INTEGRATING A GENERALIZED RATIONAL FUNCTION

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Abstract

In this paper, the author would present the solution for the following generalized integral:

$$I_N = \int \frac{dx}{1 + x^N}$$

Of which N is a natural number. The purpose of this paper is not only showing the result, but also more importantly presenting the author's process to find the solution. In particular, the author assesses the integral for cases where N is a power of two, an odd number, and an even number, respectively. Finally, those findings are applied to form the generalized result.

Key words: (Indefinite) Integral, Rational Function.

1. Introduction

This paper discusses on the generalized indefinite integral:

$$I_N = \int \frac{dx}{1+x^N} \quad (N \in \mathbb{N})$$

Firstly, the author approach simple cases such as:

For $N = 0$, $N = 1$, and $N = 2$, respectively:

$$I_0 = \int dx = x + const$$

$$I_1 = \int \frac{dx}{1+x} = \ln|1+x| + const$$

$$I_2 = \int \frac{dx}{1+x^2} = \tan^{-1} x + const$$

Additionally, two following integrals are applied throughout the paper:

$$J_1 = \int \frac{dx}{x^2 - a^2} \quad (a > 0)$$

$$J_2 = \int \frac{dx}{x^2 + a^2} \quad (a > 0)$$

In order to calculate J_1, the following fraction decomposition is performed:

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

Thus,

$$J_1 = \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right) = \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + const$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + const$$

Regarding J_2:

$$J_2 = \int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2 + 1} = \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)^2 + 1} = \frac{1}{a}\tan^{-1}\frac{x}{a} + const$$

For $N = 3$:

$$I_3 = \int \frac{dx}{1+x^3}$$

Applying $x^3 + 1 = (x+1)(x^2 - x + 1)$, it is supposed to find real numbers A, B, and C such that

$$\frac{1}{x^3+1} = \frac{Ax+B}{x^2-x+1} + \frac{C}{x+1} \quad \forall x \in \mathbb{R} \setminus \{-1\}$$

Replacing $x = 0$, $x = 1$, and $x = 2$ to the above, we get:

$$\begin{cases} B + C = 1 \\ A + B + \frac{1}{2}C = \frac{1}{2} \\ \frac{2}{3}A + \frac{1}{3}B + \frac{1}{3}C = \frac{1}{9} \end{cases} \Leftrightarrow (A; B; C) = \left(-\frac{1}{3}; \frac{2}{3}; \frac{1}{3}\right)$$

And then:

$$\begin{aligned}
I_3 &= \int \frac{dx}{1+x^3} \\
&= \frac{1}{3}\int \frac{-x+2}{x^2-x+1}dx + \frac{1}{3}\int \frac{dx}{x+1} \\
&= \frac{1}{3}\int \frac{-x+\frac{1}{2}+\frac{3}{2}}{x^2-x+1}dx + \frac{1}{3}\ln|x+1| + const \\
&= -\frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2-x+1} + \frac{1}{3}\ln|x+1| + const \\
&= -\frac{1}{6}\int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{3}\ln|x+1| + const \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1/2}{\sqrt{3}/2}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2} + const \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + const
\end{aligned}$$

Following up the introduction as presented in Section 1, the paper will discuss on particular cases of N in order to direct the generalized result. Accordingly, Section 2 to Section 6 discuss on the case that N are powers of two, i.e. $N = 2^n$ $(n \in \mathbb{N}; n \geq 2)$; Section 7 discusses on the case that N are odd numbers, i.e. $N = 2s+1$ $(s \in \mathbb{N}^+)$; Section 8 discusses on the case that N are even numbers, i.e. $N = 2r$ $(r \in \mathbb{N}; r \geq 2)$; and Section 9 presents the conclusion for the generalized integral.

2. N = 4: multiple solutions for a problem

This section discusses on multiple solutions for the case $N = 4$.

$$I_4 = \int \frac{dx}{1 + x^4}$$

2.1. Partial fraction decomposition

Factoring:

$$\begin{aligned}
&x^4 + 1 \\
&= x^4 + 2x^2 + 1 - 2x^2 \\
&= (x^2 + 1)^2 - 2x^2 \\
&= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)
\end{aligned}$$

Hence, we shall perform the following partial fraction decomposition:

$$\frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 - \sqrt{2}x + 1} + \frac{Cx + D}{x^2 + \sqrt{2}x + 1} \quad \forall x \in \mathbb{R}$$

Of which A, B, C, and D are given real numbers.

Replacing $x = 0$, $x = \sqrt{2}$ and $x = i$ (as $\mathbb{R} \subset \mathbb{C}$) to the above, we get:

$$\begin{cases} B + D = 1 \\ \sqrt{2}A + B + \frac{1}{5}\left(\sqrt{2}C + D\right) = \frac{1}{5} \\ -\frac{A}{\sqrt{2}} + \frac{Bi}{\sqrt{2}} + \frac{C}{\sqrt{2}} - \frac{Di}{\sqrt{2}} = \frac{1}{2} \end{cases} \Leftrightarrow (A; B; C; D) = \left(-\frac{1}{2\sqrt{2}}; \frac{1}{2}; \frac{1}{2\sqrt{2}}; \frac{1}{2}\right)$$

Thus,

$$\begin{aligned}
I_4 &= \int \frac{dx}{1 + x^4} \\
&= \int \frac{-\frac{1}{2\sqrt{2}}x + \frac{1}{4} + \frac{1}{4}}{x^2 - \sqrt{2}x + 1} dx + \int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{4} + \frac{1}{4}}{x^2 + \sqrt{2}x + 1} dx \\
&= \frac{1}{4\sqrt{2}} \int \frac{-2x + \sqrt{2}}{x^2 - \sqrt{2}x + 1} dx + \frac{1}{4} \int \frac{dx}{x^2 - \sqrt{2}x + 1} + \frac{1}{4\sqrt{2}} \int \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} dx \\
&\qquad + \frac{1}{4} \int \frac{dx}{x^2 + \sqrt{2}x + 1}
\end{aligned}$$

$$= -\frac{1}{4\sqrt{2}}\int \frac{d\left(x^2 - \sqrt{2}x + 1\right)}{x^2 - \sqrt{2}x + 1} + \frac{1}{4\sqrt{2}}\int \frac{d\left(x^2 + \sqrt{2}x + 1\right)}{x^2 + \sqrt{2}x + 1}$$

$$+ \frac{1}{4}\int \frac{d\left(x - \frac{1}{\sqrt{2}}\right)}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{4}\int \frac{d\left(x + \frac{1}{\sqrt{2}}\right)}{\left(x + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1}\right) + \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x - 1\right) + \tan^{-1}\left(\sqrt{2}x + 1\right)\right]$$

$$+ const$$

2.2. Analyzed method I

To confront complicated problems, we sometimes *divide and conquer*. In contrast, to deal with *simple but rough* problems, it is supposed to drive those *large but soft*. This philosophy is entirely applied in this sub-section.

In particular, before dealing with

$$I_4 = \int \frac{dx}{1 + x^4}$$

We assess the follows:

$$K_1 = \int \frac{x^3}{x^4 + 1}dx$$

$$K_2 = \int \frac{x}{x^4 + 1}dx$$

Simply and softly,

$$K_1 = \int \frac{x^3}{x^4 + 1}dx = \frac{1}{4}\int \frac{d(x^4 + 1)}{x^4 + 1} = \frac{1}{4}\ln(x^4 + 1) + const$$

$$K_2 = \int \frac{x}{x^4 + 1}dx = \frac{1}{2}\int \frac{d(x^2)}{(x^2)^2 + 1} = \frac{1}{2}\tan^{-1}(x^2) + const$$

Besides,

$$\frac{1}{x^2 - \sqrt{2}x + 1} + \frac{1}{x^2 + \sqrt{2}x + 1} = \frac{2(x^2 + 1)}{x^4 + 1}$$

Thus,

$$K_3 = \int \frac{x^2 + 1}{x^4 + 1}dx$$

$$= \frac{1}{2}\left(\int \frac{dx}{x^2 - \sqrt{2}x + 1} + \int \frac{dx}{x^2 + \sqrt{2}x + 1}\right)$$

$$= \frac{1}{2}\left[\int \frac{d\left(x-\frac{1}{\sqrt{2}}\right)}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \int \frac{d\left(x+\frac{1}{\sqrt{2}}\right)}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right)+\tan^{-1}\left(\sqrt{2}x+1\right)\right]+const$$

Due to the calculation of K_1, K_2, and K_3, we consequently enable to deal with the following form:

$$\int \frac{x^3+x+\kappa(x^2+1)}{x^4+1}dx = \int \frac{x^3+\kappa x^2+x+\kappa}{x^4+1}dx$$

Of which κ is a given real number.

As a result, if we somehow calculate the following integral

$$\int \frac{x^3+\kappa x^2+x}{x^4+1}dx$$

Then by a subtraction, we enable to find

$$I_4 = \int \frac{dx}{1+x^4}$$

This is the direction of Analyzed method I.

As $x^4+1=\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)$, if we choose $\kappa=\pm\sqrt{2}$, the problem is solved. For $\kappa=\sqrt{2}$:

$$K_4 = \int \frac{x^3+\sqrt{2}x^2+x}{x^4+1}dx$$

$$= \int \frac{x^3+\sqrt{2}x^2+x}{\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)}dx$$

$$= \int \frac{x}{x^2-\sqrt{2}x+1}dx$$

$$= \int \frac{x-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{x^2-\sqrt{2}x+1}dx$$

$$= \frac{1}{2}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx + \frac{1}{\sqrt{2}}\int \frac{dx}{x^2-\sqrt{2}x+1}$$

$$= \frac{1}{2}\int \frac{d\left(x^2 - \sqrt{2}x + 1\right)}{x^2 - \sqrt{2}x + 1} + \frac{1}{\sqrt{2}}\int \frac{d\left(x - \frac{1}{\sqrt{2}}\right)}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$= \frac{1}{2}\ln\left(x^2 - \sqrt{2}x + 1\right) + \tan^{-1}\left(\sqrt{2}x - 1\right) + const$$

By a linear combination of K_1, K_2, and K_3:

$$K_5 = \int \frac{x^3 + \sqrt{2}x^2 + x + \sqrt{2}}{x^4 + 1}dx$$

$$= K_1 + K_2 + \sqrt{2}K_3$$

$$= \frac{1}{4}\ln(x^4 + 1) + \frac{1}{2}\tan^{-1}(x^2) + \tan^{-1}\left(\sqrt{2}x - 1\right) + \tan^{-1}\left(\sqrt{2}x + 1\right) + const$$

Besides:

$$\frac{1}{4}\ln(x^4 + 1) - \frac{1}{2}\ln\left(x^2 - \sqrt{2}x + 1\right)$$

$$= \frac{1}{4}\ln(x^4 + 1) - \frac{1}{4}\ln\left(x^2 - \sqrt{2}x + 1\right)^2$$

$$= \frac{1}{4}\ln\left(\frac{x^4 + 1}{\left(x^2 - \sqrt{2}x + 1\right)^2}\right)$$

$$= \frac{1}{4}\ln\left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1}\right)$$

Returning to I_4, we get:

$$I_4 = \int \frac{dx}{1 + x^4}$$

$$= \frac{1}{\sqrt{2}}(K_5 - K_4)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{4}\ln(x^4 + 1) + \frac{1}{2}\tan^{-1}(x^2) + \tan^{-1}\left(\sqrt{2}x - 1\right) + \tan^{-1}\left(\sqrt{2}x + 1\right)\right]$$

$$- \frac{1}{\sqrt{2}}\left[\frac{1}{2}\ln\left(x^2 - \sqrt{2}x + 1\right) + \tan^{-1}\left(\sqrt{2}x - 1\right)\right] + const$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1}\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}x + 1\right) + \frac{1}{2\sqrt{2}}\tan^{-1}(x^2) + const$$

In comparison to the result as found by partial fraction decomposition, we get:

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}x+1\right)+\frac{1}{2\sqrt{2}}\tan^{-1}(x^2)$$

$$=\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right)+\tan^{-1}\left(\sqrt{2}x+1\right)\right]+const$$

$$\Leftrightarrow \tan^{-1}\left(\sqrt{2}x+1\right)-\tan^{-1}\left(\sqrt{2}x-1\right)+\tan^{-1}(x^2)=const$$

Indeed, forming the function

$$\Xi(x)=\tan^{-1}\left(\sqrt{2}x+1\right)-\tan^{-1}\left(\sqrt{2}x-1\right)+\tan^{-1}(x^2)$$

Then

$$\Xi'(x)=\frac{\sqrt{2}}{\left(\sqrt{2}x+1\right)^2+1}-\frac{\sqrt{2}}{\left(\sqrt{2}x-1\right)^2+1}+\frac{2x}{x^4+1}$$

$$=\sqrt{2}\left(\frac{1}{2x^2+2\sqrt{2}x+2}-\frac{1}{2x^2-2\sqrt{2}x+2}\right)+\frac{2x}{x^4+1}$$

$$=\frac{\sqrt{2}}{2}\times\frac{\left(-2\sqrt{2}x\right)}{x^4+1}+\frac{2x}{x^4+1}$$

$$=-\frac{2x}{x^4+1}+\frac{2x}{x^4+1}$$

$$=0$$

$$\Xi'(x)=0 \quad \forall x\in\mathbb{R}$$

$$\Rightarrow \Xi(x)=const \quad \forall x\in\mathbb{R}$$

Besides, $\Xi(0)=\frac{\pi}{2}$. Thus:

$$\Xi(x)=\frac{\pi}{2} \quad \forall x\in\mathbb{R}$$

As above clarified, the two approaches of an integral result in two different expressions. This fact is explained by the constant of integration.

