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If:

$$\Omega(\theta)=\left(\sum_{k=0}^{n}\left(2^k\tan\left(2^k\theta\right)\right)\right)+2^{n+1}\cot\left(2^{n+1}\theta\right)$$

then:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\left(\Omega(\theta)\right)^2+\left(\Omega\left(\frac{\theta}{2}\right)\right)^{-2}\right)d\theta$$

$$=\frac{5\pi+12}{12}+\frac{4\left(\sqrt{8-2\sqrt{6}-2\sqrt{2}}-\sqrt{2-\sqrt{3}}\right)}{4-\sqrt{6}-\sqrt{2}}+\frac{3\left(1-\sqrt{6}+\sqrt{12}\right)}{\sqrt{3}-\sqrt{6}}$$

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Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega(\theta)=\sum_{k=0}^{n}2^k\tan\left(\theta 2^k\right)+2^{k+1}\cot\left(\theta 2^{k+1}\right)$$

$$\therefore\prod_{k=0}^{n}\cos\left(\theta 2^k\right)=\prod_{k=0}^{n}\frac{\sin\left(\theta 2^{k+1}\right)}{2\sin\left(\theta 2^k\right)}=\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}\Rightarrow\sum_{k=0}^{n}\log\left(\cos\left(\theta 2^k\right)\right)=\log\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}$$

$$\Rightarrow\sum_{k=0}^{n}\frac{d\log\left(\cos\left(\theta 2^k\right)\right)}{d\theta}=\frac{d\log\left(\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}\right)}{d\theta}\Rightarrow\sum_{k=0}^{n}2^k\tan\left(\theta 2^k\right)=$$

$$=\cot\theta-2^{n+1}\cot\left(\theta 2^{n+1}\right)$$

$$\Rightarrow\Omega(\theta)=\cot\theta\Rightarrow M=\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\Omega^2(\theta)+\frac{1}{\Omega^2\left(\frac{\theta}{2}\right)}\right)d\theta=\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\cot^2\theta+\frac{1}{\cot^2\frac{\theta}{2}}\right)d\theta$$

$$=\left\{-\theta-\cot\theta\right\}_{\frac{\pi}{4}}^{\frac{5\pi}{120}}+\left\{-\theta+2\tan\frac{\theta}{2}\right\}_{\frac{\pi}{4}}^{\frac{5\pi}{120}}=$$

$$=-\frac{5\pi}{60}-\cot\frac{5\pi}{120}+2\tan\frac{5\pi}{240}+\frac{\pi}{2}+1-2\tan\frac{\pi}{8}$$

$$=\frac{5\pi}{12}+3-2\sqrt{2}-\cot\frac{\pi}{24}+2\tan\frac{\pi}{48}\quad\left\{\tan\frac{\pi}{8}=\frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\sqrt{2}-1\right\}$$

$$\cot\frac{\pi}{24}=\frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=$$

$$=2+\sqrt{2}+\sqrt{3}+\sqrt{6}\left\{\cos\frac{\pi}{12}=\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$

$$\tan\frac{\pi}{48}=\frac{\sin\frac{\pi}{24}}{1+\cos\frac{\pi}{24}}=\frac{\sqrt{\frac{1-\cos\frac{\pi}{12}}{2}}}{1+\sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}}=\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$M=\frac{5\pi}{12}+1-3\sqrt{2}-\sqrt{3}-\sqrt{6}+2\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$=\frac{5\pi}{12}+1+2\frac{\frac{\sqrt{2}}{2}\sqrt{4-\sqrt{6}-\sqrt{2}}}{2+\frac{\sqrt{2}}{2}\sqrt{4+\sqrt{6}+\sqrt{2}}}-3\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$$=\frac{5\pi}{12}+1+\frac{\sqrt{2}\sqrt{4-\sqrt{6}-\sqrt{2}}\left(2-\frac{\sqrt{2}}{2}\sqrt{4+\sqrt{6}+\sqrt{2}}\right)}{\frac{4-\sqrt{6}-\sqrt{2}}{2}}-3\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$$=\frac{5\pi}{12}+1+\frac{2\sqrt{8-2\sqrt{6}-2\sqrt{2}}-2\sqrt{2-\sqrt{3}}}{\frac{4-\sqrt{6}-\sqrt{2}}{2}}-3\sqrt{2}+\frac{3}{\sqrt{3}-\sqrt{6}}$$

$$=\frac{5\pi}{12}+1+4\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}-\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}+3\frac{1-\sqrt{6}+\sqrt{12}}{\sqrt{3}-\sqrt{6}}$$