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In ΔABC the following relationship holds:

$$\frac{b^2+c^2}{\left((b+c)\cos\frac{A}{2}\right)^2}+\frac{c^2+a^2}{\left((c+a)\cos\frac{A}{2}\right)^2}+\frac{a^2+b^2}{\left((a+b)\cos\frac{C}{2}\right)^2}\le 1+\frac{R}{2r}$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by Bogdan Fustei-Romania, Solution 2 by Soumava Chakraborty-Kolkata-India, Solution 3 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Bogdan Fustei-Romania

$$1+\frac{R}{2r}=\frac{1}{2}\left(\frac{bc}{l_a^2}+\frac{ac}{l_b^2}+\frac{ab}{l_c^2}\right)$$

$\frac{b^2+c^2+2bc}{2}\ge\sqrt{2bc(b^2+c^2)}$ *(Inequality between arithmetic and geometric means)*

$$\frac{(b^2+c^2+2bc)^2}{4}=2bc(b^2+c^2)\Rightarrow(b^2+c^2+2bc)^2=\frac{8bc(b^2+c^2)}{16b^2c^2}$$

$$\frac{[(b+c)^2]^2}{16b^2c^2}\ge\frac{8bc(b^2+c^2)}{2bc\cdot 8bc}=\frac{b^2+c^2}{2bc}\Leftrightarrow\frac{(b+c)^2}{4bc}\ge\sqrt{\frac{b^2+c^2}{2bc}}\,|\cdot\cos\frac{A}{2}$$

$$\frac{(b+c)^2}{4bc}\cos\frac{A}{2}\ge\cos\frac{A}{2}\sqrt{\frac{b^2+c^2}{2bc}};\ l_a=\frac{2bc}{b+c}\cos\frac{A}{2}\quad\textit{(and analogs)}$$

$$\frac{(b+c)^2}{4bc}\cdot\frac{2bc}{b+c}\cos\frac{A}{2}\ge l_a\sqrt{\frac{b^2+c^2}{2bc}}\Leftrightarrow\frac{b+c}{2}\cos\frac{A}{2}\ge l_a\sqrt{\frac{b^2+c^2}{2bc}}\quad\textit{(and analogs)}$$

$$\Rightarrow\left((b+c)\cos\frac{A}{2}\right)^2 \geq 4l_a^2\cdot\frac{(b^2+c^2)}{2bc}$$

$$\frac{b^2+c^2}{\left((b+c)\cos\frac{A}{2}\right)^2} \leq \frac{b^2+c^2}{4l_a^2\cdot\frac{(b^2+c^2)}{2bc}} = \frac{b^2+c^2}{4l_a^2}\cdot\frac{2bc}{b^2+c^2} = \frac{1}{2}\cdot\frac{bc}{l_a^2}$$ ***(and the analogs)***

Summing the above we obtain the following:

$$\sum\frac{b^2+c^2}{\left((b+c)\cos\frac{A}{2}\right)^2} \leq \frac{1}{2}\left(\frac{bc}{l_a^2}+\frac{ac}{l_b^2}+\frac{ab}{l_c^2}\right) = 1+\frac{R}{2r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

By Mustafa Tarek, $\frac{b+c}{2}\cos\frac{A}{2} \geq w_a\sqrt{\frac{b^2+c^2}{2bc}} \Rightarrow \left((b+c)\cos\frac{A}{2}\right)^2 \geq 4w_a^2\left(\frac{b^2+c^2}{2bc}\right)$

$$= 4\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}\cdot\frac{b^2+c^2}{2bc} = \frac{8s(s-a)(b^2+c^2)}{(b+c)^2}$$

$$\therefore \sum\frac{b^2+c^2}{\left((b+c)\cos\frac{A}{2}\right)^2} \leq \sum\frac{(b^2+c^2)(b+c)^2}{8s(s-a)(b^2+c^2)}$$

$$= \sum\frac{(s+s-a)^2}{8s(s-a)} = \sum\left[\frac{s^2+(s-a)^2+2s(s-a)}{8s(s-a)}\right]$$

$$= \frac{s\sum(s-b)(s-c)}{8r^2s}+\frac{1}{8s}\sum(s-a)+\frac{3}{4} = \frac{\sum\{s^2-s(b+c)+bc\}}{8r^2}+\frac{1}{8}+\frac{6}{8}$$

$$= \frac{3s^2-4s^2+s^2+4Rr+r^2}{8r^2}+\frac{7}{8} = \frac{1}{8}+\frac{7}{8}+\frac{R}{2r} = 1+\frac{R}{2r}$$ ***(Proved)***