2.3. Analyzed method II

In the previous approach, we apply the *summation*

$$\frac{1}{x^2-\sqrt{2}x+1}+\frac{1}{x^2+\sqrt{2}x+1}=\frac{2(x^2+1)}{x^4+1}$$

In this sub-section, we apply the *subtraction*

$$\frac{1}{x^2-\sqrt{2}x+1}-\frac{1}{x^2+\sqrt{2}x+1}=\frac{2\sqrt{2}x}{x^4+1}$$

Regarding Analyzed method II, integral I_4 is calculated as follow:

$$I_4 = \int \frac{dx}{x^4+1} = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2}{x^4+1}dx$$

As obtained in sub-section 2.2:

$$K_3 = \int \frac{x^2+1}{x^4+1}dx = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right)\right] + const$$

In order to calculate

$$K_6 = \int \frac{x^2}{x^4+1}dx$$

We perform the following *subtraction*:

$$K_6 = \int \frac{x^2}{x^4+1}dx$$

$$= \int \frac{x.x}{x^4+1}dx$$

$$= \frac{1}{2\sqrt{2}}\left[\int x\left(\frac{1}{x^2-\sqrt{2}x+1} - \frac{1}{x^2+\sqrt{2}x+1}\right)dx\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\int \frac{x}{x^2-\sqrt{2}x+1}dx - \int \frac{x}{x^2+\sqrt{2}x+1}dx\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\int \frac{x-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{x^2-\sqrt{2}x+1}dx - \int \frac{x+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}{x^2+\sqrt{2}x+1}dx\right]$$

$$= \frac{1}{4\sqrt{2}}\left[\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx - \int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx\right]$$

$$+ \frac{1}{4}\left[\int \frac{dx}{x^2-\sqrt{2}x+1} - \int \frac{dx}{x^2+\sqrt{2}x+1}\right]$$

$$= \frac{1}{4\sqrt{2}}\left[\int \frac{d\left(x^2-\sqrt{2}x+1\right)}{x^2-\sqrt{2}x+1} - \int \frac{d\left(x^2+\sqrt{2}x+1\right)}{x^2+\sqrt{2}x+1}\right]$$

$$+ \frac{1}{4}\left[\int \frac{d\left(x-\frac{1}{\sqrt{2}}\right)}{\left(x-\frac{1}{\sqrt{2}}\right)^2+1} - \int \frac{d\left(x+\frac{1}{\sqrt{2}}\right)}{\left(x+\frac{1}{\sqrt{2}}\right)^2+1}\right]$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + + \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right)\right]$$

$$+ const$$

And finally,

$$I_4 = \int \frac{dx}{x^4+1} = K_3 - K_6$$

$$= \frac{1}{4\sqrt{2}} \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right)\right] + const$$

This results is consistent the expression as found by partial fraction decomposition.

The integral I_4 has been solved in three distinct ways, which of those have their own pros and cons. The most logic-like approach may be partial fraction decomposition, while analyzed methods technically apply algebraic transformations. However, the three ways contain massive calculation. Therefore, we shall necessarily find a shortcut. The to-be-found new way will tremendously help deal with the generalized integral:

$$I_{2^n} = \int \frac{dx}{1+x^{2^n}} \quad (n \in \mathbb{N})$$

2.4. Sum & Sub

This method is inspired by the two algebraic equalities:

$x^2 + \frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$ and $x^2 + \frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2 + 2$

In order to calculate integral I_4, we shall respectively find the so-called *Sum & Sub* integrals:

$$\begin{cases} K_3 = \int \frac{x^2+1}{x^4+1} dx \\ K_7 = \int \frac{x^2-1}{x^4+1} dx \end{cases}$$

Accordingly:

$$K_3 = \int \frac{x^2+1}{x^4+1} dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} + const$$

And

$$K_7 = \int \frac{x^2 - 1}{x^4 + 1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + const$$

As a result:

$$I_4 = \int \frac{dx}{x^4 + 1} = \frac{1}{2}(K_3 - K_7)$$

$$= \frac{1}{4\sqrt{2}} \ln \left| \frac{x + \frac{1}{x} + \sqrt{2}}{x + \frac{1}{x} - \sqrt{2}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x - \frac{1}{x}}{\sqrt{2}} + const$$

$$= \frac{1}{4\sqrt{2}} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x - \frac{1}{x}}{\sqrt{2}} + const$$

In comparison to the result as found by partial fraction decomposition, the constant of integration form the equality:

$$\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1) - \tan^{-1} \frac{x - \frac{1}{x}}{\sqrt{2}} = \frac{\pi}{2} \quad \forall x \in \mathbb{R}$$

Moreover, the *Sum & Sub* method help solve integrals in following forms:

$$\begin{cases} I_4^0(\mu) = \int \frac{dx}{x^4 + \mu x^2 + 1} \\ I_4^2(\nu) = \int \frac{x^2}{x^4 + \nu x^2 + 1} dx \end{cases}$$

Of which, μ and ν are given real numbers.

In Section 2, we have solved a problem with four approaches. In summary, the *Sum & Sub* method seems to be the most effective way. However, other methods may be also useful in the case we have to deal with more complicated problems, such as integral I_8:

$$I_8 = \int \frac{dx}{x^8 + 1}$$

Before that, integrals I_5 and I_6 will be discussed in the next section.

3. For $N = 5$ and $N = 6$

3.1. Integral I_5

Factoring: $x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1)$

By complex numbers, we enable to further factor as follow:

$$x^5 + 1 = 0$$

$$\Leftrightarrow x^5 = -1 = \cos\pi + i\sin\pi$$

$$\Leftrightarrow x_k = \cos\frac{\pi + k2\pi}{5} + i\sin\frac{\pi + k2\pi}{5} \quad (k = \overline{0;4})$$

$$\Leftrightarrow x_k = \left(e^{i\pi/5}; e^{3i\pi/5}; -1; e^{7i\pi/5}; e^{9i\pi/5}\right) \quad (k = \overline{0;4})$$

Of which:

$$\begin{cases} x_0 + x_4 = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} \\ x_1 + x_2 = 2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2} \\ x_0 x_4 = x_1 x_2 = 1 \end{cases}$$

So,

$$x^5 + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{5} + 1\right)$$

We shall find the five real numbers $(A; B; C; D; E)$ such that

$$\frac{1}{x^5+1} = \frac{A}{x+1} + \frac{Bx^3 + Cx^2 + Dx + E}{x^4 - x^3 + x^2 - x + 1} \quad \forall x \in \mathbb{R} \setminus \{-1\}$$

Solving the above, we get:

$$(A; B; C; D; E) = \left(\frac{1}{5}; -\frac{1}{5}; \frac{2}{5}; -\frac{3}{5}; \frac{4}{5}\right)$$

Rewriting:

$$\frac{1}{x^5+1} = \frac{1}{5}\left(\frac{1}{x+1} + \frac{-x^3 + 2x^2 - 3x + 4}{x^4 - x^3 + x^2 - x + 1}\right)$$

Next, we shall find the four real numbers $(T; U; V; W)$ such that

$$\frac{-x^3 + 2x^2 - 3x + 4}{x^4 - x^3 + x^2 - x + 1} = \frac{Tx + U}{x^2 - 2x\cos\frac{\pi}{5} + 1} + \frac{Vx + W}{x^2 - 2x\cos\frac{3\pi}{5} + 1} \quad \forall x \in \mathbb{R}$$

For $x = 0$, $x = 1$ and $x = i$ (as $\mathbb{R} \subset \mathbb{C}$), we get the system:

$$\begin{cases} U + W = 4 \\ \dfrac{T + U}{2 - 2\cos\frac{\pi}{5}} + \dfrac{V + W}{2 - 2\cos\frac{3\pi}{5}} = 2 \\ -\dfrac{T}{2\cos\frac{\pi}{5}} + \dfrac{Ui}{2\cos\frac{\pi}{5}} - \dfrac{V}{2\cos\frac{3\pi}{5}} + \dfrac{Wi}{2\cos\frac{3\pi}{5}} = 2 - 2i \end{cases}$$

$$\Leftrightarrow (T; U; V; W) = \left(-2\cos\frac{\pi}{5}; 2; -2\cos\frac{3\pi}{5}; 2\right)$$

As a result:

$$\frac{-x^3 + 2x^2 - 3x + 4}{x^4 - x^3 + x^2 - x + 1} = \frac{-2x\cos\frac{\pi}{5} + 2}{x^2 - 2x\cos\frac{\pi}{5} + 1} + \frac{-2x\cos\frac{3\pi}{5} + 2}{x^2 - 2x\cos\frac{3\pi}{5} + 1} \quad \forall x \in \mathbb{R}$$

Conseuently, integral I_5 is performed as follow:

$$I_5 = \int \frac{dx}{x^5 + 1}$$

$$= \frac{1}{5}\int \left(\frac{1}{x + 1} + \frac{-x^3 + 2x^2 - 3x + 4}{x^4 - x^3 + x^2 - x + 1}\right) dx$$

$$= \frac{1}{5}\int \left(\frac{1}{x + 1} + \frac{-2x\cos\frac{\pi}{5} + 2}{x^2 - 2x\cos\frac{\pi}{5} + 1} + \frac{-2x\cos\frac{3\pi}{5} + 2}{x^2 - 2x\cos\frac{3\pi}{5} + 1}\right) dx$$

$$= \frac{1}{5}\left(\int \frac{dx}{x + 1} + \int \frac{-2x\cos\frac{\pi}{5} + 2}{x^2 - 2x\cos\frac{\pi}{5} + 1} dx + \int \frac{-2x\cos\frac{3\pi}{5} + 2}{x^2 - 2x\cos\frac{3\pi}{5} + 1} dx\right)$$

$$= \frac{1}{5}\left(\ln|x + 1| + \int \frac{-2x\cos\frac{\pi}{5} + 2}{x^2 - 2x\cos\frac{\pi}{5} + 1} dx + \int \frac{-2x\cos\frac{3\pi}{5} + 2}{x^2 - 2x\cos\frac{3\pi}{5} + 1} dx\right) + const$$

As a lemma, we assess:

$$L(\delta) = \int \frac{-2x\cos\delta + 2}{x^2 - 2x\cos\delta + 1} dx$$

$$= \int \frac{-2x\cos\delta + 2(\cos\delta)^2 + 2(\sin\delta)^2}{x^2 - 2x\cos\delta + 1} dx$$

$$= -\cos\delta \int \frac{2x + 2(\cos\delta)^2}{x^2 - 2x\cos\delta + 1} dx + 2(\sin\delta)^2 \int \frac{dx}{x^2 - 2x\cos\delta + 1}$$

$$= -\cos\delta \int \frac{d(x^2 - 2x\cos\delta + 1)}{x^2 - 2x\cos\delta + 1} dx$$

$$+ 2(\sin\delta)^2 \int \frac{dx}{x^2 - 2x\cos\delta + (\cos\delta)^2 + (\sin\delta)^2}$$

$$= -\cos\delta \ln(x^2 - 2x\cos\delta + 1) + 2(\sin\delta)^2 \int \frac{d(x - \cos\delta)}{(x - \cos\delta)^2 + (\sin\delta)^2}$$

$$= -\cos\delta \ln(x^2 - 2x\cos\delta + 1) + 2\sin\delta \tan^{-1}\frac{x - \cos\delta}{\sin\delta} + const$$

Returning to integral I_5:

$$I_5 = \frac{1}{5}\left[\ln|x + 1| + L\left(\frac{\pi}{5}\right) + L\left(\frac{3\pi}{5}\right)\right] + const$$

$$= \frac{1}{5}\left[\ln|x + 1| - \cos\frac{\pi}{5}\ln\left(x^2 - 2x\cos\frac{\pi}{5} + 1\right) + 2\sin\frac{\pi}{5}\tan^{-1}\frac{x - \cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right.$$

$$\left. - \cos\frac{3\pi}{5}\ln\left(x^2 - 2x\cos\frac{3\pi}{5} + 1\right) + 2\sin\frac{3\pi}{5}\tan^{-1}\frac{x - \cos\frac{3\pi}{5}}{\sin\frac{3\pi}{5}}\right]$$

$$+ const$$

Coincidentally, the result of integral I_3 could be written as:

$$I_3 = \int \frac{dx}{x^3 + 1}$$

$$= \frac{1}{3}\ln|x + 1| - \frac{1}{6}\ln(x^2 - x + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x - 1}{\sqrt{3}} + const$$

$$= \frac{1}{3}\left[\ln|x + 1| - \cos\frac{\pi}{3}\ln\left(x^2 - 2x\cos\frac{\pi}{3} + 1\right) + 2\sin\frac{\pi}{3}\tan^{-1}\frac{x - \cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}\right]$$

$$+ const$$

$$= \frac{1}{3}\left[\ln|x + 1| + L\left(\frac{\pi}{3}\right)\right] + const$$

This is a key point to form a conjecture for the case that N are odd numbers, i.e. $N = 2s + 1$ ($s \in \mathbb{N}^+$), which will be discussed in Section 7.

