)$$

$$= \frac{[(a+b+c)^2 + a^2][(a+b+c)^2][(a+b+c)^2 + c^2]}{abc(a+b+c)^3} \overset{(1)}{\geq} \frac{1000}{27}$$

Let $a + b + c = k > 0$;

(1) $\Leftrightarrow 27\left[k^6 + (a^2 + b^2 + c^2)k^4 + (a^2b^2 + b^2c^2 + c^2a^2)k^2 + a^2b^2c^2\right] \geq 1000abck^3$

$$\because a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq \frac{k^2}{3}$$

$\because (ab)^2 + (bc)^2 + (ca)^2 \geq abc(a + b + c) = abck$. *We must show that*

$$27\left(k^6 + \frac{k^6}{3} + abck^3 + a^2b^2c^2\right) \geq 1000abck^3 \quad (2)$$

$$\Leftrightarrow 27\left(\frac{4k^6}{3} + abck^3 + a^2b^2c^2\right) \geq 1000abck^3$$

$$\Leftrightarrow 36k^6 + 27a^2b^2c^2 \geq 973abck^3 \Leftrightarrow 36(k^3 - 27abc)\left(k^3 - \frac{abc}{36}\right) \geq 0$$

It is true because: $k^3 = (a + b + c)^3 \geq 27abc > \frac{abc}{36} \Rightarrow$ *(2) true* $\Rightarrow$ *(1) true.*

Solution 4 by Soumava Chakraborty-Kolkata-India

Let $a+b=x, b+c=y, c+a=z \therefore x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ are sides of a triangle with semiperimeter, circumradius, inradius $= s,R,r$

$$\therefore 2\sum a=\sum x=2s\Rightarrow \sum a=s \therefore c=s-x, a=s-y, b=s-z,$$

$$\therefore abc=\prod(s-c)\overset{(1)}{=}r^2s \;\&\; \sum ab=\sum(s-y)(s-z)=\sum(s^2-s(y+z)+yz)=$$

$$=3s^2-4s^2+s^2+4Rr+r^2\overset{(2)}{=}r(4R+r)$$

$$\therefore LHS=\left(\frac{s}{a}+\frac{a}{s}\right)\left(\frac{s}{b}+\frac{b}{s}\right)\left(\frac{s}{c}+\frac{c}{s}\right)=\frac{s^2}{abc}+s\sum\frac{a}{bc}+\frac{1}{s}\sum\frac{ab}{c}+\frac{abc}{s^3}$$

$$\overset{by\,(1)}{=}\frac{s^3}{r^2s}+\frac{r^2s}{s^3}+\frac{s}{r^2s}\sum a^2+\frac{\sum a^2b^2}{s(r^2s)}=$$

$$=\frac{s^2}{r^2}+\frac{r^2}{s^2}+\frac{(\sum a)^2-2\sum ab}{r^2}+\frac{(\sum ab)^2-2abc(\sum a)}{r^2s^2}$$

$$\overset{by\,(1),(2)}{=}\frac{s^2}{r^2}+\frac{r^2}{s^2}+\frac{s^2-2r(4R+r)}{r^2}+\frac{r^2(4R+r)^2-2(r^2s)s}{r^2s^2}$$

$$=\frac{s^4+r^4+s^4-s^2(8Rr+2r^2)+r^2(4R+r)^2-2r^2s^2}{r^2s^2}\geq\frac{1000}{27}$$

$$\Leftrightarrow 27(2s^4-s^2(8Rr+4r^2)+r^2(4R+r)^2+r^4)\geq 1000r^2s^2$$

$$\Leftrightarrow 54s^4-s^2(216Rr+1108r^2)+27r^2(4R+r)^2+27r^4\overset{(a)}{\geq}0$$

Now, LHS of (a) $\overset{Gerretsen}{\geq} 54(16Rr-5r^2)s^2-s^2(216Rr+1108r^2)+27r^2(4R+r)^2+27r^4$

$$=s^2(648Rr-1378r^2)+27r^2(4R+r)^2+27r^4\overset{?}{\geq}0$$

$$\Leftrightarrow s^2(648Rr-1296r^2)+27r^2(4R+r)^2+27r^4\underset{(b)}{\overset{?}{\geq}}82r^2s^2$$

$$\because 648Rr-1296r^2=648r(R-2r)\overset{Euler}{\geq}0 \therefore \text{LHS of (b)}\underset{(i)}{\overset{?}{\geq}}$$

$$(16Rr-5r^2)(648Rr-1296r^2)+27r^2(4R+r)^2+27r^4$$

$$\text{Also, RHS of (b)}\underset{(ii)}{\overset{?}{\leq}}82r^2(4R^2+4Rr+3r^2)$$

(i), (ii) $\Rightarrow$ in order to prove (b), it suffices to prove:

$$(16Rr-5r^2)(648Rr-1296r^2)+27r^2(4R+r)^2+27r^4\geq 82r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 1309R^2-3011Rr+786r^2\geq 0\Leftrightarrow(R-2r)(1309R-393r)\geq 0\rightarrow \text{true}$$

$\because R \overset{Euler}{\geq} 2r \Rightarrow$ ***(b)*** $\Rightarrow$ ***(a)*** $\Rightarrow$ ***given inequality is true (proved)***

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x \geq a, a \geq 1$***, we have this fact:*** $x + \frac{1}{x} \geq a + \frac{1}{a}$***. Hence for*** $a, b, c > 0$***, we have:***

$$\left(\frac{a+b+c}{\sqrt[3]{abc}} + \frac{\sqrt[3]{abc}}{a+b+c}\right) \geq 3 + \frac{1}{3} = \frac{10}{3} \Rightarrow \left(\frac{a+b+c}{\sqrt[3]{abc}} + \frac{\sqrt[3]{abc}}{a+b+c}\right)^3 \geq \left(\frac{10}{3}\right)^3 = \frac{1000}{27}$$

$$\Rightarrow \left(\frac{a+b+c}{a} + \frac{a}{a+b+c}\right)\left(\frac{a+b+c}{b} + \frac{b}{a+b+c}\right)\left(\frac{a+b+c}{c} + \frac{c}{a+b+c}\right) \geq \frac{1000}{27}$$

$$\Rightarrow \left(1 + \frac{b}{a} + \frac{c}{a} + \frac{a}{a+b+c}\right)\left(1 + \frac{a}{b} + \frac{c}{b} + \frac{b}{a+b+c}\right)\left(1 + \frac{a}{c} + \frac{b}{c} + \frac{c}{a+b+c}\right) \geq \frac{1000}{27}$$

Therefore, it is true.

Solution 6 by Boris Colakovic-Belgrade-Serbie

$$\left.\begin{array}{l} x = 1 + \frac{b}{a} + \frac{c}{a} = \frac{a+b+c}{a} \\ y = 1 + \frac{c}{b} + \frac{a}{b} = \frac{a+b+c}{b} \\ z = 1 + \frac{a}{c} + \frac{b}{c} = \frac{a+b+c}{c} \end{array}\right\} \quad (1)$$

From (1) inequality becomes: $\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right) \geq \frac{1000}{27}$ ***(2)***

On the other hand: $x + \frac{1}{x} \geq \frac{2}{3} + \frac{8}{9}x \Leftrightarrow \frac{1}{9}x^2 - \frac{2}{3}x + 1 \geq 0 \Leftrightarrow x^2 - 6x + 9 \geq 0 \Leftrightarrow$

$\Leftrightarrow (x-3)^2 \geq 0$ ***true***

From (2) $\Rightarrow \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right) \geq \left(\frac{2}{3} + \frac{8}{9}x\right)\left(\frac{2}{3} + \frac{8}{9}y\right)\left(\frac{2}{3} + \frac{8}{9}z\right) =$

$$= \frac{32}{81}(x+y+z) + \frac{128}{243}(xy+yz+zx) + \frac{512}{729}xyz + \frac{8}{27} \quad (3)$$

$$\left.\begin{array}{l} x+y+z = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{(a+b+c)(ab+bc+ca)}{abc} \geq 9 \\ xy+yz+zx = (a+b+c)^2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = \frac{(a+b+c)^3}{abc} \geq 27 \\ xyz = \frac{(a+b+c)^3}{abc} \geq 27 \end{array}\right\} \quad (4)$$

From (3) and (4) $\Rightarrow \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right) \geq \frac{32}{81}\underbrace{(x+y+z)}_{\geq 9} + \frac{128}{243}\underbrace{(xy+yz+zx)}_{\geq 27} +$

$+ \frac{512}{729}\underbrace{xyz}_{\geq 27} + \frac{8}{27} \geq \frac{96}{27} + \frac{384}{27} + \frac{512}{27} + \frac{8}{27} = \frac{1000}{27} \Rightarrow$ ***(2) true***

484. If $x, y, z \geq 0$ then:

$$\pi^{x^2(x^2+1)} + \pi^{y^2(y^2+1)} + \pi^{z^2(z^2+1)} \geq \pi^{xy(xy+1)} + \pi^{yz(yz+1)} + \pi^{zx(zx+1)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Klevis Liperi-Durres-Albania

$$\pi^{x^2(x^2+1)} + \pi^{y^2(y^2+1)} \overset{AM-GM}{\geq} 2\pi^{\frac{x^2(x^2+1)+y^2(y^2+1)}{2}} \overset{AM-GM}{\geq} 2\pi^{\sqrt{x^2y^2(x^2+1)(y^2+1)}} =$$

$$= 2\pi^{xy\sqrt{(x^2+1)(y^2+1)}} \overset{CBS}{\geq} 2\pi^{xy(xy+1)} \quad (1)$$

In the same way, we can prove: $\pi^{y^2(y^2+1)} + \pi^{z^2(z^2+1)} \geq 2\pi^{yz(yz+1)}$ *(2)*

And $\pi^{x^2(x^2+1)} + \pi^{z^2(z^2+1)} \geq 2\pi^{zx(zx+1)}$ *(3)*

(1)+(2)+(3) gives the desired inequality. Equality holds if $x = y = z$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $f(x) = \pi^{x^2(x^2+1)}\ (x \geq 0) \Rightarrow f'(x) = (2\log\pi)\cdot\pi^{x^4+x^2}(2x^3+x)$

$$\Rightarrow f''(x) = (4\log\pi)(2x^4+x^2)\left(\left[4x^5+2x^4+2x^3+x^2\right)\log\pi + 3x + 1\right) > 0\ (\forall x \geq 0)$$

Using T. Popoviciu's inequality:

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

$$\Leftrightarrow \sum \pi^{x^2(x^2+1)} + 3\pi^{\left(\frac{x+y+z}{3}\right)^2\left[\left(\frac{x+y+z}{3}\right)^2+1\right]} \geq$$

$$2\left\{\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]}\right\} = \left\{\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]}\right\} + \left\{\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]}\right\} \geq$$

$$\overset{(x+y)^2 \geq 4xy\ (etc)}{\geq} \left\{\sum \pi^{xy(xy+1)}\right\} + \left\{\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]}\right\}$$

$$\Rightarrow \sum \pi^{x^2}(x^2+1) \geq \left\{\sum \pi^{xy(xy+1)}\right\} + \left\{\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]}\right\} - 3\pi^{\left(\frac{x+y+z}{3}\right)^2\left[\left(\frac{x+y+z}{3}\right)^2+1\right]} \quad (*)$$

