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Find:

$$\Omega=\int_0^{-1}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\sum_{k=1}^{\infty}\frac{x^k}{k}-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx$$

Proposed by Mohamed Arahman Jama-Somalia

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\sum_{k=1}^{n}\frac{x^k}{k}-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx=$$

$$=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\sum_{n=1}^{n}\int_0^x z^k dz-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx$$

$$=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\int_0^x\frac{1-z^n}{1-z}dz+\log(1-x)\right)\right)\right)dx=$$

$$= \int_0^{-1} \left(\sum_{n=1}^{+\infty} \left(\frac{1}{n} \left(\int_0^x \frac{z^n}{z-1} dz \right) \right) \right) dx$$

$$= \int_0^{-1} \left(\int_0^x \frac{1}{z-1} \sum_{n=1}^{+\infty} \frac{z^n}{n} dz \right) dx = \int_0^{-1} \left(-\int_0^x \frac{\log(1-z)}{z-1} dz \right) dx =$$

$$= \frac{1}{2} \int_{-1}^{0} \log^2(1-x)\, dx = \frac{1}{2} \int_1^2 \log^2 x\, dx$$

$$= \frac{1}{2} \{x \log^2 x - 2(x \log x - x)\}_1^2 = (\log 2 - 1)^2 \Rightarrow \Omega = (1 - \log 2)^2$$