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Solve for natural numbers:

$$a + a^2 + a^3 + a^4 + a^5 + a^6 = b^2$$

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Solution 1 by Serban George Florin-Romania, Solution 2 by Naren Bhandari-Bajura-Nepal

Solution 1 by Serban George Florin-Romania

$$a, b \in \mathbb{N}, a + a^2 + a^3 + a^4 + a^5 + a^6 = b^2$$

If $a = 0 \Rightarrow b = 0$***, true.***

If $a \neq 0$***. Or*** $p|a, p =$ ***prime number,***

$$b^2 = a(1 + a + a^2) + a^4(1 + a + a^2), b^2 = (1 + a + a^2)(a + a^4)$$

$$b^2 = (1 + a + a^2)a(1 + a^3), b^2 = a(a + 1)(a^2 - a + 1)(a^2 + a + 1)$$

If $p|a + 1 \Rightarrow p|a \Rightarrow p|1$ ***false*** $\Rightarrow p \nmid (a + 1)$

If $p|a^2 + a + 1, p|a \Rightarrow p|a^2 \Rightarrow p|a + 1, p|a \Rightarrow \begin{matrix} p|1 \\ p = 1 \end{matrix}$ ***false*** $\Rightarrow p \nmid (a^2 + a + 1)$

If $p|a^2 - a + 1, p|a^2 \Rightarrow p|a - 1, p|a \Rightarrow \begin{matrix} p|1 \\ p = 1 \end{matrix}$ ***false*** $\Rightarrow p \nmid (a^2 - a + 1) \Rightarrow$ ***The number***

a ***is a perfect square,*** $b^2 =$ ***perfect square***

$\Rightarrow (a + 1)(a^2 - a + 1)(a^2 + a + 1) =$ ***perfect square,*** $a =$ ***perfect square*** $= k^2, k \in \mathbb{N}^*$

If $q|a+1, q$ *prime,* $q\neq 3\Rightarrow q|a^2+a$. *If* $q|a^2+a+1\Rightarrow q|1, q=1$ *false*

$\Rightarrow q\nmid(a^2+a+1)$. *If* $q|a^2-a+1$ *and* $q|a^2+a\Rightarrow q|2a-1$ *and*

$q|a+1\Rightarrow q|2a+2\Rightarrow q|3$

$\Rightarrow a+1=u^2, k^2+1=u^2, u\in\mathbb{N}, k\in\mathbb{N}^*, (u-k)(u+k)=1$

$$\begin{cases}u-k=1\\u+k=1\end{cases}\Rightarrow 2u=2, u=1, k=0 \text{ false.}$$

If $q=3, 3|a+1\Rightarrow a+1=M_3\Rightarrow k^2+1=M_3\Rightarrow k^2=M_3+2$ *false*

In conclusion, $a=b=0\in\mathbb{N}$.

Solution 2 by Naren Bhandari-Bajura-Nepal

The given expression can be written as:

$$b^2=(a+1)(a^5+a^3+a)$$

$$b=\sqrt{a(a+1)(a^4+a^2+1)}$$

Since $(a, a+1)$ *are consecutive integers, so,* $a(a+1)\neq k^2$, *where* $k\in\mathbb{N}$. *Thus* b *is a perfect square number if the following holds:*

$$\begin{cases}a(a+1)=a^4+a^2+1\\a(a^4+a^2+1)=a+1\\(a+1)(a^4+a^2+1)=a\end{cases}$$

Let's establish the lemma: Lemma: If $a, b\in\mathbb{N}$ *and* $ab=1$ *then* $a=b=1$.

WLOG, let $a=1$ *then,* $ab=b=\left(a,\frac{1}{a}\right)b$ *(by existence of reciprocal) then by associative property* $b=(ab)\frac{1}{a}=1\cdot\left(\frac{1}{a}\right)=1$ *(Proved).*

Case 1)

From $a(a+1)=a^4+a^2+1$ *we have* $a(a^3-1)=1$ *since* $a\in\mathbb{N}$, *then by the above Lemma* $a=1, a^3-1=1\Rightarrow a=1=\sqrt[3]{2}$ *which is absurd.*

Case 2)

$a(a^4+a^2+1)=a+1$ *implies* $a^3(a^2+1)=1$ *by the lemma above*

$a^3=a^2+1=1\Rightarrow a=1=0$ *which is absurd.*

Case 3)

$(a+1)(a^4+a^2+1)=a$ *gives* $(a^4+a^3)(a+1)=-1$ *which is not possible as* $a\in\mathbb{N}$, *so,* $a^m>0, \forall m\in\mathbb{Z}^+$, *or we can have if*

$a^3(a+1) = \pm 1$, ***then*** $a+1 = \mp 1$

Observation:

The given expression can be written as:

$$b^2 = (a+1)(a^5+a^3+a)$$

$$b = \sqrt{a(a+1)(a^4+a^2+1)}$$

Since $(a, a+1)$ ***are consecutive integers, so,*** $a(a+1) = 2k$, ***where*** $k \in \mathbb{N}$. ***Thus*** $b = \sqrt{2k(a^4+a^2+1)}$ ***is a perfect square if*** $2k = a^4+a^2+1$. ***Set*** $a^2 = p$ ***and further solving quadratically we arrive at***

$$p = \frac{-1 \pm \sqrt{8k-3}}{2}$$

$p \in \mathbb{N}$ ***if*** $(2m-1)^2 = 8k-3 > 1$ ***implying*** $m(m-1) = 2k-1$. ***Notice that*** $m(m-1)$ ***is always an even integer which shows*** $m(m-1) \neq 2k-1$ ***since*** $2k-1$ ***is of odd parity. The above facts show*** $\sqrt{8k-3}$ ***is not an odd perfect square number thus*** $p \neq \mathbb{N}$ ***and hence*** $a \neq \mathbb{N}$ ***shows that it doesn't exist any solution.***