3.2. *Integral I_6*

Factoring $x^6+1=(x^2+1)(x^4-x^2+1)$

And applying $I_4^0(\mu)$ for $\mu=-1$, i.e.

$$I_4^0(-1)=\int\frac{dx}{x^4-x^2+1}$$

Integral I_6 could be calculated as follow:

$$\begin{aligned}
I_6 &= \int\frac{dx}{x^6+1}\\
&= \int\frac{(x^2+1)-x^2}{x^6+1}dx\\
&= \int\frac{x^2+1}{x^6+1}dx-\int\frac{x^2}{x^6+1}dx\\
&= \int\frac{x^2+1}{(x^2+1)(x^4-x^2+1)}dx-\frac{1}{3}\int\frac{d(x^3)}{(x^3)^2+1}\\
&= \int\frac{dx}{x^4-x^2+1}-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\int\frac{(x^2+1)-(x^2-1)}{x^4-x^2+1}dx-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\int\frac{x^2+1}{x^4-x^2+1}dx-\frac{1}{2}\int\frac{x^2-1}{x^4-x^2+1}dx-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}dx-\frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}dx-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+1}-\frac{1}{2}\int\frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-3}-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\tan^{-1}\left(x-\frac{1}{x}\right)-\frac{1}{4\sqrt{3}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right|-\frac{1}{3}\tan^{-1}(x^3)+const\\
&= \frac{1}{2}\tan^{-1}\left(x-\frac{1}{x}\right)+\frac{1}{4\sqrt{3}}\ln\left(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}\right)-\frac{1}{3}\tan^{-1}(x^3)+const
\end{aligned}$$

4. $N = 8$: multiple methods combined

This section discuss on the following integral:

$$I_8 = \int \frac{dx}{x^8 + 1}$$

Factoring:

$$\begin{aligned}
&x^8 + 1 \\
&= x^8 + 2x^4 + 1 - 2x^4 \\
&= (x^4 + 1)^2 - 2x^4 \\
&= \left(x^4 - \sqrt{2}x^2 + 1\right)\left(x^4 + \sqrt{2}x^2 + 1\right) \\
&= \left(x^4 - 2x^2 \cos\frac{\pi}{4} + 1\right)\left(x^4 + 2x^2 \cos\frac{\pi}{4} + 1\right) \\
&= \left(x^4 + 2x^2 + 1 - 2x^2 \cos\frac{\pi}{4} - 2x^2\right)\left(x^4 + 2x^2 + 1 + 2x^2 \cos\frac{\pi}{4} - 2x^2\right) \\
&= \left[(x^2 + 1)^2 - 2x^2\left(1 + \cos\frac{\pi}{4}\right)\right]\left[(x^2 + 1)^2 - 2x^2\left(1 - \cos\frac{\pi}{4}\right)\right] \\
&= \left[(x^2 + 1)^2 - 4x^2\left(\cos\frac{\pi}{8}\right)^2\right]\left[(x^2 + 1)^2 - 4x^2\left(\sin\frac{\pi}{8}\right)^2\right] \\
&= \left(x^2 - 2x \cos\frac{\pi}{8} + 1\right)\left(x^2 + 2x \cos\frac{\pi}{8} + 1\right)\left(x^2 - 2x \sin\frac{\pi}{8} + 1\right)\left(x^2 \right. \\
&\qquad\qquad \left. + 2x \sin\frac{\pi}{8} + 1\right)
\end{aligned}$$

The first thoughts may relate to partial fraction decomposition. Accordingly, we shall find the eight real numbers (A, B, C, D, E, F, G, H) such that

$$\frac{1}{x^8 + 1} = \frac{Ax + B}{x^2 - 2x \cos\frac{\pi}{8} + 1} + \frac{Cx + D}{x^2 + 2x \cos\frac{\pi}{8} + 1} + \frac{Ex + F}{x^2 - 2x \sin\frac{\pi}{8} + 1} + \frac{Gx + H}{x^2 + 2x \sin\frac{\pi}{8} + 1} \quad \forall x \in \mathbb{R}$$

This approach still seems to be possible. However, massive calculation is unavoidable. Furthermore, after finding the eight real numbers, we have to deal with multiple integrals under the form

$$\int \frac{ux + v}{px^2 + qx + r}\,dx$$

Hence, this method should be temporarily delayed and we can try other approaches as detailed in Section 2. As the *Sum & Sub* is surprisingly effective, it should be first considered. In accordance with the *Sum & Sub* philosophy, we have to deal with two following integrals:

$$\begin{cases} N_1 = \int \frac{x^4+1}{x^8+1}dx \\ N_2 = \int \frac{x^4-1}{x^8+1}dx \end{cases}$$

As well as the pathway in Sub-Section 2.4:

$$N_1 = \int \frac{x^4+1}{x^8+1}dx = \int \frac{1+\frac{1}{x^4}}{x^4+\frac{1}{x^4}}dx$$

It seems possible to perform $\left(x^4+\frac{1}{x^4}\right)$ under $\left(x \pm \frac{1}{x}\right)$ or $\left(x^2 \pm \frac{1}{x^2}\right)$. However, it is noticeable that

$$d\left(x \pm \frac{1}{x}\right) = \left(1 \mp \frac{1}{x^2}\right)dx$$

And

$$d\left(x^2 \pm \frac{1}{x^2}\right) = \left(2x \mp \frac{2}{x^3}\right)dx$$

Conseequently, it is difficult to find any relations between the above and the integrated $\left(1+\frac{1}{x^4}\right)dx$. So, the *Sum & Sub* may not work alone.

With Analyzed methods, we may consider the two integrals:

$$\begin{cases} N_1 = \int \frac{x^4+1}{x^8+1}dx \\ N_3 = \int \frac{x^4}{x^8+1}dx \end{cases}$$

Applying the fraction decomposition

$$\frac{x^4+1}{x^8+1} = \frac{1}{2}\left(\frac{1}{x^4-\sqrt{2}x^2+1} + \frac{1}{x^4+\sqrt{2}x^2+1}\right)$$

Thus,

$$N_1 = \int \frac{x^4+1}{x^8+1}dx = \frac{1}{2}\left(\int \frac{dx}{x^4-\sqrt{2}x^2+1} + \int \frac{dx}{x^4+\sqrt{2}x^2+1}\right)$$

Now we can apply the *Sum & Sub*, where

$$\begin{cases} I_4^0(-\sqrt{2}) = \int \dfrac{dx}{x^4 - \sqrt{2}x^2 + 1} \\ I_4^0(\sqrt{2}) = \int \dfrac{dx}{x^4 + \sqrt{2}x^2 + 1} \end{cases}$$

Applying to Analayzed method I for integral N_3:

$$N_3 = \int \frac{x^4}{x^8+1}dx = \int \frac{x^2.x^2}{x^8+1}dx$$

$$= \frac{1}{2\sqrt{2}} \int x^2 \left(\frac{1}{x^4 - \sqrt{2}x^2 + 1} - \frac{1}{x^4 + \sqrt{2}x^2 + 1} \right) dx$$

$$= \frac{1}{2\sqrt{2}} \left(\int \frac{x^2}{x^4 - \sqrt{2}x^2 + 1} dx - \int \frac{x^2}{x^4 + \sqrt{2}x^2 + 1} dx \right)$$

$$= \frac{1}{2\sqrt{2}} \left[I_4^2(-\sqrt{2}) - I_4^2(\sqrt{2}) \right]$$

The result holds when Analyzed method II is applied:

$$\frac{x^4}{x^8+1} = \frac{x^4}{\left(x^4 - \sqrt{2}x^2 + 1\right)\left(x^4 + \sqrt{2}x^2 + 1\right)}$$

$$= \frac{1}{\sqrt{2}} \left[\frac{\left(x^6 + \sqrt{2}x^4 + x^2\right) - \left(x^6 + x^2\right)}{\left(x^4 - \sqrt{2}x^2 + 1\right)\left(x^4 + \sqrt{2}x^2 + 1\right)} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{x^2}{x^4 - \sqrt{2}x^2 + 1} - \frac{x^2(x^4+1)}{\left(x^4 - \sqrt{2}x^2 + 1\right)\left(x^4 + \sqrt{2}x^2 + 1\right)} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{x^2}{x^4 - \sqrt{2}x^2 + 1} - \frac{x^2}{2} \left(\frac{1}{x^4 - \sqrt{2}x^2 + 1} + \frac{1}{x^4 + \sqrt{2}x^2 + 1} \right) \right]$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{x^2}{x^4 - \sqrt{2}x^2 + 1} - \frac{x^2}{x^4 + \sqrt{2}x^2 + 1} \right)$$

As Analyzed methods are applied, it is noticeable that the partial fraction decomposition method still works if we find the four real numbers (A, B, C, D) such that

$$\frac{1}{x^8+1} = \frac{Ax^2 + B}{x^4 - \sqrt{2}x^2 + 1} + \frac{Cx^2 + D}{x^4 + \sqrt{2}x^2 + 1} \quad \forall x \in \mathbb{R}$$

As described in Sub-Section 2.1, the solution for the above equation is $(A; B; C; D) = \left(-\frac{1}{2\sqrt{2}}; \frac{1}{2}; \frac{1}{2\sqrt{2}}; \frac{1}{2} \right)$.

Therefore, it is said that we shall combine multiple approaches in order to deal with integral I_8. This fact is clarified when the *Sum & Sub* is applied at the second stage of the solution. Applying consequences as presented in Sub-Section 2.3, we get:

$$I_4^0(-\sqrt{2}) = \int \frac{dx}{x^4 - \sqrt{2}x^2 + 1}$$

$$I_4^0(\sqrt{2}) = \int \frac{dx}{x^4 + \sqrt{2}x^2 + 1}$$

$$I_4^2(-\sqrt{2}) = \int \frac{x^2}{x^4 - \sqrt{2}x^2 + 1} dx$$

$$I_4^2(\sqrt{2}) = \int \frac{x^2}{x^4 + \sqrt{2}x^2 + 1} dx$$

Returning to integral I_8:

$$\frac{1}{x^8 + 1} = \frac{x^4 + 1}{x^8 + 1} - \frac{x^4}{x^8 + 1}$$

$$= \frac{1}{2}\left(\frac{1}{x^4 - \sqrt{2}x^2 + 1} + \frac{1}{x^4 + \sqrt{2}x^2 + 1}\right)$$

$$+ \frac{1}{2\sqrt{2}}\left(\frac{x^2}{x^4 + \sqrt{2}x^2 + 1} - \frac{x^2}{x^4 - \sqrt{2}x^2 + 1}\right)$$

Integrating the above:

$$I_8 = \int \frac{dx}{x^8 + 1} = \frac{1}{2}\left[I_4^0(-\sqrt{2}) + I_4^0(\sqrt{2})\right] + \frac{1}{2\sqrt{2}}\left[-I_4^2(-\sqrt{2}) + I_4^2(\sqrt{2})\right]$$

In order to calculate integrals $I_4^0(\mu)$ and $I_4^2(\nu)$, we shall factor:

$$x^4 - \sqrt{2}x^2 + 1 = x^4 - 2x^2 \cos\frac{\pi}{4} + 1 = x^4 + 2x^2 + 1 - 2x^2 \cos\frac{\pi}{4} - 2x^2$$

$$= (x^2 + 1)^2 - 2x^2\left(1 + \cos\frac{\pi}{4}\right) = (x^2 + 1)^2 - 4x^2\left(\cos\frac{\pi}{8}\right)^2$$

As well as

$$x^4 + \sqrt{2}x^2 + 1 = x^4 + 2x^2 \cos\frac{\pi}{4} + 1 = x^4 + 2x^2 + 1 + 2x^2 \cos\frac{\pi}{4} - 2x^2$$

$$= (x^2 + 1)^2 - 2x^2\left(1 - \cos\frac{\pi}{4}\right) = (x^2 + 1)^2 - 4x^2\left(\sin\frac{\pi}{8}\right)^2$$

Noticeably,

$$\begin{cases} 2+\sqrt{2} = 2+2\cos\dfrac{\pi}{4} = 2\left(1+\cos\dfrac{\pi}{4}\right) = 4\left(\cos\dfrac{\pi}{8}\right)^2 \\ 2-\sqrt{2} = 2-2\cos\dfrac{\pi}{4} = 2\left(1-\cos\dfrac{\pi}{4}\right) = 4\left(\sin\dfrac{\pi}{8}\right)^2 \end{cases}$$

Denoting:

$$\begin{cases} I_4^+(\lambda) = \displaystyle\int \frac{x^2+1}{x^4+\lambda x^2+1}dx \\ I_4^-(\lambda) = \displaystyle\int \frac{x^2-1}{x^4+\lambda x^2+1}dx \end{cases}$$

Of which λ is a given real number.

We respectively obtain:

$$I_4^+\left(-\sqrt{2}\right) = \int \frac{x^2+1}{x^4-\sqrt{2}x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-\sqrt{2}}dx$$

$$= \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2-\sqrt{2}} = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\sin\frac{\pi}{8}\right)^2}$$

$$= \frac{1}{2}\csc\frac{\pi}{8}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{8}} + const$$

And

$$I_4^-\left(-\sqrt{2}\right) = \int \frac{x^2-1}{x^4-\sqrt{2}x^2+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-\sqrt{2}}dx$$

$$= \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-\left(2+\sqrt{2}\right)} = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-4\left(\cos\frac{\pi}{8}\right)^2}$$

$$= \frac{1}{4}\sec\frac{\pi}{8}\ln\left|\frac{x+\frac{1}{x}-2\cos\frac{\pi}{8}}{x+\frac{1}{x}+2\cos\frac{\pi}{8}}\right| + const$$

$$= \frac{1}{4}\sec\frac{\pi}{8}\ln\left(\frac{x^2-2x\cos\frac{\pi}{8}+1}{x^2+2x\cos\frac{\pi}{8}+1}\right) + const$$

Similarly,

$$I_4^+\left(\sqrt{2}\right) = \int \frac{x^2+1}{x^4+\sqrt{2}x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+\sqrt{2}}dx$$

$$= \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2+\sqrt{2}} = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\cos\frac{\pi}{8}\right)^2}$$

$$= \frac{1}{2}\sec\frac{\pi}{8}\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{8}} + const$$

And

$$I_4^-\left(\sqrt{2}\right) = \int \frac{x^2-1}{x^4+\sqrt{2}x^2+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+\sqrt{2}}dx$$

$$= \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-\left(2-\sqrt{2}\right)} = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-4\left(\sin\frac{\pi}{8}\right)^2}$$

$$= \frac{1}{4}\csc\frac{\pi}{8}\ln\left|\frac{x+\frac{1}{x}-2\sin\frac{\pi}{8}}{x+\frac{1}{x}+2\sin\frac{\pi}{8}}\right| + const$$

$$= \frac{1}{4}\csc\frac{\pi}{8}\ln\left(\frac{x^2-2x\sin\frac{\pi}{8}+1}{x^2+2x\sin\frac{\pi}{8}+1}\right) + const$$