In any $\Delta ABC, \frac{b+c}{2}\cos\frac{A}{2} \overset{(1)}{\geq} w_a\sqrt{\frac{b^2+c^2}{2bc}} \overset{(2)}{\geq} \sqrt{s(s-a)}$

(Mustafa Tarek's refinement)

(1) $\Leftrightarrow \frac{(b+c)^2}{4}\cos^2\frac{A}{2} \geq \left(\frac{2bc}{b+c}\right)^2\cos^2\frac{A}{2}\left(\frac{b^2+c^2}{2bc}\right)$

$$\Leftrightarrow \frac{(b+c)^2}{4} \geq \frac{4b^2c^2}{(b+c)^2}\left\{\frac{(b+c)^2-2bc}{2bc}\right\} = 2bc-\frac{4b^2c^2}{(b+c)^2}$$

$$\Leftrightarrow \left(\frac{b+c}{2}\right)^2+\left(\frac{2bc}{b+c}\right)^2-2\left(\frac{b+c}{2}\right)\left(\frac{2bc}{b+c}\right) \geq 0 \Leftrightarrow \left(\frac{b+c}{2}-\frac{2bc}{b+c}\right)^2 \geq 0 \rightarrow$$ ***true***

$\Rightarrow$ ***(1) is true, equality when*** $b=c$.

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\frac{b^2+c^2}{\left((b+c)\cos\frac{A}{2}\right)^2}+\frac{c^2+a^2}{\left((c+a)\cos\frac{B}{2}\right)^2}+\frac{a^2+b^2}{\left((a+b)\cos\frac{C}{2}\right)^2}\le 1+\frac{R}{2r}\quad (*)$$

Using inequality: $xy(x^2+y^2)\le\frac{1}{8}(x+y)^4$

We have: $LHS_{(*)}=\frac{bc(b^2+c^2)}{s(s-a)(s+c)^2}+\frac{c^2+a^2}{s(s-b)(c+a)^2}+\frac{a^2+b^2}{s(s-c)(a+b)^2}$

$$\le\frac{(b+c)^2}{8s(s-a)}+\frac{(c+a)^2}{8s(s-b)}+\frac{(a+b)^2}{8s(s-c)}$$

$$=\frac{(b+c)^2}{2(a+b+c)(b+c-a)}+\frac{(c+a)^2}{2(a+b+c)(a+c-b)}+\frac{(a+b)^2}{2(a+b+c)(a+b-c)}$$

$$=\frac{(b+c)^2(a+c-b)(a+b-c)+(c+a)^2(b+c-a)(a+b-c)+(a+b)^2(b+c-a)(a+c-b)}{2(a+b+c)(a+b-c)(b+c-a)(a+c-b)}$$

$$=\omega$$

$$RHS_{(*)}=1+\frac{abc}{(a+b-c)(b+c-a)(a+c-b)}$$

$$=\frac{(a+b-c)(b+c-a)(a+c-b)+abc}{(a+b-c)(b+c-a)(a+c-b)}$$

Let $x=a+b-c;y=b+c-a;z=a+c-b\to x+y+z=a+b+c$

$$a=\frac{x+z}{2};b=\frac{x+y}{2};c=\frac{y+z}{2}$$

We must show that: $\frac{xz(x+2y+z)^2+xy(x+y+2z)^2+yz(2x+y+z)^2}{8(x+y+z)xyz}$

$$=\frac{8xyz+(x+y)(y+z)(z+x)}{8xyz}$$

$$\leftrightarrow xz(x+2y+z)^2+xy(x+y+2z)^2+yz(2x+y+z)^2$$

$$=(x+y+z)\big(8xyz+(x+y)(y+z)(z+x)\big)$$

$$\leftrightarrow xy^3+yx^3+yz^3+zx^3+zx^3+xz^3+2(x^2y^2+y^2z^2+z^2x^2)+$$

$$+12xyz(x+y+z)=xy^3+yx^3+yz^3+zx^3+zx^3+xz^3+2(x^2y^2+y^2z^2+z^2x^2)+$$

$$+12xyz(x+y+z)$$

It is true. Hence, $LHS_{(*)}\le\omega\le RHS_{(*)}$ **Proved.**