But: But Jensen's inequality (with $g(t) = \pi^{t^2(t^2+1)}, t \geq 0$*) we have:*

$$\sum \pi^{\left(\frac{x+y}{2}\right)^2\left[\left(\frac{x+y}{2}\right)^2+1\right]} \geq 3\pi^{\left(\frac{2x+2y+2z}{6}\right)^2\left[\left(\frac{2x+2y+2z}{6}\right)^2+1\right]} = 3\pi^{\left(\frac{x+y+z}{3}\right)^2\left[\left(\frac{x+y+z}{3}\right)^2+1\right]}$$

$\Rightarrow$ *RHS (*)* $\geq \sum \pi^{xy(xy+1)} \Rightarrow$ *Proved*

485. If $a, b \geq 0$ then:

$$(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 18a^2b^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Elvin Samedov-Baku-Azerbaijan

I will prove a stronger inequality than Dan Sitaru's inequality:

$$\text{If } \{a,b\} \geq 0 \quad (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 18a^2b^2 \quad (1)$$

Proposed by Dan Sitaru

$$\text{If } \{a,b\} \geq 0 \quad (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 4\left(\sqrt{2}+1\right)^2 a^2b^2 \quad (2)$$

Proposed by E. Samedov

It is enough to prove inequality (2)

Let $\sqrt{a^2+b^2} = m, ab = n$. ***We must prove***

$$(2) \Rightarrow \sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right) \geq 2\left(\sqrt{2}+1\right)ab \Rightarrow m\left(\sqrt{m^2+2n}+m\right) \geq 2\left(\sqrt{2}+1\right)n$$

$$\Rightarrow \sqrt{m^4+2m^2n}+m^2-2\left(\sqrt{2}+1\right)n \geq 0. \text{ It is obvious,}$$

$$\begin{cases} m^2 \geq 2n \\ m^4 \geq 4n^2 \\ 2m^2n \geq 4n^2 \end{cases} \Rightarrow \begin{cases} \sqrt{m^4+2m^2n} \geq 2\sqrt{2}n \\ m^2 \geq 2n \end{cases} \Rightarrow \sqrt{m^4+2m^2n}+m^2 \geq 2n+2\sqrt{2}n$$

$$\Rightarrow \sqrt{m^4+2m^2n}+m^2-2\left(1+\sqrt{2}\right)n \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 4\left(\sqrt{2}+1\right)^2 a^2b^2. \text{ Proved.}$$

Solution 2 by Michael Sterghiou-Greece

$$(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 18a^2b^2 \quad (1)$$

(1) is homogeneous $\Rightarrow$ ***we can assume*** $ab = 1$.

(2) Let $a+b = x \geq 2$ ***as*** $a+\frac{1}{a} \geq 2$ ***for*** $a > 0$

(If $ab = 0$ ***then (1) is true). (1)***$\rightarrow (x^2-2)\left(x+\sqrt{x^2-2}\right)^2 - 18 = f(x) \geq 0$

$$f'(x) = \frac{2}{\sqrt{x^2-2}} \cdot \left(\sqrt{x^2-2}+x\right)^2\left(x^2+\sqrt{x^2-2}\cdot x-2\right) > 0 \text{ for } x \geq 2$$

Hence $f(x) \uparrow$ ***and*** $f(x) > f(2) > 0$. ***Done. Equality for*** $a = b = 0$.

Solution 3 by Nguyen Van Nho-Nghe An-Vietnam

$$\oplus\ (a-b)^2 \geq 0 \to a^2+b^2 \geq 2ab \geq 0 \to \textit{(1)}$$

$$a+b \overset{Cauchy}{\geq} 2\sqrt{ab} \text{ and } \sqrt{a^2+b^2} \geq \sqrt{2ab}$$

$$\to a+b+\sqrt{a^2+b^2} \geq \left(2+\sqrt{2}\right)\sqrt{ab} \geq 3\sqrt{ab} \geq 0$$

$$\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 9ab \geq 0 \to \textit{(2)}$$

From (1) and (2) $\to$ ***LHS*** $\geq (2ab)(9ab) = 18a^2b^2 =$ ***RHS (done)***

Equality $\leftrightarrow a=b=0$

Solution 4 by Ravi Prakash-New Delhi-India

Let $a = r\cos\alpha, b = r\sin\alpha, 0 \leq \alpha \leq \frac{\pi}{2}, r \geq 0$

$$(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \geq 18a^2b^2$$

$$\Leftrightarrow r^2[r(\cos\alpha+\sin\alpha)+r]^2 \geq 18r^4\sin^2\alpha\cos^2\alpha$$

$$\Leftrightarrow (\cos\alpha+\sin\alpha+1)^2 \geq 18\sin^2\alpha\cos^2\alpha$$

$$\Leftrightarrow \cos^2\alpha+\sin^2\alpha+1+2\cos\alpha\sin\alpha+2\cos\alpha+2\sin\alpha \geq 18\sin^2\alpha\cos^2\alpha$$

$$\Leftrightarrow 2[1+\cos\alpha\sin\alpha+\cos\alpha+\sin\alpha] \geq 18\sin^2\alpha\cos^2\alpha$$

$$\Leftrightarrow (1+\cos\alpha)(1+\sin\alpha) \geq 9(1-\cos^2\alpha)(1-\sin^2\alpha) \Leftrightarrow (1-\cos\alpha)(1-\sin\alpha) \leq \frac{1}{9}$$

Let $f(\theta) = (1-\cos\theta)(1-\sin\theta), 0 \leq \theta \leq \frac{\pi}{2}$

$$f'(\theta) = \sin\theta\,(1-\sin\theta) - \cos\theta\,(1-\cos\theta)$$

$$= \sin\theta - \cos\theta + \cos^2\theta - \sin^2\theta = (\cos\theta-\sin\theta)(\cos\theta+\sin\theta-1)$$

$$f'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}\left[\because 0 < \theta < \frac{\pi}{2}, \cos\theta+\sin\theta-1 > 0\right]$$

Now, $\max f(\theta) = \max\left\{f(0), f\left(\frac{\pi}{2}\right), f\left(\frac{\pi}{4}\right)\right\}$

$$= f\left(\frac{\pi}{4}\right) = \left(1-\frac{1}{\sqrt{2}}\right)^2 = 1+\frac{1}{2}-\sqrt{2} = 1\cdot 5-\sqrt{2} < 0.086 < \frac{1}{9}$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b > 0$

$$(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \ge \frac{(a+b)^2}{2}\left((a+b)+\frac{(a+b)}{\sqrt{2}}\right)^2$$

$$> \frac{(a+b)^2}{2}\left((a+b)+\frac{a+b}{2}\right)^2 = \frac{(a+b)^2}{2}\left(\frac{9}{4}\right)(a+b)^2 = \frac{9}{8}(a+b)^4 \ge 18a^2b^2$$

Therefore, it is true.

Solution 6 by Shahlar Maharramov-Jebrail-Azerbaijan

$\overset{A-G}{\ge} 2ab\left(2\sqrt{ab}+\sqrt{2ab}\right)^2 \overset{?}{\ge} 18a^2b^2$***. Hence*** $\sqrt{ab}=t\to ?$

$2t^4\left(2+\sqrt{2}\right)^2-18t^4 \overset{?}{\ge} 0;\ t^4\left(8+4+8\sqrt{2}-18\right)\ge 0$***. Equality when*** $ab=0$.

Solution 7 by Soumava Chakraborty-Kolkata-India

$$\forall a,b\ge 0, (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2 \overset{(1)}{\ge} 18a^2b^2$$

If $a=0, b>0$***, LHS of (1)*** $=b^2\cdot 4b^2=4b^2$ ***& RHS of (1)***$=0$ ***&*** $\because 4b^4>0, \therefore$ ***(1) is true***

If $b=0, a>0$***, LHS of (1)*** $=4a^4$ ***& RHS of (1)***$=0$ ***&*** $\because 4a^4>0, \therefore$ ***(1) is true.***

If $a=b=0$***, LHS of (1)*** $=$ ***RHS of (1)*** $(=0)\Rightarrow$ ***(1) is true.***

Now, we consider $a,b>0$***. Let*** $Q=\sqrt{\frac{a^2+b^2}{2}}, A=\frac{a+b}{2}, G=\sqrt{ab}$

Then (1) $\Leftrightarrow 2Q^2\left(2A+\sqrt{2}Q\right)^2\ge 18G^4 \Leftrightarrow \sqrt{2}Q\left(2A+\sqrt{2}Q\right)\ge 3\sqrt{2}G^2$

$$\Leftrightarrow 2AQ+\sqrt{2}Q^2 \overset{(2)}{\ge} 3G^2 \because Q\ge A\ge G, \therefore 2AQ \overset{(3)}{\ge} 2G^2$$

(2), (3) $\Rightarrow$ ***it suffices to prove:*** $\sqrt{2}Q^2\ge G^2$***. Now,*** $\sqrt{2}Q^2\ge\sqrt{2}G^2(\because Q\ge G)>G^2$

$\Rightarrow$ ***when*** $a,b>0\ \ (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2>18a^2b^2$

$\Rightarrow$ ***(1) is true. This completes the proof.***

486. **If $a,b,c\in\mathbb{R}, a^2+b^2+c^2\le 15$ then:**

$$\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}\ge 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah Al Farisi-Bechar-Algerie

$f(x)=\frac{1}{\sqrt{4+x}}$ ***is convex function then:*** $f(a^2)+f(b^2)+f(c^2)\geq 3f\left(\frac{a^2+b^2+c^2}{3}\right)=3f(5)=$

1 ***then:*** $\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}\geq 1$

Solution 2 by Amit Dutta-Jamshedpur-India

$a,b,c\in\mathbb{R}, a^2+b^2+c^2\leq 15$, ***then prove that:*** $\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}\geq 1$

Let $P=\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}; P\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{\sqrt{4+a^2}+\sqrt{4+b^2}+\sqrt{4+c^2}}$

$P\geq\frac{9}{\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4}}$ ***(1). Using Cauchy's Schwarz Inequality:***

$$[(a^2+4)+(b^2+4)+(c^2+4)][1+1+1]\geq\left(\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4}\right)^2$$

$$\Rightarrow\left(\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4}\right)\leq[(a^2+b^2+c^2+12)\cdot 3]^{\frac{1}{2}}$$

$\leq[(15+12)\cdot 3]^{\frac{1}{2}}\leq(27\times 3)^{\frac{1}{2}}\leq 9\therefore\frac{9}{\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4}}\geq 1\Rightarrow P\geq 1$ ***(Proved)***

Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

$\sqrt{4+a^2}=\sqrt{\frac{9\cdot(4+a^2)}{9}}\leq\frac{1}{3}\cdot\frac{9+4+a^2}{2}=\frac{13+a^2}{6}$***. So,*** $\frac{1}{\sqrt{4+a^2}}\geq\frac{6}{13+a^2}$

Thus, $LHS\geq 6\left(\frac{1}{13+a^2}+\frac{1}{13+b^2}+\frac{1}{13+c^2}\right)\geq 6\cdot\frac{9}{39+a^2+b^2+c^2}\geq\frac{54}{39+15}=\frac{54}{54}=1$