Returning to integrals $I_4^0\left(-\sqrt{2}\right)$, $I_4^0\left(\sqrt{2}\right)$, $I_4^2\left(-\sqrt{2}\right)$, and $I_4^2\left(\sqrt{2}\right)$:

$$I_4^0\left(-\sqrt{2}\right) = \int \frac{dx}{x^4-\sqrt{2}x^2+1} = \frac{1}{2}\left[I_4^+\left(-\sqrt{2}\right) - I_4^-\left(-\sqrt{2}\right)\right]$$

$$I_4^0\left(\sqrt{2}\right) = \int \frac{dx}{x^4+\sqrt{2}x^2+1} = \frac{1}{2}\left[I_4^+\left(\sqrt{2}\right) - I_4^-\left(\sqrt{2}\right)\right]$$

$$I_4^2\left(-\sqrt{2}\right) = \int \frac{x^2}{x^4-\sqrt{2}x^2+1}dx = \frac{1}{2}\left[I_4^+\left(-\sqrt{2}\right) + I_4^-\left(-\sqrt{2}\right)\right]$$

$$I_4^2\left(\sqrt{2}\right) = \int \frac{x^2}{x^4+\sqrt{2}x^2+1}dx = \frac{1}{2}\left[I_4^+\left(\sqrt{2}\right) + I_4^-\left(\sqrt{2}\right)\right]$$

At the final step:

$$I_8 = \int \frac{dx}{x^8+1} = \frac{1}{2}\left[I_4^0\left(-\sqrt{2}\right) + I_4^0\left(\sqrt{2}\right)\right] + \frac{1}{2\sqrt{2}}\left[-I_4^2\left(-\sqrt{2}\right) + I_4^2\left(\sqrt{2}\right)\right]$$

$$= \frac{1}{4}\left[I_4^+\left(-\sqrt{2}\right) - I_4^-\left(-\sqrt{2}\right) + I_4^+\left(\sqrt{2}\right) - I_4^-\left(\sqrt{2}\right)\right]$$

$$+ \frac{1}{4\sqrt{2}}\left[-I_4^+\left(-\sqrt{2}\right) - I_4^-\left(-\sqrt{2}\right) + I_4^+\left(\sqrt{2}\right) + I_4^-\left(\sqrt{2}\right)\right]$$

$$= \frac{2-\sqrt{2}}{8} I_4^+\left(-\sqrt{2}\right) - \frac{2+\sqrt{2}}{8} I_4^-\left(-\sqrt{2}\right) + \frac{2+\sqrt{2}}{8} I_4^+\left(\sqrt{2}\right)$$

$$- \frac{2-\sqrt{2}}{8} I_4^-\left(\sqrt{2}\right)$$

$$= \frac{1}{8}\left[\left(2-\sqrt{2}\right)\left(I_4^+\left(-\sqrt{2}\right) - I_4^-\left(\sqrt{2}\right)\right) + \left(2+\sqrt{2}\right)\left(-I_4^-\left(-\sqrt{2}\right) + I_4^+\left(\sqrt{2}\right)\right)\right]$$

It is also noticeable that:

$$\left(2-\sqrt{2}\right)\left(I_4^+\left(-\sqrt{2}\right) - I_4^-\left(\sqrt{2}\right)\right)$$

$$= 4\left(\sin\frac{\pi}{8}\right)^2 \left[\frac{1}{2}\csc\frac{\pi}{8}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{8}} - \frac{1}{4}\csc\frac{\pi}{8}\ln\left(\frac{x^2 - 2x\sin\frac{\pi}{8} + 1}{x^2 + 2x\sin\frac{\pi}{8} + 1}\right)\right]$$

$$+ const$$

$$= \sin\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{8}} + \ln\left(\frac{x^2 + 2x\sin\frac{\pi}{8} + 1}{x^2 - 2x\sin\frac{\pi}{8} + 1}\right)\right] + const$$

And similarly,

$$\left(2+\sqrt{2}\right)\left(-I_4^-\left(-\sqrt{2}\right) + I_4^+\left(\sqrt{2}\right)\right)$$

$$= 4\left(\cos\frac{\pi}{8}\right)^2 \left[-\frac{1}{4}\sec\frac{\pi}{8}\ln\left(\frac{x^2 - 2x\cos\frac{\pi}{8} + 1}{x^2 + 2x\cos\frac{\pi}{8} + 1}\right) + \frac{1}{2}\sec\frac{\pi}{8}\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{8}}\right]$$

$$+ const$$

$$= \cos\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{8}} + \ln\left(\frac{x^2 + 2x\cos\frac{\pi}{8} + 1}{x^2 - 2x\cos\frac{\pi}{8} + 1}\right)\right] + const$$

Therefore, integral I_8 could be expressed as follow:

$$I_8 = \int \frac{dx}{x^8+1}$$

$$= \frac{1}{8}\left\{\sin\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{8}} + \ln\left(\frac{x^2+2x\sin\frac{\pi}{8}+1}{x^2-2x\sin\frac{\pi}{8}+1}\right)\right]\right.$$

$$\left. + \cos\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{8}} + \ln\left(\frac{x^2+2x\cos\frac{\pi}{8}+1}{x^2-2x\cos\frac{\pi}{8}+1}\right)\right]\right\} + const$$

The above expression drives us wonder if there exist any relation between integrals I_8 and I_4, where:

$$I_4 = \int \frac{dx}{x^4+1}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} + \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + const$$

$$= \frac{1}{4}\left[\sqrt{2}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)\right] + const$$

$$= \frac{1}{4} * \frac{1}{\sqrt{2}}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} + \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)\right] + const$$

$$= \frac{1}{4}\sin\frac{\pi}{4}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{4}} + \ln\left(\frac{x^2+2x\sin\frac{\pi}{4}+1}{x^2-2x\sin\frac{\pi}{4}+1}\right)\right] + const$$

$$= \frac{1}{4}\cos\frac{\pi}{4}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{4}} + \ln\left(\frac{x^2+2x\cos\frac{\pi}{4}+1}{x^2-2x\cos\frac{\pi}{4}+1}\right)\right] + const$$

There should be an existed relation between integrals I_8 and I_4. If this relation is found, it is the key to deal with the generalized integral I_{2^n}. This fact will be discussed in next sections of this paper.

5. N = 16: integrating by deriving

This section discusses on the following integral:

$$I_{16} = \int \frac{dx}{x^{16}+1}$$

As presented in the previous section:

$$\frac{1}{x^8+1} = \frac{1}{2}\left(\frac{1}{x^4-\sqrt{2}x^2+1}+\frac{1}{x^4+\sqrt{2}x^2+1}\right) + \frac{1}{2\sqrt{2}}\left(\frac{x^2}{x^4+\sqrt{2}x^2+1}-\frac{x^2}{x^4-\sqrt{2}x^2+1}\right)$$

Applying the three methods as presented in the previous section, we got:

$$\frac{1}{x^8+1} = \frac{Ax^2+B}{x^4-\sqrt{2}x^2+1}+\frac{Cx^2+D}{x^4+\sqrt{2}x^2+1} \quad \forall x \in \mathbb{R}$$

The obtained result was $(A;B;C;D) = \left(-\frac{1}{2\sqrt{2}};\frac{1}{2};\frac{1}{2\sqrt{2}};\frac{1}{2}\right)$.

Integral I_8 was calculated based on the follows:

$$\begin{cases} I_4^0(\mu) = \int \frac{dx}{x^4+\mu x^2+1} \\ I_4^2(\nu) = \int \frac{x^2}{x^4+\nu x^2+1}dx \end{cases}$$

Of which, μ and ν are given real numbers. The above integrals could be solved effectively thanks to *Sum & Sub* method.

Regarding integral I_{16}, we perform the partial fraction decomposition:

$$\frac{1}{x^{16}+1} = \frac{Ax^4+B}{x^8-\sqrt{2}x^4+1}+\frac{Cx^4+D}{x^8+\sqrt{2}x^4+1} \quad \forall x \in \mathbb{R}$$

The result is also $(A;B;C;D) = \left(-\frac{1}{2\sqrt{2}};\frac{1}{2};\frac{1}{2\sqrt{2}};\frac{1}{2}\right)$.

Next, we further perform rewrite the above under

$$\sum \frac{a_i x^2+b_i}{x^4+\kappa x^2+1}$$

Of which, a_i, b_i, and κ are given real numbers. In particular, κ will be found directly through the factoring of $(x^{16}+1)$, a_i and b_i will be found based on further processes of partial fraction decomposition.

And thanks to the impressive *Sum & Sub*, those integrals could be simply solved. Hence, despite the massive calculation, there is somehow a way to find integral I_{16}. However, if we can find any relation between the found results of integrals I_4 and I_8, we can build a conjecture for integral I_{16}.

Indeed, we obtained

$$I_4=\int \frac{dx}{x^4+1}$$
$$=\frac{1}{4}\sin\frac{\pi}{4}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{4}}+\ln\left(\frac{x^2+2x\sin\frac{\pi}{4}+1}{x^2-2x\sin\frac{\pi}{4}+1}\right)\right]+const$$
$$=\frac{1}{4}\cos\frac{\pi}{4}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{4}}+\ln\left(\frac{x^2+2x\cos\frac{\pi}{4}+1}{x^2-2x\cos\frac{\pi}{4}+1}\right)\right]+const$$

And

$$I_8=\int \frac{dx}{x^8+1}$$
$$=\frac{1}{8}\left\{\sin\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\pi}{8}}+\ln\left(\frac{x^2+2x\sin\frac{\pi}{8}+1}{x^2-2x\sin\frac{\pi}{8}+1}\right)\right]\right.$$
$$\left.+\cos\frac{\pi}{8}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{\pi}{8}}+\ln\left(\frac{x^2+2x\cos\frac{\pi}{8}+1}{x^2-2x\cos\frac{\pi}{8}+1}\right)\right]\right\}+const$$

With regard to above integrals, if we put

$$\eta(t)=t\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2t}+\ln\left(\frac{x^2+2xt+1}{x^2-2xt+1}\right)\right]$$

Then we can re-perform

$$I_4=\int \frac{dx}{x^4+1}=\frac{1}{4}\eta\left(\sin\frac{\pi}{4}\right)+const=\frac{1}{4}\eta\left(\cos\frac{\pi}{4}\right)+const$$

As well as

$$I_8=\int \frac{dx}{x^8+1}=\frac{1}{8}\left[\eta\left(\sin\frac{\pi}{8}\right)+\eta\left(\cos\frac{\pi}{8}\right)\right]+const$$
$$=\frac{1}{8}\left[\eta\left(\cos\frac{\pi}{8}\right)+\eta\left(\cos\frac{3\pi}{8}\right)\right]+const$$

Before building the conjecture, we perform the factoring:

$$x^{16}+1=x^{16}+2x^8+1-2x^8$$
$$=(x^8+1)^2-2x^8$$
$$=\left(x^8-\sqrt{2}x^4+1\right)\left(x^8+\sqrt{2}x^4+1\right)$$
$$=\left(x^8-2x^4\cos\frac{\pi}{4}+1\right)\left(x^8-2x^4\cos\frac{3\pi}{4}+1\right)$$
$$=\left(x^8+2x^4+1-2x^4-2x^4\cos\frac{\pi}{4}\right)\left(x^8+2x^4+1-2x^4-2x^4\cos\frac{3\pi}{4}\right)$$
$$=\left[(x^4+1)^2-2x^4\left(1+\cos\frac{\pi}{4}\right)\right]\left[(x^4+1)^2-2x^4\left(1+\cos\frac{3\pi}{4}\right)\right]$$
$$=\left[(x^4+1)^2-4x^4\left(\cos\frac{\pi}{8}\right)^2\right]\left[(x^4+1)^2-4x^4\left(\cos\frac{3\pi}{8}\right)^2\right]$$
$$=\left(x^4-2x^2\cos\frac{\pi}{8}+1\right)\left(x^4+2x^2\cos\frac{\pi}{8}+1\right)\left(x^4-2x^2\cos\frac{3\pi}{8}+1\right)$$
$$\left(x^4+2x^2\cos\frac{3\pi}{8}+1\right)$$
$$=\left(x^4-2x^2\cos\frac{\pi}{8}+1\right)\left(x^4-2x^2\cos\frac{3\pi}{8}+1\right)\left(x^4-2x^2\cos\frac{5\pi}{8}+1\right)$$
$$\left(x^4-2x^2\cos\frac{7\pi}{8}+1\right)$$

It is necessary to perform the above transformation in order to put the following function:

$$\zeta(t)=x^4-2x^2t+1$$

Returning to the factoring:

For $\zeta\left(\frac{\pi}{8}\right)$:

$$\zeta\left(\frac{\pi}{8}\right)=x^4-2x^2\cos\frac{\pi}{8}+1$$
$$=x^4+2x^2+1-2x^2-2x^2\cos\frac{\pi}{8}$$
$$=(x^2+1)^2-2x^2\left(1+\cos\frac{\pi}{8}\right)$$
$$=(x^2+1)^2-4x^2\left(\cos\frac{\pi}{16}\right)^2$$
$$=\left(x^2-2x\cos\frac{\pi}{16}+1\right)\left(x^2+2x\cos\frac{\pi}{16}+1\right)$$

For $\zeta\left(\frac{3\pi}{8}\right)$:

$$\zeta\left(\frac{3\pi}{8}\right) = x^4 - 2x^2\cos\frac{3\pi}{8} + 1$$

$$= x^4 + 2x^2 + 1 - 2x^2 - 2x^2\cos\frac{3\pi}{8}$$

$$= (x^2+1)^2 - 2x^2\left(1 + \cos\frac{3\pi}{8}\right)$$

$$= (x^2+1)^2 - 4x^2\left(\cos\frac{3\pi}{16}\right)^2$$

$$= \left(x^2 - 2x\cos\frac{3\pi}{16} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{16} + 1\right)$$

