If $a, b, c > 0, a + b + c = 3$ then:

$$9 - 4(ab + bc + ca) + 3abc < \sqrt{6\left(\frac{a^6}{bc} + \frac{b^6}{ca} + \frac{c^6}{ab}\right)}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam,Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand,Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^6}{bc} + \frac{b^6}{ac} + \frac{c^6}{ab} = \frac{(a^2)^3}{bc} + \frac{(b^2)^3}{ac} + \frac{(c^2)^3}{ab} \overset{Holder}{\geq} \frac{(a^2 + b^2 + c^2)^3}{3(ab + bc + ca)} \overset{\sum ab \leq \sum a^2}{\geq} \frac{a^2 + b^2 + c^2}{3}$$

$$\Rightarrow \sqrt{6\left(\frac{a^6}{bc} + \frac{b^6}{ac} + \frac{c^6}{ab}\right)} \geq \sqrt{2(a^2 + b^2 + c^2)}$$

$$abc \leq \frac{(a + b + c)^3}{27} = 1$$

$$\Rightarrow LHS = 9 - 4(ab + bc + ca) + 3abc = (a + b + c)^2 - 4(ab + bc + ca) + 3abc$$

$$= a^2 + b^2 + c^2 - 2(ab + bc + ca) + 3abc$$

$$\overset{abc \leq 1}{\leq} \sum a^2 - 2\sum ab + 3\sum a^2 - 2\sum ab + \frac{(a + b + c)^2}{3}$$

$$= \frac{4}{3}\sum a^2 - \frac{4}{3}\sum ab = \frac{4}{3}\left[\sum a^2 - \sum ab\right]$$

We must show that: $\frac{4}{3}[m-n]<\sqrt{2m}\ \left(m=\sum a^2\geq\sum ab=n\right)$

$$\Leftrightarrow m^2+16mn-8n^2>0\Leftrightarrow (m-n)^2+18mn-9n^2>0$$

$$\Leftrightarrow (m-n)^2+9n(2m-n)>0\ \textit{(true} \because m\geq n\textit{). Proved.}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ ***and*** $a+b+c=3\Rightarrow(ab+bc+ca)\leq 3\Rightarrow\frac{1}{ab+bc+ca}\geq\frac{1}{3}$

$$\Rightarrow(ab+bc+ca)^3\geq 27\left[\sqrt[3]{(abc)^2}\right]^3=27(abc)^2$$

$$\Rightarrow ab+bc+ca\geq 3\sqrt[3]{(abc)^2}\geq 3abc$$

Consider $9+3abc-4(ab+bc+ca)<\sqrt{6\left(\frac{a^6}{bc}+\frac{b^6}{ca}+\frac{c^6}{ab}\right)}$

Iff $9+3abc<4(ab+bc+ca)+\sqrt{6\left(\frac{a^6}{bc}+\frac{b^6}{ca}+\frac{c^6}{ab}\right)}$

Iff $9+3abc<4(ab+bc+ca)+\sqrt{\frac{(a^3+b^3+c^3)^2}{ab+bc+ca}}$

Iff $9+3abc<4(ab+bc+ca)+\sqrt{2}(a^3+b^3+c^3)$

Iff $9+3abc<2(ab+bc+ca)+(a+b+c)^2+\left(\sqrt{2}-1\right)(a^3+b^3+c^3)$

and it is true because:

1. $(a+b+c)^2\geq 9$

2. $ab+bc+ca\geq 3abc$

Hence $2(ab+bc+ca)\geq 6abc>3abc$

3. $\left(\sqrt{2}-1\right)a^3+b^3+c^3>0$

and 4. $a^3+b^3+c^3\geq a^2+b^2+c^2 : a+b+c=3, a,b,c>0$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sum a=3\overset{A-G}{\geq}3\sqrt[3]{abc}\Rightarrow abc\leq 1\therefore 9-4\sum ab+3abc\leq 9-4\sum ab+3$$

$$=12-4\sum ab=\frac{12}{9}\left(\sum a\right)^2-4\sum ab\ \left(\because 9=\left(\sum a\right)^2\right)$$

$$=4\left[\frac{(\sum a)^2}{3}-\sum ab\right]=\frac{4}{3}\left(\sum a^2-\sum ab\right)=\frac{4}{3}(x-y)\ \ \textit{(where } x=\sum a^2, y=\sum ab\textit{)}$$

$\Rightarrow LHS\overset{(1)}{\leq}\frac{4}{3}(x-y)$***. Now,*** $6\sum\frac{a^6}{bc}=6\sum\frac{a^7}{abc}=\frac{6\sum a^7}{abc}$

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$\overset{Chebyshev}{\geq} \frac{6(\sum a^4)(\sum a^3)}{3abc}$ ***[WLOG, assuming*** $a \geq b \geq c \Rightarrow a^4 \geq b^4 \geq c^4$ ***and*** $a^3 \geq b^3 \geq c^3$***]***

$$\overset{A-G}{\geq} \frac{6abc(\sum a^4)}{abc} \geq \frac{6}{3}\left(\sum a^2\right)^2 = 2x^2 \Rightarrow RHS \overset{(2)}{\geq} \sqrt{2x^2}$$

(1), (2) $\Rightarrow$ ***it suffices to prove:*** $2x^2 > \frac{16}{9}(x-y)^2 \Leftrightarrow 9x^2 > 8(x^2 - 2xy + y^2) \Leftrightarrow$

$$\Leftrightarrow x^2 + 16xy - 8y^2 > 0 \Leftrightarrow t^2 + 16t - 8 > 0 \left(t = \frac{x}{y}\right)$$

$\Leftrightarrow t^2 + 16(t-1) + 8 > 0 \rightarrow$ ***true*** $\because t = \frac{\sum a^2}{\sum ab} \geq 1$

(Proved)