Solution 4 by Sagar Kumar-Patna Bihar-India

$AM\geq GM$: $\sum_{cyc}\frac{1}{\sqrt{4+a_i^2}}\geq 3\left(\frac{1}{(4+a^2)(4+b^2)(4+c^2)}\right)^{\frac{1}{6}}$ ***(1)***

Now, $(4+a^2)(4+b^2)(4+c^2)\leq\left(\frac{12+(a^2+b^2+c^2)}{3}\right)^3$***. Now,***

$4^3\leq(4+a^2)(4+b^2)(4+c^2)$ ***max possible*** $\leq 9^3$ ***(2)***

From (1): $\sum_{cyc}\frac{1}{\sqrt{4+a_i^2}}\geq 3\left(\frac{1}{(4+a^2)(4+b^2)(4+c^2)}\right)^{\frac{1}{6}}$***. From (2):***

$$\left(\frac{1}{9^3}\right)^{\frac{1}{6}}\leq\left(\frac{1}{(4+a^2)(4+b^2)(4+c^2)}\right)^{\frac{1}{6}}\leq\left(\frac{1}{4^3}\right)^{\frac{1}{6}}$$

$$\frac{1}{3} \le b \le \frac{1}{2} \le \left(\frac{3^6}{(4+a^2)(4+b^2)(4+c^2)}\right)^{\frac{1}{6}} \le \frac{3}{2} \text{ or From (1): } \sum_{cyc} \frac{1}{\sqrt{4+a_i^2}} \ge 1$$

Equality holds when: $a = b = c = \pm\sqrt{5}$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 15$, ***we have:*** $(a^2 + 4)(b^2 + 4) + (c^2 + 4) \le 27$

$$\left(\sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4}\right)^2 \le 3\left((a^2 + 4) + (b^2 + 4) + (c^2 + 4)\right) \le 81$$

$$\Rightarrow \sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4} \le 9$$

$$\Rightarrow \frac{1}{\sqrt{a^2+4}} + \frac{1}{\sqrt{b^2+4}} + \frac{1}{\sqrt{c^2+4}} \ge 1, \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = 1.$$ ***Therefore, it is true.***

Solution 6 by Soumava Chakraborty-Kolkata-India

If $a, b, c \in \mathbb{R} | \sum a^2 \le 15$, ***then*** $\sum \frac{1}{\sqrt{4+a^2}} \ge 1$

$$LHS \overset{Bergstrom}{\ge} \frac{9}{\sum\sqrt{4+a^2}} \overset{CBS}{\ge} \frac{9}{\sqrt{3}\sqrt{12+\sum a^2}} \overset{\sum a^2 \le 15}{\ge} \frac{9}{81} = 1 \text{ (Proved)}$$

487. If $x, y > 0$ then:

$$4\left(x + \frac{x+1}{y}\right)\left(y + \frac{y+1}{x}\right) \le \left(2 + x + y + \frac{1}{x} + \frac{1}{y}\right)^2$$

Proposed by Andrei Stefan Mihalcea-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Consider: $\left(2 + x + y + \frac{1}{x} + \frac{1}{y}\right)^2 - 4\left(x + \frac{x+1}{y}\right)\left(y + \frac{y+1}{x}\right) =$

$$= 4 + (x+y)^2 + \left(\frac{x+y}{xy}\right)^2 + 4(x+y) + 4\left(\frac{x+y}{xy}\right) + \frac{2(x+y)^2}{xy} -$$

$$-4\left[xy + x + 1 + y + 1 + \frac{(x+1)(y+1)}{xy}\right]$$

$$= (x+y)^2 - 4xy + \left(\frac{x+y}{xy}\right)^2 - 4\left(\frac{1}{x} + \frac{1}{y}\right) + 4\left(\frac{1}{x} + \frac{1}{y}\right) - \frac{4}{xy} +$$

$$+4(x+y) - 4(x+y) + 4 - 12 + \frac{2(x+y)^2}{xy} = (x-y)^2 + \frac{(x-y)^2}{x^2y^2} + \frac{2(x-y)^2}{xy} \ge 0$$

Solution 2 by Boris Colkakovic-Belgrade-Serbie

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\le\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2\Leftrightarrow 4(xy+x+1)(xy+y+1)\le$$

$$\le xy\left(2+x+y+\frac{x+y}{xy}\right)^2\Leftrightarrow 4((xy)^2+xy^2+xy+x^2y+xy+x+xy+y+1)\le$$

$$\le xy\left(4+(x+y)^2+\left(\frac{x+y}{xy}\right)^2+4(x+y)+\frac{4(x+y)}{xy}+\frac{2(x+y)^2}{xy}\right)\Leftrightarrow$$

$$\Leftrightarrow 4(xy)^2+4xy(x+y)+12xy+4(x+y)+4\le 4xy+xy(x+y)^2+\frac{(x+y)^2}{xy}+$$

$$+4xy(x+y)+4(x+y)+2(x+y)^2\Leftrightarrow$$

$$\Leftrightarrow xy(x+y)^2+\frac{(x+y)^2}{xy}+2(x+y)^2-8xy-4(xy)^2-4\ge 0\Leftrightarrow$$

$$\Leftrightarrow (x+y)^2\left(xy+\frac{1}{xy}\right)+2(x-y)^2-4((xy)^2+1)\ge 0\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{(xy)^2+1}{xy}\right)(x+y)^2+2(x-y)^2-4((xy)^2+1)\ge 0\Leftrightarrow$$

$$\Leftrightarrow (x+y)^2((xy)^2+1)+2xy(x-y)^2-4xy((xy)^2+1)\ge 0\Leftrightarrow$$

$$\Leftrightarrow ((xy)^2+1)[(x+y)^2-4xy]+2xy(x-y)^2\ge 0\Leftrightarrow$$

$$\Leftrightarrow ((xy)^2+1)(x-y)^2+2xy(x-y)^2\ge 0\Leftrightarrow (x-y)^2(xy+1)^2\ge 0$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\le\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2$$

If $4(xy+x+1)(xy+y+1)\le(2xy+x^2y+y^2x+x+y)\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)$

If $4(xy^2+xy+y)(x^2y+xy+x)\le$

$$(2xy+x^2y+y^2x+x+y)(2xy+x^2y+y^2x+y)=(2xy+x^2y+y^2x+x+y)^2$$

and it is true. Because: $4(xy^2+xy+y)(x^2y+xy+x)\le(xy^2+x^2y+2xy+x+y)^2$

Therefore, it's true.

Solution 4 by Trieu Tan Hung-Vietnam

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\le\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2$$

$$\Leftrightarrow 4\left(\frac{xy+x+1}{y}\right)\left(\frac{xy+y+1}{x}\right) \le \left(1+x+\frac{1}{y}+1+y+\frac{1}{x}\right)^2$$

$$\Leftrightarrow 4\left(\frac{xy+x+1}{x}\right)\left(\frac{xy+y+1}{y}\right) \le \left(\frac{xy+y+1}{y}+\frac{xy+x+1}{x}\right)^2 \quad (*)$$

Let $a = \frac{xy+x+1}{x}; b = \frac{xy+y+1}{y}$

$$(*) \Leftrightarrow 4ab \le (a+b)^2 \Leftrightarrow (a-b)^2 \ge 0 \quad (true)$$

488. If $x, y, z > 0$ then:

$$\frac{e^{x^3+y^3}}{e^{2(x+y)}} + \frac{e^{y^3+z^3}}{e^{2(y+z)}} + \frac{e^{z^3+x^3}}{e^{2(z+x)}} \ge \frac{1}{e^2}\left(x^x y^y + y^y z^z + z^z x^x\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{x}{1+x} \le \ln(1+x) \le x$, ***replacing*** x ***by*** $x-1$

$\frac{x-1}{x} \le \ln x \le x-1$. ***Let*** $f(x) = x^3 - 2x - x\ln x + 1 \ge x^3 - 2x - x(x-1) + 1$

$= x^3 - x^2 - x + 1 = (x+1)(x-1)^2 \ge 0$. ***Hence*** $f(x) \ge 0$ ***for all*** $x \ge 1$

Hence $x^3 - 2x \ge x\ln x - 1$

$$\therefore \sum_{cyc} e^{x^3+y^3-2(x+y)} \ge \sum_{cyc} e^{x\ln x - 1 + y\ln y - 1} \therefore \sum_{cyc} \frac{e^{x^3+y^3}}{e^{2(x+y)}} \ge \frac{1}{e^2}\sum_{cyc} x^x y^y$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall m > -1, e^m \ge m+1 \Rightarrow m \overset{(1)}{\ge} \ln(m+1)$$

$$\therefore \ln x = \ln\big(1+(x-1)\big) \overset{by\ (1)}{\le} x-1 \ (\because x-1 > -1)$$

$$\Rightarrow \ln x - (x-1) \le 0 \Rightarrow x\big(\ln x - (x-1)\big) \le 0$$

$\Rightarrow x\ln x \overset{(2)}{\le} x^2 - x$. ***Similarly,*** $y\ln x \overset{(3)}{\le} y^2 - y$

Now, $\frac{x^{x^3+y^3}}{e^{2(x+y)}} \ge \frac{x^x y^y}{e^2} \Leftrightarrow e^{2+x^3+y^3} \ge e^{2(x+y)} \cdot x^x y^y$

$$\Leftrightarrow 2 + x^3 + y^3 \ge 2(x+y) + x\ln x + y\ln y$$

$$\Leftrightarrow (1 + x^3 - 2x - x\ln x) + (1 + y^3 - 2y - y\ln y) \overset{(4)}{\ge} 0$$

(2) $\Rightarrow 1 + x^3 - 2x - x\ln x \ge 1 + x^3 - 2x + x - x^2$

$$= x^3 - x^2 - x + 1 = x^2(x-1) - (x-1)$$

$$= (x-1)^2(x+1) \geq 0 \Rightarrow 1 + x^3 - 2x - x\ln x \overset{(a)}{\geq} 0$$

Similarly, using (3), $1 + y^3 - 2y - y\ln y \overset{(b)}{\geq} 0$

(a)+(b)⇒(4) is true $\Rightarrow \frac{e^{x^3+y^3}}{e^{2(x+y)}} \overset{(i)}{\geq} \frac{x^x y^y}{e^2}$

Similarly, $\frac{e^{y^3+z^3}}{e^{2(y+z)}} \overset{(ii)}{\geq} \frac{y^y z^z}{e^2}$ ***&*** $\frac{e^{z^3+x^3}}{e^{2(z+x)}} \overset{(iii)}{\geq} \frac{z^z x^x}{e^2}$

(i)+(ii) +(iii)⇒ given inequality is true (proved)

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $a, b > 0$***, we get that:***

$$(a-1)^2(a+1) \geq 0 \Rightarrow a^3 - a^2 - a + 1 \geq 0 \Rightarrow a^3 + 1 - a \geq a^2$$

$\Rightarrow a^2 + \frac{1}{a} - 1 \geq 0 \Rightarrow e^{a^2+\frac{1}{a}-2} \geq a \Rightarrow e^{a^3+1-2a} \geq a^a$***. Similarly,*** $e^{b^3+1-2b} \geq b^b$

Hence, $e^{a^3+1-2a} \cdot e^{b^3+1-2b} \geq a^a b^b \Rightarrow e^{a^3+b^3+2-2(a+b)} \geq a^a b^b$

$\Rightarrow \frac{e^2 \cdot e^{a^3+b^3}}{e^{2(a+b)}} \geq a^a b^b \Rightarrow \frac{e^{a^3+b^3}}{e^{2(a+b)}} \geq \frac{a^a b^b}{e^2}$***. Hence, for*** $x, y, z > 0$***, we have:***

$\frac{e^{x^3+y^3}}{e^{2(x+y)}} + \frac{e^{y^3+z^3}}{e^{2(y+z)}} + \frac{e^{z^3+x^3}}{e^{2(z+x)}} \geq \frac{x^x y^y}{e^2} + \frac{y^y z^z}{e^2} + \frac{z^z x^x}{e^2} = \frac{1}{2}(x^x y^y + y^y z^z + z^z x^x)$ ***ok***

Therefore, it's true.