For $\zeta\left(\frac{5\pi}{8}\right)$:

$$\zeta\left(\frac{5\pi}{8}\right) = x^4 - 2x^2\cos\frac{5\pi}{8} + 1$$

$$= x^4 + 2x^2 + 1 - 2x^2 - 2x^2\cos\frac{5\pi}{8}$$

$$= (x^2+1)^2 - 2x^2\left(1 + \cos\frac{5\pi}{8}\right)$$

$$= (x^2+1)^2 - 4x^2\left(\cos\frac{5\pi}{16}\right)^2$$

$$= \left(x^2 - 2x\cos\frac{5\pi}{16} + 1\right)\left(x^2 + 2x\cos\frac{5\pi}{16} + 1\right)$$

For $\zeta\left(\frac{7\pi}{8}\right)$:

$$\zeta\left(\frac{7\pi}{8}\right) = x^4 - 2x^2\cos\frac{7\pi}{8} + 1$$

$$= x^4 + 2x^2 + 1 - 2x^2 - 2x^2\cos\frac{7\pi}{8}$$

$$= (x^2+1)^2 - 2x^2\left(1 + \cos\frac{7\pi}{8}\right)$$

$$= (x^2+1)^2 - 4x^2\left(\cos\frac{7\pi}{16}\right)^2$$

$$= \left(x^2 - 2x\cos\frac{7\pi}{16} + 1\right)\left(x^2 + 2x\cos\frac{7\pi}{16} + 1\right)$$

Thus,

$$x^{16}+1=\prod_{k=0}^{3}\left[\left(x^2-2x\cos\frac{(2k+1)\pi}{16}+1\right)\left(x^2+2x\cos\frac{(2k+1)\pi}{16}+1\right)\right]$$

$$=\prod_{k=0}^{3}\left[\left(x^2-2x\sin\frac{(2k+1)\pi}{16}+1\right)\left(x^2+2x\sin\frac{(2k+1)\pi}{16}+1\right)\right]$$

Considering fucntion $\eta(t)$:

$$\eta(t)=t\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2t}+\ln\left(\frac{x^2+2xt+1}{x^2-2xt+1}\right)\right]$$

We obtained

$$I_4=\int\frac{dx}{x^4+1}=\frac{1}{4}\eta\left(\sin\frac{\pi}{4}\right)+const=\frac{1}{4}\eta\left(\cos\frac{\pi}{4}\right)+const$$

As well as

$$I_8=\int\frac{dx}{x^8+1}=\frac{1}{8}\left[\eta\left(\sin\frac{\pi}{8}\right)+\eta\left(\cos\frac{\pi}{8}\right)\right]+const$$

$$=\frac{1}{8}\left[\eta\left(\cos\frac{\pi}{8}\right)+\eta\left(\cos\frac{3\pi}{8}\right)\right]+const$$

For integral I_4, $\frac{\pi}{4}$ is an arc that is less than $\frac{\pi}{2}$. For integral I_8, $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ are odd multiples of $\frac{\pi}{8}$ and also less than $\frac{\pi}{2}$. Coincidentally, for integral I_{16}, arcs of $\frac{\pi}{16}$, $\frac{3\pi}{16}$, $\frac{5\pi}{16}$, and $\frac{7\pi}{16}$ have appeared and are all odd multiples of $\frac{\pi}{16}$ and also less than $\frac{\pi}{2}$.

The result remains consistent through another way of factoring as follow:

$$x^4+2x^2\cos\frac{\pi}{8}+1$$

$$=x^4+2x^2+1-2x^2+2x^2\cos\frac{\pi}{8}$$

$$=(x^2+1)^2-2x^2\left(1-\cos\frac{\pi}{8}\right)$$

$$=(x^2+1)^2-4x^2\left(\sin\frac{\pi}{16}\right)^2$$

$$=\left(x^2-2x\sin\frac{\pi}{16}+1\right)\left(x^2+2x\sin\frac{\pi}{16}+1\right)$$

$$=\left(x^2-2x\cos\frac{7\pi}{16}+1\right)\left(x^2+2x\cos\frac{7\pi}{16}+1\right)$$

As well as

$$x^4 + 2x^2 \cos\frac{3\pi}{8} + 1$$
$$= x^4 + 2x^2 + 1 - 2x^2 + 2x^2 \cos\frac{3\pi}{8}$$
$$= (x^2 + 1)^2 - 2x^2\left(1 - \cos\frac{3\pi}{8}\right)$$
$$= (x^2 + 1)^2 - 4x^2\left(\sin\frac{3\pi}{16}\right)^2$$
$$= \left(x^2 - 2x\sin\frac{3\pi}{16} + 1\right)\left(x^2 + 2x\sin\frac{3\pi}{16} + 1\right)$$
$$= \left(x^2 - 2x\cos\frac{5\pi}{16} + 1\right)\left(x^2 + 2x\cos\frac{5\pi}{16} + 1\right)$$

Based on above presentation, we enable to build the conjecture:

$$I_{16} = \int \frac{dx}{x^{16} + 1}$$
$$= \frac{1}{16}\left[\eta\left(\sin\frac{\pi}{16}\right) + \eta\left(\sin\frac{3\pi}{16}\right) + \eta\left(\sin\frac{5\pi}{16}\right) + \eta\left(\sin\frac{7\pi}{16}\right)\right] + const$$
$$= \frac{1}{16}\left[\eta\left(\cos\frac{\pi}{16}\right) + \eta\left(\cos\frac{3\pi}{16}\right) + \eta\left(\cos\frac{5\pi}{16}\right) + \eta\left(\cos\frac{7\pi}{16}\right)\right] + const$$

Once the conjecture appears, what we have to do is proving that conjecture is true, in other words, we have to derive the predicted result. That is the reason why this section is named *"integrating by deriving"*. Indeed, forming:

$$F_{16}(x) = \frac{1}{16}\left[\eta\left(\cos\frac{\pi}{16}\right) + \eta\left(\cos\frac{3\pi}{16}\right) + \eta\left(\cos\frac{5\pi}{16}\right) + \eta\left(\cos\frac{7\pi}{16}\right)\right]$$
$$= \frac{1}{16}\sum_{k=0}^{3}\cos\frac{(2k+1)\pi}{16}\left[2\tan^{-1}\frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{16}}\right.$$
$$\left. + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{16} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{16} + 1}\right)\right]$$

$$= \frac{1}{4}\sum_{k=0}^{3}\left(\cos\frac{(2k+1)\pi}{16}\right)^2\left[\frac{1}{2\cos\frac{(2k+1)\pi}{16}}\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{16}}\right.$$

$$\left.+\frac{1}{4\cos\frac{(2k+1)\pi}{16}}\ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{16}+1}{x^2-2x\cos\frac{(2k+1)\pi}{16}+1}\right)\right]$$

Deriving the function $F_{16}(x)$:

$$F'_{16}(x) = \frac{1}{4}\sum_{k=0}^{3}\left(\cos\frac{(2k+1)\pi}{16}\right)^2\left[\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+4\left(\cos\frac{(2k+1)\pi}{16}\right)^2}\right.$$

$$\left.-\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-4\left(\cos\frac{(2k+1)\pi}{16}\right)^2}\right]$$

$$= \frac{1}{4}\sum_{k=0}^{3}\left(\cos\frac{(2k+1)\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+\left(4\left(\cos\frac{(2k+1)\pi}{16}\right)^2-2\right)x^2}\right.$$

$$\left.-\frac{x^2-1}{x^4+1+\left(2-4\left(\cos\frac{(2k+1)\pi}{16}\right)^2\right)x^2}\right]$$

$$= \frac{1}{4}\sum_{k=0}^{3}\left(\cos\frac{(2k+1)\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{(2k+1)\pi}{8}}\right.$$

$$\left.-\frac{x^2-1}{x^4+1-2x^2\cos\frac{(2k+1)\pi}{8}}\right]$$

$$= \frac{1}{4}\left\{\left(\cos\frac{\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{\pi}{8}} - \frac{x^2-1}{x^4+1-2x^2\cos\frac{\pi}{8}}\right]\right.$$

$$+\left(\cos\frac{3\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{3\pi}{8}} - \frac{x^2-1}{x^4+1-2x^2\cos\frac{3\pi}{8}}\right]$$

$$+\left(\cos\frac{5\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{5\pi}{8}} - \frac{x^2-1}{x^4+1-2x^2\cos\frac{5\pi}{8}}\right]$$

$$\left.+\left(\cos\frac{7\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{7\pi}{8}} - \frac{x^2-1}{x^4+1-2x^2\cos\frac{7\pi}{8}}\right]\right\}$$

Due to properties $(\cos\theta)^2 = \left[\sin\left(\frac{\pi}{2}-\theta\right)\right]^2$ and $\cos(\pi-\theta) = -\cos\theta$, we continue to perform the derivative as follow:

$$F'_{16}(x) = \frac{1}{4}\left\{\left(\cos\frac{\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{\pi}{8}} + \frac{1-x^2}{x^4+1-2x^2\cos\frac{\pi}{8}}\right]\right.$$

$$+\left(\cos\frac{3\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1+2x^2\cos\frac{3\pi}{8}} + \frac{1-x^2}{x^4+1-2x^2\cos\frac{3\pi}{8}}\right]$$

$$+\left(\sin\frac{3\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1-2x^2\cos\frac{3\pi}{8}} + \frac{1-x^2}{x^4+1+2x^2\cos\frac{3\pi}{8}}\right]$$

$$\left.+\left(\sin\frac{\pi}{16}\right)^2\left[\frac{x^2+1}{x^4+1-2x^2\cos\frac{\pi}{8}} + \frac{1-x^2}{x^4+1+2x^2\cos\frac{\pi}{8}}\right]\right\}$$

$$= \frac{1}{4}\left[\frac{\left(\cos\frac{\pi}{16}\right)^2 (x^2+1) + \left(\sin\frac{\pi}{16}\right)^2 (1-x^2)}{x^4+1+2x^2\cos\frac{\pi}{8}}\right.$$

$$+ \frac{\left(\cos\frac{\pi}{16}\right)^2 (1-x^2) + \left(\sin\frac{\pi}{16}\right)^2 (x^2+1)}{x^4+1-2x^2\cos\frac{\pi}{8}}$$

$$+ \frac{\left(\cos\frac{3\pi}{16}\right)^2 (x^2+1) + \left(\sin\frac{3\pi}{16}\right)^2 (1-x^2)}{x^4+1+2x^2\cos\frac{3\pi}{8}}$$

$$\left.+ \frac{\left(\cos\frac{3\pi}{16}\right)^2 (1-x^2) + \left(\sin\frac{3\pi}{16}\right)^2 (x^2+1)}{x^4+1-2x^2\cos\frac{3\pi}{8}}\right]$$

$$= \frac{1}{4}\left[\frac{1+x^2\cos\frac{\pi}{8}}{x^4+1+2x^2\cos\frac{\pi}{8}} + \frac{1-x^2\cos\frac{\pi}{8}}{x^4+1-2x^2\cos\frac{\pi}{8}} + \frac{1+x^2\cos\frac{3\pi}{8}}{x^4+1+2x^2\cos\frac{3\pi}{8}}\right.$$

$$\left.+ \frac{1-x^2\cos\frac{3\pi}{8}}{x^4+1-2x^2\cos\frac{3\pi}{8}}\right]$$

In the above, we have applied properties $(\cos\theta)^2 + (\sin\theta)^2 = 1$ and $(\cos\theta)^2 - (\sin\theta)^2 = \cos(2\theta)$.

Returning to $F'_{16}(x)$:

$$F'_{16}(x) = \frac{1}{4}\left[\frac{1}{x^4+1+2x^2\cos\frac{\pi}{8}} + \frac{1}{x^4+1-2x^2\cos\frac{\pi}{8}}\right.$$

$$+ x^2\cos\frac{\pi}{8}\left(\frac{1}{x^4+1+2x^2\cos\frac{\pi}{8}} - \frac{1}{x^4+1-2x^2\cos\frac{\pi}{8}}\right)$$

$$+ \frac{1}{x^4+1+2x^2\cos\frac{3\pi}{8}} + \frac{1}{x^4+1-2x^2\cos\frac{3\pi}{8}}$$

$$\left.+ x^2\cos\frac{3\pi}{8}\left(\frac{1}{x^4+1+2x^2\cos\frac{3\pi}{8}} - \frac{1}{x^4+1-2x^2\cos\frac{3\pi}{8}}\right)\right]$$

$$=\frac{1}{4}\left[\frac{2(x^4+1)}{x^8+1-2x^4\cos\frac{\pi}{4}}-\frac{4x^4\left(\cos\frac{\pi}{8}\right)^2}{x^8+1-2x^4\cos\frac{\pi}{4}}+\frac{2(x^4+1)}{x^8+1-2x^4\cos\frac{3\pi}{4}}\right.$$

$$\left.-\frac{4x^4\left(\cos\frac{\pi}{8}\right)^2}{x^8+1-2x^4\cos\frac{3\pi}{4}}\right]$$

$$=\frac{1}{2}\left[\frac{x^4+1}{x^8+1-2x^4\cos\frac{\pi}{4}}-\frac{2x^4\left(\cos\frac{\pi}{8}\right)^2}{x^8+1-2x^4\cos\frac{\pi}{4}}+\frac{x^4+1}{x^8+1+2x^4\cos\frac{\pi}{4}}\right.$$

$$\left.-\frac{2x^4\left(\cos\frac{\pi}{8}\right)^2}{x^8+1+2x^4\cos\frac{\pi}{4}}\right]$$

$$=\frac{1}{2}\left[\frac{x^4+1-2x^4\left(\cos\frac{\pi}{8}\right)^2}{x^8+1-2x^4\cos\frac{\pi}{4}}+\frac{x^4+1-2x^4\left(\cos\frac{3\pi}{8}\right)^2}{x^8+1+2x^4\cos\frac{\pi}{4}}\right]$$

$$=\frac{1}{2}\left[(x^4+1)\left(\frac{1}{x^8-\sqrt{2}x^4+1}+\frac{1}{x^8+\sqrt{2}x^4+1}\right)\right.$$

$$\left.-2x^4\left(\frac{\left(\cos\frac{\pi}{8}\right)^2}{x^8-\sqrt{2}x^4+1}+\frac{\left(\cos\frac{3\pi}{8}\right)^2}{x^8+\sqrt{2}x^4+1}\right)\right]$$

