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Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1-\varepsilon}\left(\frac{(1-x^2)^2\log x}{1-x^6}\right)dx\right)$$

Proposed by Jesu Mhe-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie, Solution 2 by Abdul Mukhtar-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{0}^{1-\varepsilon}\frac{(1-x^2)^2\log x}{1-x^6}dx = \int_0^1 \frac{(x^2-1)^2}{1-x^6}\log x\,dx = \int_0^1 (x^2-1)^2\log x\sum_{n\geq 0}x^{6n}\,dx$$

$$= \sum_{n\geq 0}\int_0^1 x^{6n+4}\log x + x^{6n}\log x - 2x^{6n+2}\log x\,dx$$

$$= \sum_{n\geq 0}\frac{2}{(6n+3)^2} - \frac{1}{(6n+5)^2} - \frac{1}{(6n+1)^2}$$

$$\int_0^1 x^k\log x\,dx = -\frac{1}{(k+1)^2}$$

$$=\frac{1}{36}\sum_{n\ge0}\frac{2}{\left(n+\frac{1}{2}\right)^2}-\frac{1}{\left(n+\frac{5}{6}\right)^2}-\frac{1}{\left(n+\frac{1}{6}\right)^2}=\frac{1}{36}\left\{2\varphi_1\left(\frac{1}{2}\right)-\varphi\left(\frac{5}{6}\right)-\varphi_1\left(\frac{1}{6}\right)\right\}$$

$$=\frac{1}{36}\left\{2\cdot\frac{\pi^2}{2}-\left(\varphi_1\left(1-\frac{1}{6}\right)+\varphi_1\left(\frac{1}{6}\right)\right)\right\}=\frac{1}{36}\left\{\pi^2-\frac{\pi^2}{\sin^2\left(\frac{\pi}{6}\right)}\right\}=-\frac{\pi^2}{12}$$

$$\varphi_1(1-x)+\varphi_1(x)=\frac{\pi^2}{\sin^2(\pi x)}$$

Solution 2 by Abdul Mukhtar-Nigeria

$$I=\int_0^1\frac{(1-x^2)^2\ln x}{1-x^6}dx$$

$$I=\sum_{n=0}^{\infty}\int_0^1(1-x^2)^2x^{6n}\ln x\,dx$$

$$I=-\left[\sum_{n=0}^{\infty}\frac{1}{(6n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{(6n+5)^2}\right]+2\sum_{n=0}^{\infty}\frac{1}{(6n+3)^2}$$

$$I=-\sum_{n=-\infty}^{\infty}\frac{1}{(6n+1)^2}+\frac{1}{9}\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2}$$

$$I=\frac{\pi}{36}Res\left(\frac{\cot(\pi s)}{\left(s+\frac{1}{6}\right)^2}\right)-\frac{\pi}{36}Res\left(\frac{\cot(\pi s)}{\left(s+\frac{1}{2}\right)^2}\right)$$

$$I=\frac{\pi}{36}\frac{d}{ds}(\cot(\pi s))-\frac{\pi}{36}\frac{d}{ds}(\cot(\pi s))$$

where $s=-\frac{1}{6}$ ***and*** $s=-\frac{1}{2}$

$$I=\frac{\pi}{36}(-4\pi)-\frac{\pi}{36}(-\pi)$$

$$I=-\frac{4\pi^2}{36}+\frac{\pi^2}{36}$$

$$I=\frac{-4\pi^2+\pi^2}{36}$$

$$I=\frac{-36\pi^2}{36}$$

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$$I=-\frac{\pi^2}{12}$$