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$\left.\begin{matrix} \frac{e^{x^3}}{e^{2x}} \overset{?}{\geq} \frac{1}{e} \cdot e^{x\ln(x)} \\ e^{x^3-2x} \geq e^{x\ln(x)-1} \end{matrix}\right\}$***. So: let be the function***

$$f(x) = x^3 - x\ln(x) + 1 - 2x;\ f(x) = x^3 - 2x + 1 - x \cdot \ln(x)$$

$$]0, +\infty[$$

$$\lim_{n\to 0} f(x) = 1, \lim_{n\to+\infty} f(x) = \lim_{n\to+\infty}\left[x^3\left(1 - \underbrace{\frac{2}{x^2}}_{0} + \underbrace{\frac{1}{x^3}}_{0} + \underbrace{\frac{\ln(x)}{x^2}}_{0}\right)\right] = +\infty$$

$$f'(x) = 3x^2 - 2 - [\ln(x) + 1]$$

$$f'(x) = 3x^2 - \ln(x) - 3, f(1) = 1 + 1 - 2 = 0$$

$$f'(1) = 0$$

x	0	1	$+\infty$
$f'(x)$	$-\ -\ -\ -\ -\ -\ -\ -\ -\ -$	0	$+\ +\ +\ +\ +\ +\ +\ +\ +$
$f(x)$	1	0	$+\infty$

$$\forall x\in\,]0,+\infty[:f(x)\geq 0;\ x^3-2x-\ln(x)+1\geq 0;\ x^3-2x\geq x\ln(x)-1$$

$$e^{x^3-2x}\geq e^{x\ln(x)}\cdot e^{-1};\ \frac{e^{x^3}}{e^{2x}}\geq\frac{x^x}{e},\frac{e^{y^3}}{e^{2y}}\geq\frac{y^y}{e}\Rightarrow\frac{e^{x^3+y^3}}{e^{2x}\cdot e^{2y}}\geq\frac{x^x\cdot y^y}{e^2}$$

$\frac{e^{x^3+y^3}}{e^{2(x+y)}}\geq\frac{1}{e^2}x^x\cdot y^y$. So: $\frac{e^{x^3+y^3}}{e^{2(x+y)}}+\frac{e^{y^3+z^3}}{e^{2(y+z)}}+\frac{e^{z^3+x^3}}{e^{2(z+x)}}\geq\frac{1}{e^2}(x^x\cdot y^y+y^y\cdot z^z+z^z\cdot x^x)$

489. Let $a,b,c\in(0;+\infty)$. Prove:

$$\left(\frac{a^2-ab+b^2}{b^2+bc+c^2}\right)^3+\left(\frac{b^2-bc+c^2}{c^2+ca+a^2}\right)^3+\left(\frac{c^2-ca+a^2}{a^2+ab+b^2}\right)^3\geq\frac{1}{9}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Do Duc Huy-Vietnam

Let $a,b,c>0$. We have: $a^2-ab+b^2\geq\frac{1}{3}(a^2+ab+b^2)\Leftrightarrow 2(a-b)^2\geq 0$ (true)

$$\text{So, }\sum\left(\frac{a^2-ab+b^2}{b^2+bc+c^2}\right)^3\geq 3\cdot\frac{a^2-ab+b^2}{b^2+bc+c^2}\cdot\frac{b^2-bc+c^2}{c^3+ca+a^2}\cdot\frac{c^2-ca+a^2}{a^2+ab+b^2}$$

$$\geq 3\cdot\frac{1}{3}\cdot\frac{a^2+ab+b^2}{b^2+bc+c^2}\cdot\frac{1}{3}\cdot\frac{b^2+bc+c^2}{c^2+ca+a^2}\cdot\frac{c^2+ca+a^2}{a^2+ab+b^2}\cdot\frac{1}{3}=\frac{1}{9}\Rightarrow \text{Q.E.D.}$$

$$\text{“}=\text{”}\Leftrightarrow a=b=c$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$, we have: $\frac{a^2+ab+b^2}{b^2+bc+c^2}\geq\frac{3ab}{b^2+bc+c^2},\frac{b^2+bc+c^2}{c^2+ca+a^2}\geq\frac{3bc}{c^2+ca+a^2}$,

$\frac{c^2+ca+a^2}{a^2+ab+b^2}\geq\frac{3ca}{a^2+ab+b^2}$. Hence, $3\left(\frac{(a^2+ab+b^2)}{b^2+bc+c^2}+\frac{(b^2+bc+c^2)}{c^2+ca+a^2}+\frac{(c^2+ca+a^2)}{a^2+ab+b^2}\right)$

$$\geq\frac{b^2+bc+c^2+bab}{b^2+bc+c^2}+\frac{c^2+ca+a^2+bbc}{c^2+ca+a^2}+\frac{a^2+ab+b^2+bca}{a^2+ab+b^2}$$

$$\Rightarrow\frac{a^2+ab+b^2}{b^2+bc+c^2}+\frac{b^2+bc+c^2}{c^2+ca+a^2}+\frac{c^2+ca+ab}{a^2+ab+b^2}$$

$$\geq \frac{(b^2+bc+c^2)+bab}{3(b^2+b+c^2)}+\frac{c^2+ca+a^2+bbc}{3(c^2+ca+a^2)}+\frac{a^2+ab+b^2+bca}{3(a^2+ab+b^2)}$$

$$=1+\frac{2ab}{b^2+bc+c^2}+\frac{2bc}{c^2+ca+ca^2}+\frac{2ca}{a^2+ab+b^2}$$

$$\Rightarrow \frac{a^2-ab+b^2}{b^2+bc+c^2}+\frac{b^2-bc+c^2}{c^2+ca+a^2}+\frac{c^2-ca+a^2}{a^2+ab+b^2}\geq 1$$

$$\Rightarrow \left(\frac{a^2-ab+b^2}{b^2+bc+c^2}+\frac{b^2-bc+c^2}{c^2+ca+a^2}+\frac{c^2-ca+a^2}{a^2+ab+b^2}\right)^3\geq 1$$

$$\Rightarrow \frac{1}{9}\left(\frac{a^2-ab+b^2}{b^2+bc+c^2}+\frac{b^2-bc+c^2}{c^2+ca+ca^2}+\frac{c^2+a+a^2}{a^2+ab+b^2}\right)^3\geq \frac{1}{9}$$

$$\Rightarrow \left(\frac{a^2-ab+b^2}{b^2+bc+c^2}\right)^3+\left(\frac{b^2-bc+c^2}{c^2+ca+c^2}\right)^3+\left(\frac{c^2-ca+a^2}{a^2+ab+b^2}\right)^3\geq \frac{1}{9}.$$ ***Therefore, it is true.***

490. If $a,b,c>0$ then:

$$\sum_{cyc}\left(\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}+\frac{a+b}{2\sqrt{ab}}\right)\geq 6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}=\left(1+\left(\frac{2\sqrt{ab}}{a+b}-1\right)\right)^{\frac{a+b}{2\sqrt{ab}}}\overset{Bernoulli}{\geq}1+\left(\frac{2\sqrt{ab}}{a+b}-1\right)\left(\frac{a+b}{2\sqrt{ab}}\right)$$

$$\left[\because \frac{2\sqrt{ab}}{a+b}-1>-1\ \&\ \frac{a+b}{2\sqrt{ab}}\overset{A-G}{\geq}1\right]==2-\frac{a+b}{2\sqrt{ab}}\Rightarrow\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}\overset{(1)}{\geq}2-\frac{a+b}{2\sqrt{ab}}$$

Similarly, $\left(\frac{2\sqrt{bc}}{b+c}\right)^{\frac{b+c}{2\sqrt{bc}}}\overset{(2)}{\geq}2-\frac{b+c}{2\sqrt{bc}}$ & $\left(\frac{2\sqrt{ca}}{c+a}\right)^{\frac{c+a}{2\sqrt{ca}}}\overset{(3)}{\geq}2-\frac{c+a}{2\sqrt{ca}}$

(1) + (2) + (3) $\Rightarrow LHS\geq 6-\sum\frac{a+b}{2\sqrt{ab}}+\sum\frac{a+b}{2\sqrt{ab}}=6$ ***(Proved)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $f(t)=t^{\frac{1}{t}}+\frac{1}{t}\left(t=\frac{a+b}{2\sqrt{ab}};t\geq 1\right)\Rightarrow f'(t)=\frac{\sqrt[t]{t}(1-\log t)-1}{t^2};f'(t)=0\Leftrightarrow 1-\log t=\frac{1}{\sqrt[t]{t}}$

$$\Leftrightarrow 1 = \log t + \frac{1}{\sqrt[t]{t}} \Leftrightarrow t = 1$$

(Because: $g(t) = \log t + \frac{1}{\sqrt[t]{t}}\ (t \ge 1) \Rightarrow g'(t) = t^{-\frac{1}{t}-2}\left(t^{\frac{1}{t}+1} + \log t - 1\right) \ge 0\ \ (t \ge 1)$

$\Rightarrow g(t) \ge g(1) = 1$*). Hence,*

$$t \ge 1 \Leftrightarrow f'(t) \ge 0 \Rightarrow f(t) \nearrow [1; +\infty) \Rightarrow f(t) \ge f(1) = 2$$

$\Rightarrow t^{\frac{1}{t}} + \frac{1}{t} \ge 2\ \ (t \ge 1) \Rightarrow \left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2ab}{a+b}} + \frac{2ab}{a+b} \ge 2$ *(etc)*$\Rightarrow LHS \overset{(*)}{\ge} 2 + 2 + 2 = 6$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$\left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b;\ a + b \ge 2\sqrt{ab} \Rightarrow \frac{a+b}{2\sqrt{ab}} \ge 1$*. Let be the function:*

$$f(n) = \left(\frac{1}{x}\right)^x + x: x \in [1, +\infty[;\ f(n) = e^{x\ln\left(\frac{1}{x}\right)} + x = e^{-x\ln(x)} + x$$

$$f(1) = 2, \lim_{x\to\infty}\left(f(n)\right) = +\infty;\ f'(n) = -(\ln(n) + 1)\cdot e^{-x\ln(x)} + 1$$

$$f'(n) = 0 \Rightarrow x = 1, f(1) = 2$$

x	1 $\qquad\qquad$ $+\infty$
$f'(x)$	0 + + + + + + + + + + + + + + + + + + +
$f(x)$	2 $\nearrow$ $+\infty$

$\forall x \in [1, +\infty[: f(x) \ge 2$*. So:* $\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} + \frac{a+b}{2\sqrt{ab}} \ge 2$