$F'_{16}(x)$

$$=\frac{1}{2}\left[\frac{2(x^4+1)(x^8+1)}{x^{16}+1}\right.$$

$$\left.-\frac{2x^4\left(\left(\cos\frac{\pi}{8}\right)^2\left(x^8+\sqrt{2}x^4+1\right)+\left(\cos\frac{3\pi}{8}\right)^2\left(x^8-\sqrt{2}x^4+1\right)\right)}{x^{16}+1}\right]$$

$$F'_{16}(x)=\frac{(x^4+1)(x^8+1)-x^4(x^8+1)-\sqrt{2}x^8\left[\left(\cos\frac{\pi}{8}\right)^2-\left(\cos\frac{3\pi}{8}\right)^2\right]}{x^{16}+1}$$

$$F'_{16}(x) = \frac{x^8 + 1 - \sqrt{2}x^8\left[\left(\cos\frac{\pi}{8}\right)^2 - \left(\sin\frac{\pi}{8}\right)^2\right]}{x^{16} + 1}$$

$$F'_{16}(x) = \frac{x^8 + 1 - \sqrt{2}x^8 \cos\frac{\pi}{4}}{x^{16} + 1}$$

$$F'_{16}(x) = \frac{1}{x^{16} + 1}$$

Thus,

$$I_{16} = \int \frac{dx}{x^{16} + 1} = F_{16}(x) + const$$

In a full expression:

$$I_{16} = \int \frac{dx}{x^{16} + 1}$$

$$= \frac{1}{16}\sum_{k=0}^{3} \cos\frac{(2k+1)\pi}{16}\left[2\tan^{-1}\frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{16}} + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{16} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{16} + 1}\right)\right] + const$$

It is noticeable that:

$$I_4 = \int \frac{dx}{x^4 + 1} = \frac{1}{4}\cos\frac{\pi}{4}\left[2\tan^{-1}\frac{x - \frac{1}{x}}{2\cos\frac{\pi}{4}} + \ln\left(\frac{x^2 + 2x\cos\frac{\pi}{4} + 1}{x^2 - 2x\cos\frac{\pi}{4} + 1}\right)\right] + const$$

And

$$I_8 = \int \frac{dx}{x^8 + 1}$$

$$= \frac{1}{8}\sum_{k=0}^{1} \cos\frac{(2k+1)\pi}{8}\left[2\tan^{-1}\frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{8}} + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{8} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{8} + 1}\right)\right] + const$$

Those results bring back a key point for us to build a generalized conjecture, which will be discussed in the nest section.

6. $\mathrm{N} = 2^{\mathrm{n}}$: The Symphony No. 2^{n}

6.1. Overtune

Based on calculated integrals I_4, I_8, and I_{16} as presented in previous sections, we enable to predict a conjecture for the generalized problem, i.e.

$$I_{2^n} = \int \frac{dx}{1+x^{2^n}}$$
$$= \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left\{\cos\frac{(2k+1)\pi}{2^n}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right]\right\} + const$$

As an overtune, we shall discuss: *why* $2^{n-2}-1$?

Firstly, the process of factoring gives us:

$$x^{2^n}+1=0 \Leftrightarrow x^{2^n}=-1 \Leftrightarrow x^{2^n}=\cos\pi + i\sin\pi$$

Of which ω ($\omega \in \mathbb{C}$) are roots of equation $\omega^{2^n} = -1$, we get:

$$\omega_k = \cos\frac{\pi+k2\pi}{2^n} + i\sin\frac{\pi+k2\pi}{2^n} \quad (k=\overline{0;2^n-1})$$
$$= \cos\frac{(2k+1)\pi}{2^n} + i\sin\frac{(2k+1)\pi}{2^n} \quad (k=\overline{0;2^n-1})$$

By definition,

$$\omega_{2^n-1-k} = \cos\frac{[2(2^n-1-k)+1]\pi}{2^n} + i\sin\frac{[2(2^n-1-k)+1]\pi}{2^n}$$
$$\omega_{2^n-1-k} = \cos\frac{(2^{n+1}-1-2k)\pi}{2^n} + i\sin\frac{(2^{n+1}-1-2k)\pi}{2^n}$$
$$\omega_{2^n-1-k} = \cos\left[2\pi-\frac{(2k+1)\pi}{2^n}\right] + i\sin\left[2\pi-\frac{(2k+1)\pi}{2^n}\right]$$
$$\omega_{2^n-1-k} = \cos\frac{(2k+1)\pi}{2^n} - i\sin\frac{(2k+1)\pi}{2^n}$$

Consequently,

$$\begin{cases} \omega_k + \omega_{2^n-1-k} = 2\cos\dfrac{(2k+1)\pi}{2^n} \\ \omega_k\omega_{2^n-1-k} = \left[\cos\dfrac{(2k+1)\pi}{2^n}\right]^2 + \left[\sin\dfrac{(2k+1)\pi}{2^n}\right]^2 = 1 \end{cases} \quad (k=\overline{0;2^n-1})$$

According to Vieta's theorem, ω_k and ω_{2^n-1-k} are two roots of the equation:

$$x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1 = 0$$

Thus,

$$x^{2^n} + 1 = \prod_{k=0}^{2^{n-1}-1}\left[x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1\right]$$

In the above process of factoring, there is a total of 2^n elements, from element $k = 0$ to element $k = 2^n - 1$. In those 2^n elements, it is optional to choose any element k and then element $2^n - 1 - k$. Therefore, no matter how element k is chosen, either from element $k = 0$ to element $k = 2^{n-1} - 1$ (i.e. k belongs to the front half of total 2^n elements), or from element $k = 2^{n-1}$ to element $k = 2^n - 1$ (i.e. k belongs to the back half of total 2^n elements), the result of determining k and $2^n - 1 - k$ remains unchanged. This symmetric property is just like $C_n^k = C_n^{n-k}$.

This is so-called the *Symmetric proposition for* 2^n *elements*. As illustrated above, k is chosen in the front half, i.e. from element $k = 0$ to element $k = 2^{n-1} - 1$.

Besides, putting

$$\mathrm{P}_k(x) = x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1 \quad (k = \overline{0; 2^n - 1})$$

Then by definition,

$$\mathrm{P}_{2^{n-1}-1-k}(x) = x^2 - 2x\cos\frac{[2(2^{n-1} - 1 - k) + 1]\pi}{2^n} + 1$$

$$\mathrm{P}_{2^{n-1}-1-k}(x) = x^2 - 2x\cos\frac{(2^n - 1 - 2k)\pi}{2^n} + 1$$

$$\mathrm{P}_{2^{n-1}-1-k}(x) = x^2 - 2x\cos\left[\pi - \frac{(2k+1)\pi}{2^n}\right] + 1$$

$$\mathrm{P}_{2^{n-1}-1-k}(x) = x^2 + 2x\cos\frac{(2k+1)\pi}{2^n} + 1$$

Applying *Symmetric proposition for* 2^{n-1} *elements*, for k chosen from the front half (i.e. from element $k = 0$ to element $k = 2^{n-2} - 1$), we can rewrite thee factoring result as follow:

$$x^{2^n}+1=\prod_{k=0}^{2^{n-2}-1}\left[x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1\right]\left[x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1\right]$$

Above discussions are the first answer for the question: *why* $2^{n-2}-1$*?*

Secondly, looking back to calculated integrals I_4, I_8, and I_{16} and considering the function

$$\eta(t)=t\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2t}+\ln\left(\frac{x^2+2xt+1}{x^2-2xt+1}\right)\right]$$

Then

$$I_4=\int\frac{dx}{x^4+1}=\frac{1}{4}\eta\left(\sin\frac{\pi}{4}\right)+const=\frac{1}{4}\eta\left(\cos\frac{\pi}{4}\right)+const$$

$$I_8=\int\frac{dx}{x^8+1}=\frac{1}{8}\left[\eta\left(\sin\frac{\pi}{8}\right)+\eta\left(\cos\frac{\pi}{8}\right)\right]+const$$

$$=\frac{1}{8}\left[\eta\left(\cos\frac{\pi}{8}\right)+\eta\left(\cos\frac{3\pi}{8}\right)\right]+const$$

And

$$I_{16}=\int\frac{dx}{x^{16}+1}$$

$$=\frac{1}{16}\left[\eta\left(\sin\frac{\pi}{16}\right)+\eta\left(\sin\frac{3\pi}{16}\right)+\eta\left(\sin\frac{5\pi}{16}\right)+\eta\left(\sin\frac{7\pi}{16}\right)\right]+const$$

$$=\frac{1}{16}\left[\eta\left(\cos\frac{\pi}{16}\right)+\eta\left(\cos\frac{3\pi}{16}\right)+\eta\left(\cos\frac{5\pi}{16}\right)+\eta\left(\cos\frac{7\pi}{16}\right)\right]+const$$

Hence, we could build a conjecture for the generalized integral as follow:

$$I_{2^n}=\int\frac{dx}{1+x^{2^n}}=\frac{1}{2^n}\sum_{k=0}^{k_0}\eta(\sin\alpha_k)+const=\frac{1}{2^n}\sum_{k=0}^{k_0}\eta(\cos\alpha_k)+const$$

Besides the fact that $k\geq 0$, the supermum of k should satisfy that

$$\alpha_k=\frac{(2k+1)\pi}{2^n}<\frac{\pi}{2}\Leftrightarrow 2k+1<2^{n-1}$$

In other words, let k_0 be the *strongest* number such that $k\leq k_0$, we get:

$$2k_0+1=2^{n-1}-1\Leftrightarrow 2k_0=2^{n-1}-2\Leftrightarrow k_0=2^{n-2}-1$$

This is why the summation is from element $k=0$ to $k=k_0=2^{n-2}-1$.

Conseuently, the generalized conjecture is true for $2^{n-2}-1\geq 0\Leftrightarrow n\geq 2$.

6.2. *The Symphony*

I. Molto Allegro

We shall prove that:

$$I_{2^n}=\int\frac{dx}{1+x^{2^n}}$$
$$=\frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left\{\cos\frac{(2k+1)\pi}{2^n}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}}\right.\right.$$
$$\left.\left.+\ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right]\right\}+const\quad(n\in\mathbb{N},n\geq 2)$$

Denoting:

$$\begin{cases}I_4^+(\lambda)=\int\frac{x^2+1}{x^4+\lambda x^2+1}dx\\ I_4^-(\lambda)=\int\frac{x^2-1}{x^4+\lambda x^2+1}dx\end{cases}$$

Of which λ is a given real number.

For $\lambda=-2\cos\theta$:

$$I_4^+(-2\cos\theta)=\int\frac{x^2+1}{x^4-2x^2\cos\theta+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-2\cos\theta}dx$$
$$=\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2-2\cos\theta}=\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\sin\frac{\theta}{2}\right)^2}$$
$$=I_4^+(-2\cos\theta)=\frac{1}{2}\csc\frac{\theta}{2}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\theta}{2}}+const$$

And

$$I_4^-(-2\cos\theta) = \int \frac{x^2-1}{x^4-2x^2\cos\theta+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-2\cos\theta}dx$$

$$= \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2-2\cos\theta} = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-4\left(\cos\frac{\theta}{2}\right)^2}$$

$$= \frac{1}{4}\sec\frac{\theta}{2}\ln\left|\frac{x+\frac{1}{x}-2\cos\frac{\theta}{2}}{x+\frac{1}{x}+2\cos\frac{\theta}{2}}\right| + const$$

$$= \frac{1}{4}\sec\frac{\theta}{2}\ln\left(\frac{x^2-2x\cos\frac{\theta}{2}+1}{x^2+2x\cos\frac{\theta}{2}+1}\right) + const$$

Forming the function $F_{2^n}(x)$ as follow:

$$F_{2^n}(x)$$

$$= \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left\{\cos\frac{(2k+1)\pi}{2^n}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right]\right\} + const$$

Of which

$$\alpha_k = \frac{(2k+1)\pi}{2^n} \quad \left(k=\overline{0;2^{n-2}-1}\right)$$

Where arcs α_k are all odd multiples of $\frac{\pi}{2^n}$ and less than $\frac{\pi}{2}$. Thanks to above results of $I_4^+(-2\cos\theta)$ and $I_4^-(-2\cos\theta)$, the function $F_{2^n}(x)$ could be transformed as:

$$F_{2^n}(x)$$

$$= \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left[2\cos\frac{(2k+1)\pi}{2^n}\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \cos\frac{(2k+1)\pi}{2^n}\ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right] + const$$

$F_{2^n}(x)$

$$= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[2 \sin \frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2 \sin \frac{(2k+1)\pi}{2^n}} + \cos \frac{(2k+1)\pi}{2^n} \ln \left| \frac{x + \frac{1}{x} + 2 \cos \frac{(2k+1)\pi}{2^n}}{x + \frac{1}{x} - 2 \cos \frac{(2k+1)\pi}{2^n}} \right| \right] + const$$

$F_{2^n}(x)$

$$= \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \left[\frac{1}{2} \left(\sin \frac{(2k+1)\pi}{2^n} \right)^2 \csc \frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2 \sin \frac{(2k+1)\pi}{2^n}} + \frac{1}{4} \left(\cos \frac{(2k+1)\pi}{2^n} \right)^2 \sec \frac{(2k+1)\pi}{2^n} \ln \left| \frac{x + \frac{1}{x} + 2 \cos \frac{(2k+1)\pi}{2^n}}{x + \frac{1}{x} - 2 \cos \frac{(2k+1)\pi}{2^n}} \right| \right] + const$$

II. *Andante*

This movement presents the derivative calculation of $F_{2^n}(x)$:

