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**Prove that**

$$\lim_{n\to\infty} n\ln(n)\prod_{p\le n}\frac{p}{\sqrt[p-1]{p}(p+1)}=\frac{\pi^2}{6}$$

**The product on the left-hand side extends over all prime numbers $p\le n$**

***Proposed by Prem Kumar-India***

***Solution by proposer***

***We are going to use Merten's 3rd theorem:*** $\lim_{n\to\infty}\frac{1}{\ln n}\prod_{p\le n}\frac{p}{p-1}=e^{\gamma}$

***A well-known result:*** $\gamma=\lim_{n\to\infty}\left(\ln-\sum_{p<n}\frac{\ln p}{p-1}\right)\Rightarrow e^{\gamma}=\lim_{n\to\infty}\frac{n}{\sqrt[p-1]{p}}$

***And Euler's product formula for*** $s=2$

$$\lim_{n\to\infty}\prod_{p\le n}\frac{p^2}{p^2-1}=\zeta(2)$$

$$LHS=\lim_{n\to\infty} n\ln n\prod_{p\le n}\frac{p}{\sqrt[p-1]{p}(p+1)}\times\frac{p}{p}\times\frac{p-1}{p-1}$$

$$=\lim_{n\to\infty}\prod_{p\le n}\frac{n}{\sqrt[p-1]{p}}\times\frac{1}{\frac{1}{\ln n}\times\frac{p}{p-1}}\times\frac{p^2}{p^2-1}=e^{\gamma}\times e^{-\gamma}\times\zeta(2)=\frac{\pi^2}{6}=RHS$$

***Hence proved.***