So: $\sum_{cyc}\left(\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} + \frac{a+b}{2\sqrt{ab}}\right) \ge 3\cdot 2;\ \sum_{cyc}\left(\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} + \frac{a+b}{2\sqrt{ab}}\right) \ge 6$

Solution 4 by Michael Sterghiou-Greece

$\sum_{cyc}\left(\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} + \frac{a+b}{2\sqrt{ab}}\right) \ge 6$ **(1)** *- homogeneous, so assume* $\sum_{cyc}\frac{a+b}{2\sqrt{ab}} = 3$

Let $\frac{a+b}{2\sqrt{ab}} = x \ge 1$ *(AM-GM) (1) can be written as* $\sum_{cyc}\left[\left(\frac{1}{x}\right)^x + x\right] \ge 6$ *(2)*

(also $y = \frac{b+c}{2\sqrt{bc}}, z = \frac{c+d}{2\sqrt{ca}}$*, both* ≥ 1*). The function* $\left(\frac{1}{x}\right)^x + x$ *is convex on* $[1, \infty)$ *as*

$$f''(x) = x^{-x}\left[(\ln x + 1)^2 \frac{1}{x}\right] \geq 0 \quad (3)$$

To see this, write (3) as $x(\ln x + 1)^2 - 1 \geq 0$ *(as* $x^{-x} > 0$*) which for* $x < 1$ *does not add:* $x < 1$ *and* $(\ln x + 1)^2 < 1$ *while it does for* $x \geq 1$*.*

Apply Jensen in (2) with $x + y + z = 3$ *we get* $\sum_{cyc}\left(\frac{1}{x}\right)^x + x \geq 3 \cdot (1 + 1) = 6$*. Done.*

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $0 < x \leq 1$*, we get this fact:* $x^{(x+1)} \geq x^{2x} \Rightarrow x^{\left(\frac{x+1}{x}\right)} \geq x^2 \Rightarrow \sqrt{x^{\left(\frac{x+1}{x}\right)}} \geq x$

$\Rightarrow 2\sqrt{x^{\left(\frac{x+1}{x}\right)}} \geq 2x \Rightarrow x^{\frac{x+1}{x}} + 1 \geq 2x \Rightarrow x^{\frac{1}{x}} + \frac{1}{x} \geq 2: 0 < x \leq 1$*, fact. Hence for* $a, b, c > 0$*,*

we have: $a + b \geq 2\sqrt{ab}, b + c \geq 2\sqrt{bc}, c + a \geq 2\sqrt{ca}$

and $\left[\left(\frac{2\sqrt{ab}}{a+b}\right)^{\left(\frac{a+b}{2\sqrt{ab}}\right)} + \frac{a+b}{2\sqrt{ab}}\right] + \left[\left(\frac{2\sqrt{bc}}{b+c}\right)^{\left(\frac{b+c^2}{2\sqrt{bc}}\right)} + \frac{b+c}{2\sqrt{bc}}\right] + \left[\left(\frac{2\sqrt{ca}}{c+a}\right)^{\left(\frac{c+a}{2\sqrt{ca}}\right)} + \frac{c+a}{2\sqrt{ca}}\right] \geq$

$\geq 2 + 2 + 2 = 6$ *ok. Because:* $\frac{a+b}{2\sqrt{ab}}, \frac{b+c}{2\sqrt{bc}}, \frac{c+a}{2\sqrt{ca}} \geq 1, \frac{2\sqrt{ab}}{a+b}, \frac{2\sqrt{bc}}{b+c}, \frac{2\sqrt{ca}}{c+a} \leq 1$

Therefore, it is to be true.

491. If $a, b, c > 0, a + b + c = 6$ **then:**

$$\frac{31 - 6b - 6c}{a^2 + b + c - 5} + \frac{31 - 6c - 6a}{b^2 + c + a - 5} + \frac{31 - 6a - 6b}{c^2 + a + b - 5} \leq 7$$

Proposed by Iuliana Trașcă-Romania

Solution by Serban George Florin-Romania

$$\sum_{a,b,c} \frac{31 - 6b - 6c}{a^2 + b + c - 5} = \sum_{a,b,c} \frac{31 - 6(b + c)}{a^2 + b + c - 5} = \sum_{a,b,c} \frac{31 - 6(6 - a)}{a^2 + 6 - a - 5} =$$

$$\sum \frac{6a - 5}{a^2 - a + 1}, \begin{matrix} a + b + c = 6 \\ a, b, c > 0 \end{matrix}, \begin{matrix} a + b = 6 - c > 0 \\ \Rightarrow c < 6 \end{matrix}$$

$b + c = 6 - a > 0 \Rightarrow a < 6, a + c = 6 - b > 0 \Rightarrow b < 6 \Rightarrow a, b, c < 0 \Rightarrow a, b, c \in (0, 6)$

$$\frac{6a - 5}{a^2 - a + 1} \leq -\frac{a + 9}{3}, (\forall) a \in (0, 6)$$

$$a^2 - a + 1 > 0, \Delta = -3 < 0, (\forall) a \in \mathbb{R} \Rightarrow 18a - 15 \leq (-a + 9)(a^2 - a + 1)$$

$$18a-15\le -a^3+a^2-a+9a^2-9a+9$$

$$-a^3+10a^2-28a+24\ge 0;\ -a^3+2a^2+8a^2-16a-12a+24\ge 0$$

$$-a^2(a-2)+8a(a-2)-12(a-2)\ge 0;\ (a-2)(-a^2+8a-12)\ge 0$$

$$(a-2)(-a^2+2a+6a-12)\ge 0;\ (a-2)[-a(a-2)+6(a-2)]\ge 0$$

$$(a-2)(a-2)(6-a)\ge 0,(a-2)^2(6-a)\ge 0$$

$$(a-2)^2\ge 0, 6-a>0\Rightarrow a<6, (\forall)a\in(0,6)\ \textit{true}$$

$$\Rightarrow \sum\frac{31-6b-6c}{a^2+b+c-5}\le\sum\frac{-a+9}{3}=-\frac{1}{3}\sum a+\sum 3=$$

$$=-\frac{1}{3}\cdot 6+9=-2+9=7\ \textit{true.}$$

492. **If $a,b,c>1, m,n>0$ then:**

$$\frac{\log_a^2 b}{m\log_b c+n\log_c a}+\frac{\log_b^2 c}{m\log_c a+n\log_a b}+\frac{\log_c^2 a}{m\log_a b+n\log_b c}\ge\frac{3}{m+n}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$LHS=\frac{\log_a^2 b}{m\cdot\log_b c+n\cdot\log_c a}+\frac{\log_b^2 c}{m\cdot\log_c a+n\cdot\log_a b}+\frac{\log_c^2 a}{m\cdot\log_a b+n\cdot\log_b c}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(\log_a b+\log_b c+\log_c a)^2}{m(\log_b c+\log_c a+\log_a b)+n(\log_c a+\log_a b+\log_b c)}$$

$$=\frac{\frac{\ln b}{\ln a}+\frac{\ln c}{\ln b}+\frac{\ln a}{\ln c}}{m+n}\overset{AM-GM}{\ge}\frac{3\sqrt[3]{\frac{\ln b}{\ln a}\cdot\frac{\ln c}{\ln b}\cdot\frac{\ln a}{\ln c}}}{m+n}=\frac{3}{m+n}=RHS$$

Solution 2 by Sagar Kumar-Patna Bihar-India

Applying Bergström: $\frac{(\log_a b)^2}{m\log_b c+n\log_c a}+\frac{(\log_b c)^2}{m\log_c a+n\log_a b}+\frac{(\log_c c)^2}{m\log_a b+n\log_b c}$

$$\ge\frac{(\log_a b+\log_b c+\log_c a)^2}{(\log_b c+\log_c a+\log_a b)(m+n)}\ge\frac{\log_a b+\log_b c+\log_c a}{(m+n)}$$

$$\ge\frac{3\left((\log_a b)(\log_b c)(\log_c a)\right)^{\frac{1}{3}}}{m+n}\ (AM\ge GM)\quad\ge\frac{3}{m+n}\ \textit{(proved)}$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\log_a b = x > 0, \log_b c = y > 0, \log_c a = z > 0$$

$$A = \frac{x^2}{my+nz} + \frac{y^2}{mz+nx} + \frac{z^2}{mx+ny} \geq \frac{3}{m+n}$$

$$A \geq \frac{(x+y+z)^2}{m(x+y+z)+n(x+y+z)} = \frac{(x+y+z)^2}{(x+y+z)(m+n)} = \frac{x+y+z}{m+n} \geq \frac{3}{m+n}$$

Because $\log_a b + \log_b c + \log_c a \geq 3\sqrt[3]{\log_a b \cdot \log_b c \cdot \log_c a} = 3$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$LHS \overset{Bergstrom}{\geq} \frac{\left(\sum\frac{\log b}{\log a}\right)^2}{(m+n)\left(\sum\frac{\log b}{\log a}\right)} = \frac{\left(\sum\frac{\log b}{\log a}\right)}{m+n} \overset{A-G}{\geq} \frac{3\sqrt[3]{\frac{\log b}{\log a}\cdot\frac{\log c}{\log b}\cdot\frac{\log a}{\log c}}}{m+n} = \frac{3}{m+n}$$

493. Let $a, b, c \in (0; +\infty) \wedge a^3 + b^3 + c^3 = 3$. Prove:

$$\left(a + 2\sqrt{b} + \sqrt[3]{c}\right)\left(b + 2\sqrt{c} + \sqrt[3]{a}\right)\left(c + 2\sqrt{a} + \sqrt[3]{b}\right) \leq 64$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Marian Ursărescu-Romania

We must show: $\sqrt[3]{\left(a + 2\sqrt{b} + \sqrt[3]{c}\right)\left(b + 2\sqrt{c} + \sqrt[3]{a}\right)\left(c + 2\sqrt{a} + \sqrt[3]{b}\right)} \leq 4$ *(1)*

But $\sqrt[3]{\left(a + 2\sqrt{b} + \sqrt[3]{c}\right)\left(b + 2\sqrt{c} + \sqrt[3]{a}\right)\left(c + 2\sqrt{a} + \sqrt[3]{b}\right)} \leq \frac{a+b+c+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}$ *(2)*

Form (1)+(2) we must show:

$$a + b + c + 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) + \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \leq 12 \quad (3)$$

From Hölder's inequality we have: $a^3 + b^3 + c^3 \geq \frac{(a+b+c)^3}{9} \Leftrightarrow$

$$(a+b+c)^3 \leq 27 \Leftrightarrow a + b + c \leq 3 \quad (4)$$

From Cauchy's inequality: $\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \leq 3(a+b+c) \leq 9 \Rightarrow$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3 \quad (5)$$

Again, from Hölder's inequality $\Rightarrow$

$$\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{9} \leq a + b + c \leq 3 \Rightarrow \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \leq 3 \quad (6)$$

From (4)+(5)+(6)$\Rightarrow$

$$a+b+c+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\le 12\Rightarrow \textit{(3) is true.}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Since $a^3+b^3+c^3=3$*, where* $a,b,c>0$*. We get* $a+b+c\le 3$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\le 3;\ \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\le 3$$

Hence $\left(a+2\sqrt{b}+\sqrt[3]{c}\right)\left(b+2\sqrt{c}+\sqrt[3]{a}\right)\left(c+2\sqrt{a}+\sqrt[3]{b}\right)$

$$\le\left(\frac{a+2\sqrt{b}+\sqrt[3]{c}+b+2\sqrt{c}+\sqrt[3]{a}+c+2\sqrt{a}+\sqrt[3]{b}}{3}\right)^3$$

$$=\left(\frac{a+b+c+2\sqrt{a}+2\sqrt{b}+2\sqrt{c}+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\right)^3\le\left(\frac{3+6+3}{3}\right)^3=(4)^3=64$$

Therefore, it is true.

