R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solve for real numbers:

$$(\cos 2x)^{15} \cdot (\cos 4x)^{6} \cdot \cos 6x = \cos^{192} x$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$(\cos 2x)^{15} \cdot (\cos 4x)^{6} \cdot \cos 6x = \cos^{192} x \quad (*)$$

We have: $\cos 2x = 2\cos^2 x - 1; \cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 8\cos^2 x - 1$$

Let $t = \cos^2 x; (0 \le t \le 1)$

**** If*** $0 \le t < 0{,}146447$ ***we have:*** $LHS_{(*)} < 0 \le \cos^{192} x \Rightarrow$ ***(*) no roots.***

**** If*** $1 \ge t \ge 0,146447$ ***we have:*** $\cos 2x \le \cos^4 x \Leftrightarrow (\cos^2 x - 1)^2 \ge 0 \Leftrightarrow (t-1)^2 \ge 0$

(true) $\Rightarrow (\cos 2x)^{15} \le (\cos^4 x)^{15} = \cos^{60} x$ ***(1)***

$$\cos 4x \le \cos^{16} x \Leftrightarrow 8\cos^4 x - 8\cos^2 x + 1 \le \cos^{16} x$$

$$\Leftrightarrow (t-1)^2(t^6 + 2t^5 + 3t^4 + 4t^3 + 5t^2 + 6t - 1) \ge 0$$

(True because: $0,146447 \le t \le 1$***)***

$$\Rightarrow |\cos 4x| \le \cos^{16} x \Rightarrow |\cos 4x|^6 \le (\cos^{16} x)^6 = (\cos x)^{96}$$

$$\cos 6x \le \cos^{36} x \; (\Rightarrow |\cos 6x| \le \cos^{36} x)$$

$$\Leftrightarrow 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1 \le \cos^{36} x$$

$$\Leftrightarrow t^{18} - 32t^3 + 48t^2 - 18t + 1 \ge 0$$

$$\Leftrightarrow (t-1)^2(t^{16} + 2t^{15} + 3t^{14} + 4t^{13} + 5t^{12} + 6t^{11} + 7t^{10} + 8t^9 +$$

$$+9t^8 + 10t^7 + 11t^6 + 12t^5 + 13t^4 + 14t^3 + 15t^2 - 16t + 1) \ge 0 \; \textit{(True)}$$

$$\Rightarrow LHS_{(*)} \le \cos^{60} x \cdot \cos^{90} x \cdot \cos^{36} x = \cos^{192} x$$

Equality $\Leftrightarrow x = k\pi \quad (k \in \mathbb{Z})$

Solution 2 by Soumava Chakraborty-Kolkata-India

If $\cos 2x = 0$, *then* $LHS = 0 \Rightarrow RHS = 0 \Rightarrow \cos x = 0 \Rightarrow \cos 2x = 2\cos^2 x - 1 = -1$, *a contradiction* $\Rightarrow \cos 2x \overset{(1)}{\ne} 0$

If $\cos 4x = 0$, *then* $LHS = RHS = 0 \Rightarrow \cos x = 0 \Rightarrow \cos 2x = -1 \Rightarrow$

$\Rightarrow \cos 4x = 2\cos^2 2x - 1 = 1$, *a contradiction* $\Rightarrow \cos 4x \overset{(2)}{\ne} 0$

If $\cos 6x = 0$, *then* $LHS = RHS = 0 \Rightarrow \cos x = 0 \Rightarrow \cos 2x = -1 \Rightarrow$

$\Rightarrow \cos 6x = 4\cos^3 2x - 3\cos 2x = -4 + 3 = -1$, *a contradiction* $\Rightarrow \cos 6x \overset{(3)}{\ne} 0$

If $\cos x = 0$, *then* $RHS = 0 \Rightarrow LHS = 0$. *But (1), (2), (3)* $\Rightarrow LHS \ne 0 \Rightarrow \cos x \overset{(4)}{\ne} 0$

(4)$\Rightarrow RHS > 0 \Rightarrow LHS > 0 \Rightarrow (\cos 2x)^{14}(\cos 4x)^6(\cos 2x \cos 6x) > 0$

$\Rightarrow \cos 2x \cos 6x \overset{(5)}{>} 0$. *Now, given equation* $\Leftrightarrow$

$$\sqrt[24]{(\cos 2x)^{15}(\cos 4x)^6 \cos 6x} \overset{(i)}{=} \cos^8 x \; (\because LHS > 0)$$

Now, $\because (\cos 2x)^{14} \overset{by\,(1)}{>} 0, (\cos 4x)^6 \overset{by\,(2)}{>} 0, \cos 2x \cos 6x \overset{by\,(5)}{>} 0$

$$\therefore \sqrt[24]{(\cos 2x)^{15}(\cos 4x)^6 \cos 6x} =$$

$$= \sqrt[24]{(\cos 2x)^{14}(\cos 4x)^6(\cos 2x \cos 6x) \cdot 1 \cdot 1 \cdot 1}$$

$$\overset{weighted\ GM \le weighted\ AM}{\le} \frac{14\cos 2x + 6\cos 4x + \cos 2x \cos 6x + 3}{24}$$

$$= \frac{14t + 6(2t^2-1) + (4t^3-3t)t + 3}{24} \quad (t = \cos 2x) \Rightarrow \textit{LHS of (i)} \le \frac{14t + 6(2t^2-1) + t(4t^3-3t) + 3}{24}$$

$$\Rightarrow \cos^8 x \le \frac{14t + 6(2t^2-1) + t(4t^3-3t) + 3}{24}$$

$$\Rightarrow \frac{(1+t)^4}{16} \le \frac{14t + 6(2t^2-1) + t(4t^3-3t) + 3}{24}$$

$$\Rightarrow \frac{3(1+t)^4 - 28t - 12(2t^2-1) - 2t(4t^3-3t) - 6}{48} \le 0$$

$$\Rightarrow -(t-1)^2(5t^2-2t-9) \le 0 \Rightarrow (t-1)^2(5t^2-2t-9) \overset{(ii)}{\ge} 0$$

$$\because -1 \le t \le 1, \therefore 5t^2 - 2t - 9 \le 5 + 2 - 9 < 0$$

$\therefore$ ***(ii)*** $\Rightarrow (t-1)^2 \le 0$. ***But*** $(t-1)^2 \ge 0$

$$\therefore (t-1)^2 = 0 \Rightarrow t = 1 \Rightarrow \cos 2x = 1 \Rightarrow 2x = 2n\pi (n \in \mathbb{Z}) \Rightarrow x = n\pi\ (n \in \mathbb{Z})$$

(Answer)