$F_{2^n}(x)$

$$= \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \left[\frac{1}{2} \left(\sin \frac{(2k+1)\pi}{2^n} \right)^2 \csc \frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2 \sin \frac{(2k+1)\pi}{2^n}} + \frac{1}{4} \left(\cos \frac{(2k+1)\pi}{2^n} \right)^2 \sec \frac{(2k+1)\pi}{2^n} \ln \left| \frac{x + \frac{1}{x} + 2 \cos \frac{(2k+1)\pi}{2^n}}{x + \frac{1}{x} - 2 \cos \frac{(2k+1)\pi}{2^n}} \right| \right] + const$$

Then,

$F'_{2^n}(x)$

$$= \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \left[\left(\sin \frac{(2k+1)\pi}{2^n} \right)^2 \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 4 \left(\sin \frac{(2k+1)\pi}{2^n} \right)^2} + \left(\cos \frac{(2k+1)\pi}{2^n} \right)^2 \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 4 \left(\cos \frac{(2k+1)\pi}{2^n} \right)^2} \right]$$

$$F'_{2^n}(x)$$

$$= \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \left[\left(\sin \frac{(2k+1)\pi}{2^n} \right)^2 \frac{(x^2+1)}{x^4 + 1 - 2x^2 \cos \frac{(2k+1)\pi}{2^{n-1}}} + \left(\cos \frac{(2k+1)\pi}{2^n} \right)^2 \frac{(1-x^2)}{x^4 + 1 - 2x^2 \cos \frac{(2k+1)\pi}{2^{n-1}}} \right]$$

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \frac{1 - x^2 \cos \frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 - 2x^2 \cos \frac{(2k+1)\pi}{2^{n-1}}}$$

Let

$$\beta_k = \frac{(2k+1)\pi}{2^{n-1}} \quad \left(k = \overline{0; 2^{n-2}-1}\right)$$

By definition,

$$\beta_{2^{n-2}-1-k} = \frac{[2(2^{n-2}-1-k)+1]\pi}{2^{n-1}}$$

$$\beta_{2^{n-2}-1-k} = \frac{(2^{n-1}-1-2k)\pi}{2^{n-1}}$$

$$\beta_{2^{n-2}-1-k} = \pi - \frac{(2k+1)\pi}{2^{n-1}}$$

$$\beta_{2^{n-2}-1-k} = \pi - \beta_k$$

Which implies:

$$\cos(\beta_{2^{n-2}-1-k}) = -\cos \beta_k$$

Applying *Symmetic proposition for* 2^{n-2} *elements*, choosing k and then determining $2^{n-2}-1-k$ does not depend on the position of k in a set of 2^{n-2} elements. Accordingly, this set could be separated into two halves, the front contains 2^{n-3} elements from $k = 0$ to $k = 2^{n-3}-1$, and the back contains 2^{n-3} remaining elements from $k = 2^{n-3}$ to $k = 2^{n-2}-1$.

Following up the above transformation:

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \left[\frac{1 - x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} + \frac{1 + x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} \right]$$

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \left[\frac{1}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} + \frac{1}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} + x^2 \cos\frac{(2k+1)\pi}{2^{n-1}} \left(\frac{1}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} - \frac{1}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} \right) \right]$$

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \frac{2(x^4+1) - 4x^4 \left(\cos\frac{(2k+1)\pi}{2^{n-1}}\right)^2}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

$$F'_{2^n}(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 + x^4 \left[1 - 2\left(\cos\frac{(2k+1)\pi}{2^{n-1}}\right)^2\right]}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

$$F'_{2^n}(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 - x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

Here come the arcs of

$$\gamma_k = \frac{(2k+1)\pi}{2^{n-2}} \quad \left(k = \overline{0; 2^{n-3}-1}\right)$$

III. Menuetto

This movement reveals whether there is any relations between α_k, β_k, and γ_k.

Putting

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m}-1} \frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}$$

Forming $\Psi_m(x)$, we may rewrite:

$$F'_{2^n}(x) = \Psi_2(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \frac{1 - x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}$$

As well as

$$F'_{2^n}(x) = \Psi_3(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 - x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

It is importantly noticeable that:

$$F'_{2^n}(x) = \Psi_2(x) = \Psi_3(x)$$

Before using this notification, it is necessary to ensure that $m \in \mathbb{N}$ and:

$$\begin{cases} m \geq 2 \\ 2^{n-m} - 1 \geq 0 \end{cases} \Leftrightarrow \begin{cases} m \geq 2 \\ 2^{n-m} \geq 1 \end{cases} \Leftrightarrow \begin{cases} m \geq 2 \\ n - m \geq 0 \end{cases} \Leftrightarrow 2 \leq m \leq n$$

Due to $\Psi_2(x) = \Psi_3(x)$, there should be a question on the relation between $\Psi_m(x)$ and $\Psi_{m+1}(x)$. If this relation actually exists, it is the key to clinch the generalized integral.

Expressing:

$$\Psi_{m+1}(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1 - x^{2^m} \cos\frac{(2k+1)\pi}{2^{n-m}}}{x^{2^{m+1}} + 1 - 2x^{2^m} \cos\frac{(2k+1)\pi}{2^{n-m}}}$$

Now, we shall prove that $\Psi_m(x) = \Psi_{m+1}(x)$ for $m \in \mathbb{N}$ and $2 \leq m < n$.

Denoting:

$$\phi_k = \frac{(2k+1)\pi}{2^{n-m+1}} \quad (k = \overline{0; 2^{n-m} - 1})$$

By definition:

$$\phi_{2^{n-m}-1-k} = \frac{[2(2^{n-m} - 1 - k) + 1]\pi}{2^{n-m+1}}$$

$$\phi_{2^{n-m}-1-k} = \frac{(2^{n-m+1} - 1 - 2k)\pi}{2^{n-m+1}}$$

$$\phi_{2^{n-m}-1-k} = \pi - \frac{(2k+1)\pi}{2^{n-m+1}}$$

$$\phi_{2^{n-m}-1-k} = \pi - \phi_k$$

Which implies:

$$\cos(\phi_{2^{n-m}-1-k}) = -\cos\phi_k$$

Applying *Symmetric proposition for* 2^{n-m} *elements*, we may transform:

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m}-1} \frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \left[\frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + \frac{1 + x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}\right]$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \left[\frac{1}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + \frac{1}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}} \left(\frac{1}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} - \frac{1}{x^{2^m} - 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}\right)\right]$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \frac{2\left(x^{2^{m+1}} + 1\right) - 4x^{2^m}\left(\cos\frac{(2k+1)\pi}{2^{n-m+1}}\right)^2}{x^{2^{m+1}} + 1 - 2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1 + x^{2^m}\left[1 - 2\left(\cos\frac{(2k+1)\pi}{2^{n-m+1}}\right)^2\right]}{x^{2^{m+1}} + 1 - 2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1 - x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}{x^{2^{m+1}} + 1 - 2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \Psi_{m+1}(x)$$

So, we have proved that $\Psi_m(x) = \Psi_{m+1}(x)$ for $m \in \mathbb{N}$ and $2 \leq m < n$.

This results in:

$$F'_{2^n}(x) = \Psi_2(x) = \Psi_3(x) = \cdots = \Psi_{n-1}(x) = \Psi_n(x) \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Therefore, QED is obtained as per the expression of $\Psi_n(x)$. Indeed:

$$\Psi_n(x) = \frac{1}{2^{n-n}} \sum_{k=0}^{2^{n-n}-1} \frac{1 - x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2^{n-n+1}}}{x^{2^n} + 1 - 2x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2^{n-n+1}}}$$

$$\Psi_n(x) = \sum_{k=0}^{0} \frac{1 - x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2}}{x^{2^n} + 1 - 2x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2}}$$

$$\Psi_n(x) = \frac{1 - x^{2^{n-1}}\cos\frac{\pi}{2}}{x^{2^n} + 1 - 2x^{2^{n-1}}\cos\frac{\pi}{2}}$$

$$\Psi_n(x) = \frac{1}{x^{2^n} + 1}$$

As

$$F'_{2^n}(x) = \Psi_n(x) = \frac{1}{x^{2^n} + 1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Which means:

$$F_{2^n}(x) = \int \frac{dx}{x^{2^n} + 1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

QED.

IV. Allegro assai

The proof is granted upon the completion of Movement III – *Menuetto*. The conclusion is presented in Movement IV – *Allegro assai*.

We have proved that

$$F_{2^n}(x) = \int \frac{dx}{x^{2^n}+1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Which means

$$I_{2^n} = \int \frac{dx}{1+x^{2^n}}$$

$$= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left\{ \cos\frac{(2k+1)\pi}{2^n} \left[2\tan^{-1} \frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1} \right) \right] \right\} + const \quad (n \in \mathbb{N}, n \geq 2)$$

This is such a beautiful expression, which could also be written as

$$I_{2^n} = \int \frac{dx}{1+x^{2^n}}$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left\{ \cos\frac{(2k+1)\pi}{2^n} \left[2\tan^{-1} \frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1} \right) \right] \right\} + const$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[2\cos\frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \cos\frac{(2k+1)\pi}{2^n} \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1} \right) \right] + const$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[2\sin\frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2\sin\frac{(2k+1)\pi}{2^n}} + \cos\frac{(2k+1)\pi}{2^n} \ln\left| \frac{x + \frac{1}{x} + 2\cos\frac{(2k+1)\pi}{2^n}}{x + \frac{1}{x} - 2\cos\frac{(2k+1)\pi}{2^n}} \right| \right] + const$$

Furthermore, we may denote:

$$\begin{cases} f_{(\theta)}(x) = 2\tan^{-1}\frac{x - \frac{1}{x}}{2\sin\theta} \\ g_{(\theta)}(x) = \ln\left|\frac{x + \frac{1}{x} + 2\cos\theta}{x + \frac{1}{x} - 2\cos\theta}\right| \end{cases}$$

Then the result of I_{2^n} could be expressed as:

$$I_{2^n} = \int \frac{dx}{1 + x^{2^n}}$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[\sin\frac{(2k+1)\pi}{2^n} f_{\left(\frac{(2k+1)\pi}{2^n}\right)}(x) + \cos\frac{(2k+1)\pi}{2^n} g_{\left(\frac{(2k+1)\pi}{2^n}\right)}(x) \right] + const$$

Of which, n is an integer and $n \geq 2$.

6.3. Encore

Looking back, what we can find may be the beauty of the generalized integral as well as its solution. Of which, the key for us to reach the treasure should be function $\Psi_m(x)$. Accordingly, the relation $\Psi_m(x) = \Psi_{m+1}(x)$ is just like a bridge to heaven.

Interestingly, although $\Psi_m(x)$ is an x-variable function, the gold is found thanks to m. We seemed to perform the *integration by substitution* from variable x to variable m in order to reach the goal. Indeed, the technique of *integration by substitution* has been performed in this paper. However, the *substitution* from *x* to *m* is truly the game-changing move.

Besides, with the relation between $\Psi_m(x)$ and $\Psi_{m+1}(x)$, we may comment that the generalized integral in this section is calculated based on *integration by substitution* and *mathematical induction*.

7. Generalization: $\mathbf{N = 2s + 1}$ $(\mathbf{s} \in \mathbb{N}^+)$

This section discusses on the generalized integral I_{2s+1}:

$$I_{2s+1} = \int \frac{dx}{x^{2s+1}+1} \quad (s \in \mathbb{N}^+)$$

In the introducing section, integral I_3 was calculated as follow:

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\left(\int \frac{dx}{x+1} + \int \frac{-x+2}{x^2-x+1}dx\right)$$
$$= \frac{1}{3}\left(\int \frac{dx}{x+1} + \int \frac{-2x\cos\frac{\pi}{3}+2}{x^2-2x\cos\frac{\pi}{3}+1}dx\right)$$

In Sub-Section 3.2, integral I_5 was calculated as follow:

$$I_5 = \int \frac{dx}{x^5+1}$$
$$= \frac{1}{5}\left(\int \frac{dx}{x+1} + \int \frac{-x^3+2x^2-3x+4}{x^4-x^3+x^2-x+1}dx\right)$$
$$= \frac{1}{5}\left(\int \frac{dx}{x+1} + \int \frac{-2x\cos\frac{\pi}{5}+2}{x^2-2x\cos\frac{\pi}{5}+1}dx + \int \frac{-2x\cos\frac{3\pi}{5}+2}{x^2-2x\cos\frac{3\pi}{5}+1}dx\right)$$

Based on above results, we may build a conjecture for integral I_{2s+1} as follow:

$$I_{2s+1} = \int \frac{dx}{x^{2s+1}+1}$$
$$= \frac{1}{2s+1}\left(\int \frac{dx}{x+1} + \int \frac{\sum_{p=0}^{2s-1}(-1)^p(2s-p)x^p}{\sum_{q=0}^{2s}(-1)^q x^q}dx\right)$$
$$= \frac{1}{2s+1}\left(\int \frac{dx}{x+1} + \sum_{j=0}^{s-1}\int \frac{-2x\cos\frac{(2j+1)\pi}{2s+1}+2}{x^2-2x\cos\frac{(2j+1)\pi}{2s+1}+1}dx\right)$$

Therefore, it is supposed to prove the following so-called $2s+1$ *equality*:

$$\frac{1}{x+1} + \sum_{j=0}^{s-1}\frac{-2x\cos\frac{(2j+1)\pi}{2s+1}+2}{x^2-2x\cos\frac{(2j+1)\pi}{2s+1}+1} = \frac{2s+1}{x^{2s+1}+1} \quad \forall x \in \mathbb{R} \setminus \{-1\}$$

Let complex numbers ω_j be roots of the equation $x^{2s+1}+1=0$. Factoring:

$$\omega_j^{2s+1} = -1 = \cos\pi + i\sin\pi$$

$$\Leftrightarrow \omega_j = \cos\frac{(2j+1)\pi}{2s+1} + i\sin\frac{(2j+1)\pi}{2s+1} \quad (j = \overline{0;2s})$$