494. Let $a,b,c,d\in(0;+\infty)\wedge abcd=1$. Prove:

$$\frac{d}{a^5+b^5+c^5+d}+\frac{c}{a^5+b^5+d^5+c}+\frac{b}{a^5+d^5+c^5+b}+\frac{a}{d^5+b^5+c^5+a}\le 1$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a^5+b^5+c^5+d\overset{Chebyshev}{\ge}\frac{1}{3}(a^2+b^2+c^2)(a^3+b^3+c^3)+d$$

$$\overset{A-G}{\ge} abc(a^2+b^2+c^2)+d\overset{abcd=1}{=}abc(a^2+b^2+c^2)+d(abcd)$$

$$=abc(a^2+b^2+c^2+d^2)=\frac{a^2+b^2+c^2+d^2}{d}\quad\left(\because abc=\frac{1}{d}\right)$$

$$\Rightarrow\frac{d}{a^5+b^5+c^5+d}\overset{(1)}{\le}\frac{d^2}{a^2+b^2+c^2+d^2}.\ \textit{Similarly,}\ \frac{a}{b^5+c^5+d^5+a}\overset{(2)}{\le}\frac{a^2}{a^2+b^2+c^2+d^2}$$

$$\frac{b}{c^5+d^5+a^5+b}\overset{(3)}{\le}\frac{b^2}{a^2+b^2+c^2+d^2}\ \&\ \frac{c}{d^5+a^5+b^5+c}\overset{(4)}{\le}\frac{c^2}{a^2+b^2+c^2+d^2}$$

$$\textit{(1)+(2)+(3)+(4)}\Rightarrow LHS\le\frac{\sum a^2}{\sum a^2}=1\quad\textit{(proved)}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c,d>0$ *and* $abcd=1$*, we have:*

1. $\frac{a^5+b^5+c^5}{a^5+b^5+c^5+d} \geq \frac{a^2+b^2+c^2}{a^2+b^2+c^2+d^2}$

2. $\frac{b^5+c^5+d^5}{b^5+c^5+d^5+a} \geq \frac{b^2+c^2+d^2}{b^2+c^2+d^2+a^2}$

3. $\frac{c^5+d^5+a^5}{c^5+d^5+a^5+b} \geq \frac{c^2+d^2+a^2}{c^2+d^2+a^2+b^2}$

4. $\frac{d^5+a^5+b^5}{d^5+a^5+b^5+c} \geq \frac{d^2+a^2+b^2}{d^2+a^2+b^2+c^2}$

Hence $\frac{a^5+b^5+c^5}{a^5+b^5+c^5+d} + \frac{b^5+c^5+d^5}{b^5+c^5+d^5+a} + \frac{c^5+d^5+a^5}{c^5+d^5+a^5+b} + \frac{d^5+a^5+b^5}{d^5+a^5+b^5+c} \geq \frac{3(a^2+b^2+c^2+d^2)}{(a^2+b^2+c^2+d^2)} = 3$

Hence $\frac{(-d)}{a^5+b^5+c^5+d} + \frac{(-a)}{b^5+c^5+d^5+a} + \frac{(-b)}{c^5+d^5+a^5+b} + \frac{(-c)}{d^5+a^5+b^5+c} \geq (-1)$

That is $\frac{d}{a^5+b^5+c^5+d} + \frac{a}{b^5+c^5+d^5+a} + \frac{b}{c^5+d^5+a^5+b} + \frac{c}{d^5+a^5+b^5+c} \leq 1$

Therefore, it is true. Remark: $\frac{a^5+b^5+c^5}{a^5+b^5+c^5+d} \geq \frac{a^2+b^2+c^2}{a^2+b^2+c^2+d^2}$

If $a^5d^2 + b^5d^2 + c^5d^2 \geq a^2d + b^2d + c^2d$

If $a^5d + b^5d + c^5d \geq a^2 + b^2 + c^2$

If $\frac{a^4}{bc} + \frac{b^4}{ac} + \frac{c^4}{ab} \geq a^2 + b^2 + c^2$

If $\frac{(a^2+b^2+c^2)^2}{(bc+ac+ab)} \geq (a^2 + b^2 + c^2)$

If $a^2 + b^2 + c^2 \geq ab + bc + ca$

495. Let $a, b, c > 0$ and $\sum ab = 1$. Show that:

$$4 + 3\sum a^3b \leq \sum a^2 + 4\left(\sum a\right)\left(\sum a^2b\right)$$

Proposed by Andrei Ștefan Mihalcea-Romania

Solution by Serban George Florin-Romania

$$\sum a \cdot \sum a^2b = (a+b+c)(a^2b+b^2c+c^2a) = a^3b +$$

$$+ab^2c + a^2c^2 + a^2b^2 + b^3c + abc^2 + a^2bc + b^2c^2 + a^2c^2 = \sum a^3b + \sum a^2b^2 + abc(a+b+c)$$

$$\sum ab = 1 \Rightarrow \left(\sum ab\right)^2 = 1^2, \sum a^2b^2 + 2abc(a+b+c) = 1 \Rightarrow \sum a^2b^2 = 1 - 2abc(a+b+c)$$

$$\sum a \cdot \sum a^2b = \sum a^3b + 1 - 2abc(a+b+c) + abc(a+b+c) = \sum a^3b + 1 - abc(a+b+c)$$

$$4+3\sum a^3b\le\sum a^2+4\left(\sum a\right)\cdot\left(\sum a^2b\right)$$

$$\Rightarrow 4+3\sum a^3b\le\sum a^2+4\sum a^3b+4-4abc(a+b+c)\Rightarrow\sum a^2+\sum a^3b\ge 4abc(a+b+c)$$

$$\sum a^2(1+ab)\ge 4abc(a+b+c)$$

$$ab+bc+ac=1\overset{Ma\ge Mg}{\ge}3\sqrt[3]{a^2b^2c^2}\Rightarrow 1\ge 27a^2b^2c^2\Rightarrow abc\le\frac{1}{3\sqrt{3}}$$

$$\sum a^2(1+ab)=\sum a^2+\sum a^3b\ge 4abc(a+b+c)$$

$$\sum a^3b=\sum\frac{a^2}{\frac{1}{ab}}\overset{Bergstrom}{\ge}\frac{(a+b+c)^2}{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}=\frac{abc(a+b+c)^2}{a+b+c}=abc(a+b+c)$$

$$\sum a^2\ge 3abc(a+b+c),(a+b+c)^2\le 3(a^2+b^2+c^2)$$

$$\Rightarrow\sum a^2\ge\frac{(a+b+c)^2}{3}\ge 3abc(a+b+c)\Rightarrow(a+b+c)^2\ge 9abc(a+b+c)$$

$$\Rightarrow a+b+c\ge 9abc, a+b+c\overset{Ma\ge Mg}{\ge}3\sqrt[3]{abc}\ge 9abc$$

$$3\sqrt[3]{abc}\ge 9abc,\sqrt[3]{abc}\ge 3abc, abc\ge 27a^3b^3c^3\Rightarrow a^2b^2c^2\le\frac{1}{27}, abc\le\frac{1}{3\sqrt{3}}, \textit{true}$$

$$\Rightarrow\sum a^2(1+ab)=\sum a^2+\sum a^3b\ge 3abc(a+b+c)+abc(a+b+c)=$$

$$=4abc(a+b+c), \textit{true.}$$

496. If $a,b,c>0, a+b+c=1$ then:

$$8\prod(1+ab-c)\le\prod(1+2a-a^2)$$

Proposed by Andrei Ștefan Mihalcea-Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

Because $a,b,c>0$ and $a+b+c=1$, we have:

$$8(1+ab-c)(1+bc-a)(1+ca-b)\le(1+2a+a^2)(1+2b-b^2)(1+2c-c^2)$$

$$\textit{If } 8(a+b+ab)(b+c+bc)(c+a+ca)\le$$

$$\le\big(1+a+a(1-a)\big)\big(1+b+b(1-b)\big)\big(1+c+c(1-c)\big)$$

$$\textit{If }\big(2(a+b+ab)\big)\big(2(b+c+bc)\big)\big(2(c+a+ca)\big)\le$$

$$\le(1+a+ab+ac)(1+b+ab+bc)(1+c+bc+ac)$$

$$=(a+b+ab+a+c+ac)(a+b+ab+b+c+bc)(b+c+bc+c+a+ca)$$

If $(2x)(2y)(2z) \le (x+y)(y+z)(z+x)$, *where* $a+b+ab: x, b+c+bc: y$ *and* $c+a+ca: z$ *and it is true.*

Because $(x+y)(y+z)(z+x) = x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + 2xyz \ge 8xyz$. *Therefore, this problem is true.*

497. If $x, y, z > 0$ then:

$$\sum_{cyc} \frac{x}{y} + 4\sum_{cyc} \frac{y}{x+y} + 4\sum_{cyc} \frac{xy}{(x+y)^2} \ge 12$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Must show that: $\frac{x}{y} + 4\cdot\frac{y}{x+y} + 4\cdot\frac{xy}{(x+y)^2} \ge 4$ *(etc)*

$$\Leftrightarrow x(x+y)^2 + 4y^2(x+y) + 4xy^2 \ge 4y(x+y)^2 \Leftrightarrow x^3 - 2x^2y + xy^2 \ge 0$$

$$\Leftrightarrow x(x^2 - 2xy + y^2) \ge 0 \Leftrightarrow x(x-y)^2 \ge 0 \quad \textit{(true)}$$

$$\Rightarrow \sum \frac{x}{y} + 4\sum \frac{y}{x+y} + 4\sum \frac{xy}{(x+y)^2} \ge 4+4+4 = 12 \textit{ Proved.}$$

