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ROMANIAN MATHEMATICAL MAGAZINE

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Prove without softs:

$$\left|\int_0^2 (\sqrt{x}\cdot\sin(\pi x))dx\right| < \int_0^1 (|\sqrt{x+1}-\sqrt{x}|\cdot\sin(\pi x))dx$$

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$$\left|\int_0^2 \sqrt{x}\sin(\pi x)\,dx\right| < \int_0^1 (|\sqrt{x+1}-\sqrt{x}|\sin(\pi x))\,dx$$

$$\left|\int_0^2 \sqrt{x}\sin(\pi x)dx\right| = \left|\int_0^1 \sqrt{x}\sin(\pi x)\,dx + \underbrace{\int_1^2 \sqrt{x}\sin(\pi x)\,dx}_{y}\right| \quad \textit{(1)}$$

$$\left.\begin{matrix} x-1=t \Rightarrow \begin{matrix} x=t+1 \\ dx=dt \end{matrix} \\ x=1 \Rightarrow t=0 \\ x=2 \Rightarrow t=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y=\int_0^1 \sqrt{t+1}\sin(\pi(t+1))dt \\ \sin(\pi t+\pi)=\sin\pi t\underbrace{\cos\pi}_{-1}+\underbrace{\sin\pi}_{0}\cos(\pi t)=-\sin\pi t \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow y=\int_0^1 -\sqrt{t+1}\sin(\pi t)\,dt \text{ or } y=-\int_0^1 \sqrt{x+1}\sin\pi x\,dx$$

$$\textit{(1)}\Rightarrow \left|\int_0^2 \sqrt{x}\sin(\pi x)\,dx\right| = \left|\int_0^1 \sqrt{x}\sin(\pi x)\,dx - \int_0^1 \sqrt{x+1}\sin(\pi x)\,dx\right| =$$

$$= \left|\int_0^1 \left(\sqrt{x} - \sqrt{x+1}\sin \pi x\, dx\right)\right| \le \int_0^1 \left|\left(\sqrt{x} - \sqrt{x+1}\right)\sin(\pi x)\right| dx =$$

$$\left.\begin{array}{l} = \int_0^1 \left|\sqrt{x+1} - \sqrt{x}\right| \cdot |\sin(\pi x)| dx \\ x \in [0;1] \Rightarrow \pi x \in [0,\pi] \Rightarrow \sin(\pi x) > 0 \Rightarrow |\sin(\pi x)| = \sin \pi x \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left|\int_0^2 \sqrt{x}\sin(\pi x)\, dx\right| < \int_0^1 \left|\sqrt{x+1} - \sqrt{x}\right| \sin \pi x\, dx$$