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In ΔABC the following relationships holds:

$$n_a + g_a + \sqrt{2r_b r_c} \le 2\sqrt{3}m_a,\ n_a \text{ – Nagel's cevian}$$

$$\frac{n_a + g_a + \sqrt{2r_b r_c}}{w_a\sqrt{3}} \le \frac{b}{c} + \frac{c}{b},\ g_a \text{ – Gergonne's cevian}$$

Proposed by Bogdan Fustei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc)$$

$$\Rightarrow s(b^2+c^2) - 2sbc = an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_ar_a \Rightarrow n_a^2 \overset{(1)}{=} s^2 - 2h_ar_a$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - (b^3+c^3) = ag_a^2 + a(s^2 - s(2s-a) + bc)$$

$$= ag_a^2 + a(-s^2 + as + bc)$$

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$$\Rightarrow ag_a^2 = as^2 + \frac{(b^2+c^2)(\sum a) - 2(b^3+c^3) - a^2(\sum a) - 2abc}{2}$$

$$= as^2 + \frac{ab^2 + ac^2 + b^3 + bc^2 + b^2c + c^3 - 2(b^3+c^3) - a^3 - a^2b - a^2c - 2abc}{2}$$

$$= as^2 + \frac{a(b-c)^2 - (a^3+b^3+c^3) + b^2c + bc^2 - a^2b - a^2c}{2}$$

$$= as^2 + \frac{a(b-c)^2 - a^2(\sum a) - (b+c)(b^2 - bc + c^2) + bc(b+c)}{2}$$

$$= as^2 + \frac{a(b-c)^2 - 2sa^2 - (2s-a)(b-c)^2}{2}$$

$$= as^2 + \frac{2a(b-c)^2 - 2sa^2 - 2s(b^2+c^2-2b)}{2} = as^2 + a(b-c)^2 - s\sum a^2 + 2sbc$$

$$\Rightarrow g_a^2 \overset{(2)}{=} (b-c)^2 + s^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a}$$

(1)+(2)$\Rightarrow n_a^2 + g_a^2 + 2r_br_c$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - 2\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) + 2\frac{\Delta^2}{(s-b)(s-c)}$$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - \frac{4s(s-a)(s-b)(s-c)}{a(s-a)} +$$

$$+ \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{\sum a^2 + 4(s-b)(s-c) - 2bc}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{a^2 - (b-c)^2 + a^2 + (b^2+c^2-2bc)}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left(\frac{2a^2}{a}\right) = (b-c)^2 + 4s(s-a)$$

$$= (b-c)^2 + (b+c+a)(b+c-a)$$

$$= (b-c)^2 + (b+c)^2 - a^2 = 2b^2 + 2c^2 - a^2 = 4m_a^2$$

$$\therefore n_a^2 + g_a^2 + 2r_br_c \overset{(3)}{=} 4m_a^2$$

Now, $n_a + g_a + \sqrt{2r_br_c} \overset{CBS}{\leq} \sqrt{3}\sqrt{n_a^2 + g_a^2 + 2r_br_c} \overset{by\,(3)}{=} \sqrt{3}\sqrt{4m_a^2} = 2\sqrt{3}m_a$

$$\therefore n_a + g_a + \sqrt{2r_br_c} \overset{(4)}{\leq} 2\sqrt{3}m_a$$

Again, $\frac{n_a+g_a+\sqrt{2r_br_c}}{w_a\sqrt{3}} \overset{by\ (4)}{\leq} \frac{2m_a}{w_a} \overset{Tsintsifas}{\leq} \frac{b^2+c^2}{bc} = \frac{b}{c} + \frac{c}{b}$

$\therefore \frac{n_a+g_a+\sqrt{2r_br_c}}{w_a\sqrt{3}} \leq \frac{b}{c} + \frac{c}{b}$ ***(Proved)***

Solution 2 by Tran Hong-Dong Thap-Vietnam

Fustei's Identities: $n_a^2 + g_a^2 + 2r_br_c = 4m_a^2$

$$\rightarrow n_a + g_a + \sqrt{2r_br_c} \overset{B.C.S.}{\leq} \sqrt{3(n_a^2 + g_a^2 + 2r_br_c)} = \sqrt{3 \cdot 4m_a^2} = 2\sqrt{3} \cdot m_a \quad (*)$$

$$\frac{n_a + g_a + \sqrt{2r_br_c}}{w_a\sqrt{3}} \overset{(*)}{\leq} \frac{2\sqrt{3}m_a}{w_a\sqrt{3}} = \frac{2m_a}{w_a}$$

We must show that: $\frac{2m_a}{w_a} \leq \frac{b}{c} + \frac{c}{b}$

$$\leftrightarrow 4(bc)^2m_a^2 \leq w_a^2(b^2 + c^2)^2$$

$$\leftrightarrow 4(bc)^2 \cdot \frac{2b^2 + 2c^2 - a^2}{4} \leq \frac{4bcs(s-a)}{(b+c)^2}(b^2 + c^2)^2$$

$$\leftrightarrow bc(b+c)^2(2b^2 + 2c^2 - a^2) \leq 4s(s-a)(b^2 + c^2)^2$$

$$\leftrightarrow bc(b+c)^2(2b^2 + 2c^2 - a^2) \leq (a+b+c)(b+c-a)(b^2 + c^2)^2$$

$$\leftrightarrow (a+b+c)(b+c-a)(b^2 + c^2)^2 - bc(b+c)^2(2b^2 + 2c^2 - a^2) \geq 0$$

$$\leftrightarrow (b^2 - c^2)^2(b^2 + c^2) + a^2(b-c)^2(b^2 + bc + c^2) \geq 0$$

$$\leftrightarrow (b-c)^2[(b+c)^2(b^2 + c^2) + a^2(b^2 + bc + c^2)] \geq 0$$

It is true. Equality $\leftrightarrow b = c.$