Solution 2 by Șerban George Florin-Romania

$$\frac{x}{y} + \frac{4y}{x+y} + \frac{4xy}{(x+y)^2} \ge 4, (\forall)x, y > 0;\ \frac{x}{y} = t \Rightarrow t + \frac{4}{t+1} + \frac{4t}{(t+1)^2} - 4 \ge 0$$

$$\frac{t(t+1)^2 + 4(t+1) + 4t - 4(t+1)^2}{(t+1)^2} \ge 0, (\forall)t > 0$$

$$\Rightarrow \frac{t^3 + 2t^2 + t + 4t + 4 + 4t - 4t^2 - 8t - 4}{(t+1)^2} \ge 0$$

$$\frac{t^3 - 2t^2 + t}{(t+1)^2} \ge 0, \frac{t(t-1)^2}{(t+1)^2} \ge 0, \textit{ true } (\forall)t > 0$$

$$\Rightarrow \sum_{cyc}\left(\frac{x}{y} + \frac{4y}{x+y} + \frac{4xy}{(x+y)^2}\right) = \sum_{cyc}\frac{x}{y} + 4\sum_{cyc}\frac{y}{x+y} + 4\sum_{cyc}\frac{xy}{(x+y)^2} \ge 4+4+4 = 12 \textit{ true}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality} \Leftrightarrow \sum\frac{x}{y} + 4\sum\left\{\frac{y}{x+y} + \frac{xy}{(x+y)^2}\right\} - 12 \ge 0$$

$$\Leftrightarrow \sum\frac{x}{y} + 4\sum\left\{\frac{y}{x+y}\left(1 + \frac{x}{x+y}\right)\right\} - 12 \ge 0$$

$$\Leftrightarrow \sum \frac{x}{y} + 4\sum\left\{\frac{y(2x+y)}{(x+y)^2} - 1\right\} \geq 0 \Leftrightarrow \sum \frac{x}{y} + 4\sum\left[\frac{2xy + y^2 - (x+y)^2}{(x+y)^2}\right] \geq 0$$

$$\Leftrightarrow \sum \frac{x}{y} + 4\sum\left\{\frac{-x^2}{(x+y)^2}\right\} \geq 0 \Leftrightarrow \sum \frac{x}{y} \geq \sum \frac{4x^2}{(x+y)^2}$$

$$\Leftrightarrow \sum\left\{\frac{x}{y} - \frac{4x^2}{(x+y)^2}\right\} \geq 0 \Leftrightarrow \sum\left[x\left\{\frac{1}{y} - \frac{4x}{(x+y)^2}\right\}\right] \geq 0$$

$$\Leftrightarrow \sum\left[x\left\{\frac{(x+y)^2 - 4xy}{y(x+y)^2}\right\}\right] \geq 0 \Leftrightarrow \sum\left\{\frac{x(x-y)^2}{y(x+y)^2}\right\} \geq 0 \rightarrow \textbf{\textit{true (proved)}}$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$ ***give*** $a = x + y, b = y + z, c = z + x$

Hence $x = \frac{a+c-b}{2}, y = \frac{a+b-c}{2}, z = \frac{b+c-a}{2}$ ***and we have as follows***

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 4\left(\frac{x}{x+z} + \frac{y}{y+x} + \frac{z}{z+x}\right) + 4\left(\frac{xy}{(x+y)^2} + \frac{yz}{(y+z)^2} + \frac{zx}{(z+x)^2}\right)$$

$$= \frac{a+c-b}{a+b-c} + \frac{a+b-c}{b+c-a} + \frac{b+c-a}{a+c-b} + 2\left(\frac{a+c-b}{c} + \frac{a+b-c}{a} + \frac{b+c-a}{b}\right) +$$

$$+ \frac{(a+c-b)}{a^2} + \frac{(a+b-c)(b+c-a)}{b^2} + \frac{(b+c-a)(a+c-b)}{c^2}$$

$$= \left[\frac{a+c-b}{a+b-c} + \frac{(a+c-b)(a+b-c)}{a^2}\right] + \left[\frac{a+b-d}{b+c-a} \cdot \frac{(a+b-c)(b+c-a)}{b^2}\right] +$$

$$+ \left[\frac{b+c-a}{a+c-b} + \frac{(b+c-a)(a+c-b)}{c^2}\right] + 2\left(\frac{a+c-b}{c} + \frac{a+b-c}{a} + \frac{b+c-a}{b}\right)$$

$$\geq 2\left[\frac{b+c-b}{a} + \frac{a+b-c}{a}\right] + 2\left[\frac{a+b-d}{b} + \frac{b+c-a}{b}\right] + 2\left[\frac{b+c-a}{c} + \frac{a+c-b}{c}\right]$$

$$= 2\left[\frac{a+c-b+a+b-c}{a}\right] + 2\left[\frac{a+b-c+b+c-a}{b}\right] + 2\left[\frac{b+c-a+a+c-b}{c}\right]$$

$$= 2\left[\frac{2a}{a} + \frac{2b}{b} + \frac{2c}{c}\right] = 2[2+2+2] = 2 \times 6 = 12$$ ***ok. Therefore, it is true.***

498. If $a, b, c > 0$ then:

$$\frac{e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c}}{e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}} \geq 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $f(x) = e^{3x} (x > 0) \Rightarrow f'(x) = 3e^{3x} \Rightarrow f''(x) = 9e^{3x} > 0\ (\forall x > 0)$

Using T. Popoviciu's inequality we have:

$$f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \geq 2\left[f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right)\right]$$

$$\Leftrightarrow e^{3a} + e^{3b} + e^{3c} + 3e^{3\cdot\frac{a+b+c}{3}} \geq 2\left[e^{3\cdot\frac{a+b}{2}} + e^{3\cdot\frac{b+c}{2}} + e^{3\cdot\frac{c+a}{2}}\right] = 2\cdot\sum e^{3\cdot\frac{a+b}{2}}$$

But: $\frac{a+b}{2} \overset{AM-GM}{\geq} \sqrt{ab}$ **(etc)** $\Rightarrow 2\cdot\sum e^{3\cdot\frac{a+b}{2}} \geq 2\cdot\sum e^{3\sqrt{ab}}$

$\Rightarrow e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c} \geq 2\sum e^{3\sqrt{ab}}$. **Proved**

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $x, y, z, m, n, a, b, c > 0$**, we get that:**

$(x - y) + (y - z) + (z - x) = 0$ **and** $x^2 + y^2 + z^2 \geq xy + yz + zx$

Hence: $(x^2 + y^2 + z^2)[(x - y) + (y - z) + (z - x)] \geq (xy + yz + zx)[(x - y) + (y - z) + (z - x)]$

and make: $x^2(x - y) + y^2(y - z) + z^2(z - x) \geq xy(y - z) + yz(z - x) + zx(x - y)$

$$\Rightarrow x^3 - x^2y + y^3 - y^2z + z^3 - z^2x \geq xy^2 - xyz + yz^2 - xyz + zx^2 + xyz$$

$$\Rightarrow x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2z + z^2x + x^2z + z^2y + y^2x$$

and if we give $x = e^a, y = e^b$ **and** $z = e^c$**, we have as follows:**

$$(e^a)^3 + (e^b)^3 + (e^c)^3 + 3e^ae^be^c = e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c}$$

$$\geq e^{2a+b} + e^{2b+c} + e^{2c+a} + e^{2a+c} + e^{2c+b} + e^{2b+a}$$

$$= \left[e^{2a+b} + e^{2b+a}\right] + \left[e^{2b+c} + e^{2c+b}\right] + \left[e^{2c+a} + e^{2a+c}\right]$$

$$\geq 2\left[\sqrt{e^{3(a+b)}} + \sqrt{e^{3(b+c)}} + \sqrt{e^{3(c+a)}}\right] \geq 2\left(e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}\right)$$

Hence: $\frac{e^{3a}+e^{3b}+e^{3c}+3e^{(a+b+c)}}{e^{3\sqrt{ab}}+e^{3\sqrt{bc}}+e^{3\sqrt{ca}}} \geq 2$ **ok**

Because $\sqrt{e^{3(m+n)}} \geq \sqrt{e^{2(3\sqrt{mn})}} = \left(e^{\left(2(3\sqrt{mn})\right)}\right)^{\frac{1}{2}} = e^{3\sqrt{mn}}$ **ok. Therefore, it is true.**

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c}}{e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}} \overset{(1)}{\geq} 2$$

$$\text{(1)} \Leftrightarrow e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c} \overset{(2)}{\geq} 2\left(e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}\right)$$

Now, by $G \le A$, RHS of (2)

$$\overset{(a)}{\le} 2\left[\left(e^{\frac{a+b}{2}}\right)^3 + \left(e^{\frac{b+c}{2}}\right)^3 + \left(e^{\frac{c+a}{2}}\right)^3\right] = 2\left[\left(e^{\frac{a}{2}}e^{\frac{b}{2}}\right)^3 + \left(e^{\frac{b}{2}}e^{\frac{c}{2}}\right)^3 + \left(e^{\frac{c}{2}}e^{\frac{a}{2}}\right)^3\right]$$

$$= 2(x^3y^3 + y^3z^3 + z^3x^3)\left(x = e^{\frac{a}{2}}, y = e^{\frac{b}{2}}, z = e^{\frac{c}{2}}\right)$$

Also, LHS of (2) $= x^6 + y^6 + z^6 + 3x^2y^2z^2$

$$(\because x^2 = e^a \Rightarrow e^{3a} = (e^a)^3 = x^6, etc) \overset{Schur}{\ge} \sum x^4y^2 + \sum x^2y^4 = \sum (x^4y^2 + x^2y^4)$$

$\overset{A-G}{\ge} \sum(2x^3y^3) = 2\sum x^3y^3 \overset{by\ (a)}{\ge} RHS$ ***of (2) ⇒ (2)⇒(1) is true (proved)***

Solution 4 by Sudhir Jha-Kolkata-India

$\frac{e^{3a}+e^{3b}+e^{3c}}{3} \ge \sqrt[3]{e^{3a}\cdot e^{3b}\cdot e^{3c}}$ ***(equality holds for*** $a = b = c$***)***

$\Rightarrow e^{3a} + e^{3b} + e^{3c} \ge 3e^{a+b+c}$ ***(equality holds for*** $a = b = c$***)***

$\Rightarrow e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c} \ge 2\left(3e^{a+b+c}\right)$ ***(equality holds for*** $a = b = c$***)***

$$\Rightarrow e^{3a} + e^{3b} + e^{3c} + 3e^{a+b+c} \ge 2\left(e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}\right)$$

(equality holds for $a = b = c$***)***

$\because 3e^{a+b+c} \ge e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}$ ***(equality holds for*** $a = b = c$***)***

$\Rightarrow \frac{e^{3a}+e^{3b}+e^{3c}+3e^{a+b+c}}{e^{3\sqrt{ab}}+e^{3\sqrt{bc}}+e^{3\sqrt{ca}}} \ge 2$ ***(equality holds for*** $a = b = c$***). Proved. We must prove that:***

$$3e^{a+b+c} \ge e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}$$

We know that: $a + b + c \ge \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$

$$\Rightarrow 3e^{a+b+c} \ge e^{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} + e^{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} + e^{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}$$

$\Rightarrow 3e^{a+b+c} \ge e^{3\sqrt{ab}} + e^{3\sqrt{bc}} + e^{3\sqrt{ca}}$ ***∵[taking*** $ab = bc = ca$***]***

∵[The equality holds for $a = b = c$***]***

499. **If** $x, y, z > 0, xyz = 1$ **then:**

$$\sum_{cyc}\left(\left(\frac{x}{y} + \frac{y}{x}\right)z + \frac{16}{(x+y)^2} - 6x\right) \ge 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\sum\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right]\geq 0$$

$$\Leftrightarrow\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right]+\left[\left(\frac{y}{z}+\frac{z}{y}\right)+\frac{16}{(y+z)^2}-6y\right]+$$

$$+\left[\left(\frac{x}{z}+\frac{z}{x}\right)y+\frac{16}{(z+x)^2}-6z\right]\geq 0$$

$$\Leftrightarrow\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z+\left(\frac{y}{z}+\frac{z}{y}\right)x+\frac{16}{(y+z)^2}-6x+\left(\frac{x}{z}+\frac{z}{x}\right)y+$$

$$+\frac{16}{(z+x)^2}-6y\geq 0\Leftrightarrow\sum\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z\right]\geq 0$$