By the the property of complex conjugates:

$$\omega_{2s-j} = \cos\frac{(4s-2j+1)\pi}{2s+1} + i\sin\frac{(4s-2j+1)\pi}{2s+1}$$

$$= \cos\frac{[4s+2-(2j+1)]\pi}{2s+1} + i\sin\frac{[4s+2-(2j+1)]\pi}{2s+1}$$

$$= \cos\left[2\pi - \frac{(2j+1)\pi}{2s+1}\right] + i\sin\left[2\pi - \frac{(2j+1)\pi}{2s+1}\right]$$

$$= \cos\frac{(2j+1)\pi}{2s+1} - i\sin\frac{(2j+1)\pi}{2s+1}$$

Therefore, we get:

$$\begin{cases} \omega_j + \omega_{2s-j} = 2\cos\frac{(2j+1)\pi}{2s+1} \\ \omega_j\omega_{2s-j} = 1 \\ \omega_s = -1 \end{cases}$$

In other words,

$$x^2 - 2x\cos\frac{(2j+1)\pi}{2s+1} + 1 = (x-\omega_j)(x-\omega_{2s-j})$$

Generalizing:

$$x^{2s+1} + 1 = \prod_{j=0}^{2s}(x-\omega_j)$$

Considering function $L(\delta)$ as defined in Sub-Section 3.1:

$$L(\delta) = \int \frac{-2x\cos\delta + 2}{x^2 - 2x\cos\delta + 1}dx$$

The partial fraction decomposition method requires to find two real numbers A and B such that:

$$\frac{-2x\cos\frac{(2j+1)\pi}{2s+1} + 2}{x^2 - 2x\cos\frac{(2j+1)\pi}{2s+1} + 1} = \frac{A}{x-\omega_j} + \frac{B}{x-\omega_{2s-j}} \quad \forall x \in \mathbb{C} \setminus \{\omega_j;\omega_{2s-j}\}$$

The above results in:

$$A(x-\omega_{2s-j}) + B(x-\omega_j) \equiv -2x\cos\frac{(2j+1)\pi}{2s+1} + 2$$

$$\Leftrightarrow (A+B)x - \left(A\omega_{2s-j} + B\omega_j\right) \equiv -2x\cos\frac{(2j+1)\pi}{2s+1} + 2$$

Which leads to the system of equations:

$$\begin{cases} A+B = -2\cos\dfrac{(2j+1)\pi}{2s+1} \\ A\omega_{2s-j} + B\omega_j = -2 \end{cases} \Leftrightarrow (A;B) = \left(-\omega_j; -\omega_{2s-j}\right)$$

Returning to prove the $2s+1$ *equality*:

$$\frac{1}{x+1} + \sum_{j=0}^{s-1} \frac{-2x\cos\dfrac{(2j+1)\pi}{2s+1} + 2}{x^2 - 2x\cos\dfrac{(2j+1)\pi}{2s+1} + 1}$$

$$= \frac{1}{x+1} + \sum_{j=0}^{s-1} \frac{-2x\cos\dfrac{(2j+1)\pi}{2s+1} + 2}{\left(x-\omega_j\right)\left(x-\omega_{2s-j}\right)}$$

$$= \frac{-\omega_s}{x-\omega_s} + \sum_{j=0}^{s-1}\left(\frac{-\omega_j}{x-\omega_j} + \frac{-\omega_{2s-j}}{x-\omega_{2s-j}}\right)$$

$$= \sum_{j=0}^{2s} \frac{-\omega_j}{x-\omega_j}$$

$$= \sum_{j=0}^{2s}\left(1 - \frac{x}{x-\omega_j}\right)$$

$$= 2s+1 - x\sum_{j=0}^{2s}\frac{1}{x-\omega_j}$$

$$= 2s+1 - x\left[\frac{\frac{dy}{dx}\prod_{j=0}^{2s}\left(x-\omega_j\right)}{\prod_{j=0}^{2s}\left(x-\omega_j\right)}\right]$$

$$= 2s+1 - \frac{x\dfrac{dy}{dx}\left(x^{2s+1}+1\right)}{x^{2s+1}+1}$$

$$= 2s+1 - (2s+1)\frac{x^{2s+1}}{x^{2s+1}+1}$$

$$= \frac{2s+1}{x^{2s+1}+1}$$

QED. This leads to the following result of the generalized integral I_{2s+1}:

$$I_{2s+1} = \int \frac{dx}{x^{2s+1}+1}$$

$$= \frac{1}{2s+1}\left[\int \frac{dx}{x+1} + \sum_{j=0}^{s-1} \int \frac{-2x\cos\frac{(2j+1)\pi}{2s+1}+2}{x^2-2x\cos\frac{(2j+1)\pi}{2s+1}+1}dx\right]$$

$$= \frac{1}{2s+1}\left\{\int \frac{dx}{x+1} + \sum_{j=0}^{s-1} L\left[\frac{(2j+1)\pi}{2s+1}\right]\right\}$$

$$= \frac{1}{2s+1}\left\{\ln|x+1| + \sum_{j=0}^{s-1}\left[-\cos\frac{(2j+1)\pi}{2s+1}\ln\left(x^2-2x\cos\frac{(2j+1)\pi}{2s+1}+1\right) + 2\sin\frac{(2j+1)\pi}{2s+1}\tan^{-1}\frac{x-\cos\frac{(2j+1)\pi}{2s+1}}{\sin\frac{(2j+1)\pi}{2s+1}}\right]\right\} + const$$

On the other hand, the above generalized result could be rewritten under complex logarithms as follow:

$$I_{2s+1} = \int \frac{dx}{x^{2s+1}+1}$$

$$= \frac{1}{2s+1}\int\left[\frac{1}{x+1} + \sum_{j=0}^{s-1}\frac{-2x\cos\frac{(2j+1)\pi}{2s+1}+2}{x^2-2x\cos\frac{(2j+1)\pi}{2s+1}+1}\right]dx$$

$$= \frac{1}{2s+1}\int\left(\sum_{j=0}^{2s}\frac{-\omega_j}{x-\omega_j}\right)dx$$

$$= \frac{1}{2s+1}\sum_{j=0}^{2s}\left[-\omega_j\ln\left(x-\omega_j\right)\right] + const$$

Noticeably, the above expression is also true for $s = 0$ and help build a conjecture for integral I_{2r} which will be discussed in the next section.

8. Generalization: $\mathbf{N = 2r}$ $(\mathbf{r} \in \mathbb{N}; \mathbf{r} \geq \mathbf{2})$

This section discusses on the generalized integral I_{2r}:

$$I_{2r} = \int \frac{dx}{x^{2r} + 1} \quad (\mathbf{r} \in \mathbb{N}; \mathbf{r} \geq \mathbf{2})$$

Reminding the partial fraction decomposition for integral I_4 as presented in Sub-Section 2.1:

$$\frac{1}{x^4 + 1} = \frac{1}{4}\left(\frac{-\sqrt{2}x + 2}{x^2 - \sqrt{2}x + 1} + \frac{\sqrt{2}x + 2}{x^2 + \sqrt{2}x + 1}\right)$$

$$= \frac{1}{4}\left(\frac{-2x\cos\frac{\pi}{4} + 2}{x^2 - 2x\cos\frac{\pi}{4} + 1} + \frac{-2x\cos\frac{3\pi}{4} + 2}{x^2 - 2x\cos\frac{3\pi}{4} + 1}\right)$$

Besides, based on results of integrals I_8 and I_{16} as presented in Section 4 and Section 5, respectively, we enable to predict the so-called $2r$ *equality* as follow:

$$\sum_{l=0}^{r-1} \frac{-2x\cos\frac{(2l+1)\pi}{2r} + 2}{x^2 - 2x\cos\frac{(2l+1)\pi}{2r} + 1} = \frac{2r}{x^{2r} + 1} \quad \forall x \in \mathbb{R}$$

Let complex numbers ω_l be roots of the equation $x^{2r} + 1 = 0$. We get:

$$\omega_l^{2r} = -1 = \cos\pi + i\sin\pi$$

$$\Leftrightarrow \omega_l = \cos\frac{(2l+1)\pi}{2r} + i\sin\frac{(2l+1)\pi}{2r} \quad (l = \overline{0; 2r-1})$$

By the the property of complex conjugates:

$$\omega_{2r-1-l} = \cos\frac{(4r-2l-1)\pi}{2r} + i\sin\frac{(4r-2l-1)\pi}{2r}$$

$$= \cos\frac{[4r-(2l+1)]\pi}{2r} + i\sin\frac{[4r-(2l+1)]\pi}{2r}$$

$$= \cos\left[2\pi - \frac{(2l+1)\pi}{2r}\right] + i\sin\left[2\pi - \frac{(2l+1)\pi}{2r}\right]$$

$$= \cos\frac{(2l+1)\pi}{2r} - i\sin\frac{(2l+1)\pi}{2r}$$

The above results in:

$$\begin{cases} \omega_l + \omega_{2r-1-l} = 2\cos\frac{(2l+1)\pi}{2r} \\ \omega_l\omega_{2r-1-l} = 1 \end{cases}$$

In other words,

$$x^2 - 2x\cos\frac{(2l+1)\pi}{2r} + 1 = (x-\omega_l)(x-\omega_{2r-1-l})$$

Generalizing:

$$x^{2r} + 1 = \prod_{j=0}^{2r-1}(x-\omega_l)$$

Similar to the calculation as performed in Section 7:

$$\sum_{l=0}^{r-1}\frac{-2x\cos\frac{(2l+1)\pi}{2r}+2}{x^2-2x\cos\frac{(2l+1)\pi}{2r}+1}$$

$$=\sum_{l=0}^{r-1}\frac{-2x\cos\frac{(2l+1)\pi}{2r}+2}{(x-\omega_l)(x-\omega_{2r-1-l})}$$

$$=\sum_{l=0}^{r-1}\left(\frac{-\omega_l}{x-\omega_l}+\frac{-\omega_{2r-1-l}}{x-\omega_{2r-1-l}}\right)$$

$$=\sum_{l=0}^{2r-1}\frac{-\omega_l}{x-\omega_l}$$

$$=\sum_{l=0}^{2r-1}\left(1-\frac{x}{x-\omega_l}\right)$$

$$=2r - x\sum_{l=0}^{2r-1}\frac{1}{x-\omega_l}$$

$$=2r - x\left[\frac{\frac{dy}{dx}\prod_{l=0}^{2r-1}(x-\omega_l)}{\prod_{l=0}^{2r-1}(x-\omega_l)}\right]$$

$$=2r - \frac{x\frac{dy}{dx}(x^{2r}+1)}{x^{2r}+1}$$

$$=2r - \frac{2rx^{2r}}{x^{2r}+1}$$

$$=\frac{2r}{x^{2r}+1}$$

QED. This leads to the following result of the generalized integral I_{2r}:

$$
\begin{aligned}
I_{2r} &= \int \frac{dx}{x^{2r}+1} \\
&= \frac{1}{2r}\int \left[\sum_{l=0}^{r-1} \frac{-2x\cos\frac{(2l+1)\pi}{2r}+2}{x^2-2x\cos\frac{(2l+1)\pi}{2r}+1}\right]dx \\
&= \frac{1}{2r}\sum_{l=0}^{r-1} L\left[\frac{(2l+1)\pi}{2r}\right] \\
&= \frac{1}{2r}\sum_{l=0}^{r-1}\left[-\cos\frac{(2l+1)\pi}{2r}\ln\left(x^2-2x\cos\frac{(2l+1)\pi}{2r}+1\right)\right. \\
&\qquad \left. + 2\sin\frac{(2l+1)\pi}{2r}\tan^{-1}\frac{x-\cos\frac{(2l+1)\pi}{2r}}{\sin\frac{(2l+1)\pi}{2r}}\right] + const
\end{aligned}
$$

The above result is true for N are even numbers, including the case of integral I_{2^n} as presented in Section 6. On the other hand, the above generalized result could be rewritten under complex logarithms as follow:

$$
\begin{aligned}
I_{2r} &= \int \frac{dx}{x^{2r}+1} \\
&= \frac{1}{2r}\int \left[\sum_{l=0}^{r-1} \frac{-2x\cos\frac{(2l+1)\pi}{2r}+2}{x^2-2x\cos\frac{(2l+1)\pi}{2r}+1}\right]dx \\
&= \frac{1}{2r}\int \left(\sum_{l=0}^{2r-1}\frac{-\omega_l}{x-\omega_l}\right)dx \\
&= \frac{1}{2r}\sum_{l=0}^{2r-1}\left[-\omega_l\ln(x-\omega_l)\right] + const
\end{aligned}
$$

The above result is also true for $r = 1$, which results in an interesting consequence:

$$
\frac{i}{2}\ln\left(\frac{x+i}{x-i}\right) - \tan^{-1}x = const
$$

9. Conclusion

From the results as obtained in Section 7 and Section 8, we get the following generalized integral:

$$I_N = \int \frac{dx}{1+x^N} \quad (N \in \mathbb{N}^+)$$
$$= \frac{1}{N}\sum_{z=0}^{N-1}[-\omega_z \ln(x-\omega_z)] + const$$

Of which ω_z are roots of the equation $x^N = -1$, i.e.

$$\omega_z = \cos\frac{(2z+1)\pi}{N} + i\sin\frac{(2z+1)\pi}{N} = e^{\frac{(2z+1)i\pi}{N}} \quad (z = \overline{0; N-1})$$

Besides, for positive integer N, we can also develop the integral as follow:

$$I_{-N} = \int \frac{dx}{1+x^{-N}} = \int \frac{x^N}{1+x^N}dx = 1 - \int \frac{dx}{1+x^N} = 1 - I_N$$

In other words, we enable to solve for the case that N are integers $(N \in \mathbb{Z})$.

The final result of the generalized integral is fairly obvious with a solution which is not so complicated. However, in order to find that result, the author has dealt with a process of calculation as detailed from Section 2 to Section 6. Introducing this paper, the author wishes to share the message *"glory presents not only at the destination, but also throughout the journey"*.