We must show that: $\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z\geq 0$ ***(etc)***

$$\Leftrightarrow\frac{x^2+y^2}{(xy)^2}+\frac{16}{(x+y)^2}-\frac{6}{xy}\geq 0\ \ (\because xyz=1)$$

$$\Leftrightarrow(x^2+y^2)(x+y)^2+16(xy)^2-6xy(x+y)^2\geq 0$$

$$\Leftrightarrow x^4-4x^3y+6x^2y^2-4xy^3+y^4\geq 0\Leftrightarrow(x-y)^4\geq 0\ \ \textit{(true)}$$

Equality $\Leftrightarrow x=y=z=1.$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\left(\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right)\geq 0$$

$$LHS=\sum\left(\left(\frac{x}{y}+\frac{y}{x}\right)\frac{1}{xy}+\frac{16}{(x+y)^2}-\frac{6}{yz}\right)\ \ (\because xyz=1)$$

$$=\sum\left(\left(\frac{x}{y}+\frac{y}{x}\right)\frac{1}{xy}+\frac{16}{(x+y)^2}-\frac{6x}{xyz}\right)=\sum\left(\left(\frac{x^2+y^2}{xy}\right)\frac{1}{xy}+\frac{16}{(x+y)^2}-\frac{6z}{xyz}\right)$$

$$=\sum\left(\frac{x^2+y^2}{x^2y^2}+\frac{16}{(x+y)^2}-\frac{6}{xy}\right)=\sum\left(\frac{(x^2+y^2)(x+y)^2+16x^2y^2-6xy(x+y)^2}{x^2y^2(x+y)^2}\right)$$

$$=\sum\left(\frac{(x-y)^4}{x^2y^2(x+y)^2}\right)\geq 0.\ \textit{(Proved)}$$

Solution 3 by Mohamed Rakkane-Qued Zem-Morocco

$$\frac{x^2+y^2}{xy}\times z+\frac{16}{(x+y)^2}-6x=\frac{(x^2+y^2)z(x+y)^2+16xy-6xy(x+y)^2}{xy(x+y)^2}$$

$$=\frac{\frac{1}{z}\left((x^2+2xy+y^2)(x^2+y^2)+(4xy)^2-6xy(x+y)^2\right)}{xy(x+y)^2}=$$

$$=\frac{\frac{1}{z}(x^4-4x^3y+6(xy)^2-4xy^3+y^4)}{xy(x+y)^2}=\frac{\frac{1}{z}}{xy(x+y)^2}(x-y)^4\ge 0$$

$$(x,y,z\ge 0)$$

$\sum\left[\left(\frac{x}{y}+\frac{y}{x}\right)z-6x+\frac{16}{(x+y)^2}\right]\ge 0$ = ***when*** $x=y=z=1$ ***proved.***

Solution 4 by Boris Colakovic-Belgrade-Serbie

$$\sum_{cyc}\left(\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right)\ge 0 \quad \textbf{\textit{(1)}}$$

(1)$\Leftrightarrow \left(\frac{x}{y}+\frac{y}{x}\right)z+\left(\frac{y}{z}+\frac{z}{y}\right)x+\left(\frac{z}{x}+\frac{x}{z}\right)y+16\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)-$

$$-6(x+y+z)\ge 0\Leftrightarrow \frac{x^2+y^2}{xy}\cdot\frac{1}{xy}+\frac{y^2+z^2}{yz}\cdot\frac{1}{yz}+\frac{z^2+x^2}{zx}\cdot\frac{1}{zx}+$$

$$+16\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)-6(x+y+z)\ge 0\Leftrightarrow$$

$$\Leftrightarrow 2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+16\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)-$$

$$-6\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\ge 0\Leftrightarrow\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{16}{(x+y)^2}-\frac{6}{xy}\right)+\left(\frac{1}{y^2}+\frac{1}{z^2}+\frac{16}{(y+z)^2}-\frac{6}{yz}\right)+$$

$+\left(\frac{1}{z^2}+\frac{1}{x^2}+\frac{16}{(z+x)^2}-\frac{6}{zx}\right)\ge 0$. ***Substitutions*** $\frac{1}{x}=a;\frac{1}{y}=b;\frac{1}{z}=c$

$$\left(a^2+b^2+\frac{16}{\left(\frac{1}{a}+\frac{1}{b}\right)^2}-6ab\right)+\left(b^2+c^2+\frac{16}{\left(\frac{1}{b}+\frac{1}{c}\right)^2}-6bc\right)+$$

$$+\left(c^2+a^2+\frac{16}{\left(\frac{1}{c}+\frac{1}{a}\right)^2}-6ca\right)\ge 0\Leftrightarrow$$

$$\Leftrightarrow\left[(a+b)^2-8ab+\frac{16a^2b^2}{(a+b)^2}\right]+\left[(b+c)^2-8bc+\frac{16b^2c^2}{(b+c)^2}\right]+$$

$$+\left[(c+a)^2-8ca+\frac{16c^2a^2}{(c+a)^2}\right]\ge 0$$

$$\left[(a+b)-\frac{4ab}{a+b}\right]^2+\left[(b+c)-\frac{4bc}{b+c}\right]^2+\left[(c+a)-\frac{4ca}{c+a}\right]^2\geq 0.$$ ***Indeed, true.***

500. If $x,y,z>0, \sqrt{xy}+\sqrt{yz}+\sqrt{zx}=3$ then:

$$\left(x-\sqrt{x}\right)+\left(y-\sqrt{y}\right)+\left(z-\sqrt{z}\right)+2\left(\sqrt[4]{xyz}-1\right)\geq 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $u=\sqrt{x}; v=\sqrt{y}; w=\sqrt{z}\ (u,v,w>0)$

$$\Rightarrow uv+vw+wu=3; x=u^2; y=v^2; z=w^2$$

We must show that: $u^2+v^2+w^2-(u+v+w)+2\left(\sqrt{uvw}-1\right)\geq 0$

$$\Leftrightarrow [(u+v+w)^2-6]-(u+v+w)+2\sqrt{uvw}-2\geq 0$$

$$\Leftrightarrow (u+v+w)^2-(u+v+w)+2\sqrt{uvw}-8\geq 0$$

$$\Leftrightarrow 2\sqrt{uvw}\geq 8+(u+v+w)-(u+v+w)^2$$

(Let $p=u+v+w; q=uv+vw+wu=3; r=uvw$***)***

$$\Leftrightarrow 2\sqrt{r}\geq 8+p-p^2 \quad (*)$$

$\because$ ***We have:*** $p=u+v+w\geq\sqrt{3(uv+vw+wu)}=\sqrt{3\cdot 3}=3$

If $p\geq\frac{1}{2}\left(1+\sqrt{33}\right)>3\Rightarrow 8+p-p^2\leq 0<2\sqrt{r}\Rightarrow$ ***(*) true***

If $3\leq p<\frac{1}{2}\left(1+\sqrt{33}\right)$ ***then:*** $r\geq\frac{\left(8+p-p^2\right)^2}{4}$***. But by Schur's we have:***

$r\geq\frac{p(4q-p^2)}{9}=\frac{p(12-p^2)}{9}$***. We need to prove:*** $\frac{p(12-p^2)}{9}\geq\frac{\left(8+p-p^2\right)^2}{4}$

$\Leftrightarrow (p-3)\left(p^3+\frac{13}{9}p^2-\frac{32}{3}p-\frac{64}{3}\right)\leq 0$***. It is true because:***

$$3\leq p<\frac{1}{2}\left(1+\sqrt{33}\right)\Rightarrow\begin{cases} p-3\geq 0\\ p^3+\frac{13}{9}p^2-\frac{32}{3}p-\frac{64}{3}<\left[\frac{1}{2}\left(1+\sqrt{33}\right)\right]^3+\frac{13}{9}\left[\frac{1}{2}\left(1+\sqrt{33}\right)\right]^2-96\approx -52<0\end{cases}$$

$\Rightarrow$ ***(*) true*** $\Rightarrow$ ***proved.***

Solution 2 by Michael Sterghiou-Greece

If $x,y,z>0\wedge\sum_{cyc}\sqrt{xy}=3$ ***then:***

$\left[\sum_{cyc}\left(x-\sqrt{x}\right)\right]+2\left(\sqrt[4]{xyz}-1\right)\geq 0$ ***(1). Let*** $\sqrt{x}=a,\sqrt{y}=b,\sqrt{z}=c$***. We have:***

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$\sum_{cyc} ab = 3(:= q)$ ***(1)***$\rightarrow \left(\sum_{cyc} a^2\right) - \left(\sum_{cyc} a\right) + 2\sqrt{abc} - 2 \geq 0$ ***(2). Let*** $p = \sum_{cyc} a$

and $r = abc$ ***(2)***$\rightarrow p^2 - 2q - p + 2\sqrt{r} - 2 \overset{q=3}{\geq} p^2 - p - 8 + 2\sqrt{r} \geq 0$ ***(3)***

$A = p^2 \geq 3q \rightarrow p \geq 3$ ***and*** $3 = q \geq 3 \cdot r^{\frac{2}{3}} \rightarrow r \leq 1$***. From 3rd degree Schur***

$q = 3 \leq \frac{p^3+9r}{4p} \rightarrow r \geq \frac{p}{9}(12 - p^2)$***. Assume*** $p^2 \leq 12$ ***then:***

(3)$\rightarrow p^2 - p - 8 + 2\sqrt{\frac{p}{9}(12 - p^2)} \geq 0$ ***or as*** $p \geq 3$ ***the stronger***

$$p^2 - p - 8 + 2\sqrt{\frac{12 - p^2}{3}} \geq 0$$

For the last $f(p)$ ***we have*** $f''(p) = 2 \cdot \frac{2}{\sqrt{3}}\left(\frac{p^2}{(12-p^2)^{\frac{3}{2}}} + \frac{1}{(12-p^2)^{\frac{1}{2}}}\right) < 0$ ***because***

it reduces to $\frac{p^2}{(12-p^2)^{\frac{3}{2}}} + \frac{1}{(12-p^2)^{\frac{1}{2}}} - \sqrt{3} \geq 0$ ***or*** $\frac{p^2}{(12-p^2)^{\frac{3}{2}}} - \sqrt{3} \geq 0$

or $(p-3)(p+3)(3p^4 - 80p^2 + 576) \geq 0$ ***which holds for*** $p \geq 3$***. Hence*** $f'(p) \downarrow$ ***with one root at most which is a possible extreme (max).*** $f'(p) = 2p - 1 - \frac{2p}{\sqrt{3}(12-p^2)^{\frac{1}{2}}}$

$f'(3) = 3, f'(p) = -\infty; p \rightarrow \sqrt{12}$***, so*** $\exists p_0 \in \left[3, \sqrt{12}\right]: f'(p_0) = 0$ ***hence*** $f(p) \uparrow$ ***for*** $3 \leq p \leq p_0$ ***and*** $f(p) \downarrow$ ***for*** $p_0 \leq p \leq \sqrt{12} \Rightarrow f(p) > f\left(\sqrt{12}\right) > 0, f(p) > f(3) = 0,$ $f(p) < f(p_0)$***. In the case where*** $p^2 \geq 12$ ***(3)*** $\rightarrow p^2 - p - 8 \geq 0$ ***(stronger) which for*** $p \geq \sqrt{12}$ ***holds true. Done!***

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru