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Prove the following relation

$$\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cot(\pi x)-\frac{1}{x}\right)\sin(\pi x)\,e^{-\pi x}dx=\frac{1}{4}\left(\pi\left(2\sqrt{5\sqrt{2}-7}-\pi+3\right)+2\right)$$

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Solution by Tobi Joshua-Nigeria

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cot\pi x-\frac{1}{x}\right)\sin\pi x\,e^{-\pi x}dx$$

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cos\pi x\,e^{-\pi x}-\frac{\sin\pi x\;e^{-\pi x}}{x}\right)dx$$

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)(\pi\cos\pi x\,e^{-\pi x})-\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{1}{\sqrt{x}}+\frac{1}{x}\right)\left(\frac{\sin\pi x e^{-\pi x}}{x}\right)dx$$

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)(\pi\cos\pi x\,e^{-\pi x})-\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x\sqrt{x}}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x e^{-\pi x}}{x^2}\right)dx \Rightarrow \textit{applying by part}$$

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}\right)(\pi\cos\pi x\,e^{-\pi x})dx-\int_0^\infty\left(\frac{(\pi\cos\pi x\,e^{-\pi x})}{x}\right)dx-\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x\sqrt{x}}\right)dx+\frac{\sin\pi x\,e^{-\pi x}}{x}\Big|_0^\infty-\pi\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{(\pi\cos\pi x\,e^{-\pi x})}{x}\right)dx$$

$$I=\int_0^\infty\left(1-\frac{1}{\sqrt{x}}\right)(\pi\cos\pi x\,e^{-\pi x})dx-(1+\pi)\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x\sqrt{x}}\right)dx+\pi$$

$$A=\int_0^\infty\left((\pi\cos\pi x\,e^{-\pi x})-\frac{(\pi\cos\pi x\,e^{-\pi x})}{\sqrt{x}}\right)dx-\int_0^\infty(1+\pi)\left(\frac{\sin\pi x\,e^{-\pi x}}{x}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x\,e^{-\pi x}}{x\sqrt{x}}\right)dx+\pi$$

$$A=\frac{1}{2}-Re\,\pi\int_0^\infty\frac{\left(e^{-\pi x(i+1)}\right)}{\sqrt{x}}dx-(1+\pi)\,L\left\{\frac{\sin\pi x}{x}\right\}_{s=\pi}+Im\int_0^\infty\frac{\left(e^{-\pi x(-i+1)}\right)}{x\sqrt{x}}dx+\pi$$

$$A=\frac{1}{2}-Re\frac{\sqrt{\pi}}{\sqrt{i+1}}\int_0^\infty x^{\frac{-1}{2}}(e^{-x})dx-(1+\pi)\left(\frac{\pi}{4}\right)+$$

$$+Im\left(\pi^{\frac{3}{2}}\right)\sqrt{(1-i)}\int_0^\infty x^{-\frac{3}{2}}(e^{-x})\,dx+\pi$$

$$A=\frac{1}{2}-Re\,\frac{\sqrt{\pi}}{\sqrt{i+1}}\Gamma\left(\frac{1}{2}\right)-(1+\pi)\left(\frac{\pi}{4}\right)+Im\left(\pi^{\frac{1}{2}}\right)\sqrt{(1-i)}\Gamma\left(-\frac{1}{2}\right)+\pi$$

$$A=\frac{1}{2}-Re\frac{\pi}{\sqrt{i+1}}-(1+\pi)\left(\frac{\pi}{4}\right)-2Im(\pi)\sqrt{(1-i)}+\pi$$

$$A=\frac{1}{2}-\pi Re\frac{1}{\left(\sqrt{\sqrt{2}}\right)}\left[\cos\left(\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{8}\right)\right]-(1+\pi)\left(\frac{\pi}{4}\right)-$$

$$-2Im(\pi)\left(\sqrt{\sqrt{2}}\right)\left[\cos\left(\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{8}\right)\right]+\pi$$

$$A=\frac{1}{2}-\pi\frac{1}{\left(\sqrt{\sqrt{2}}\right)}\left[\frac{\sqrt{2+\sqrt{2}}}{2}\right]-(1+\pi)\left(\frac{\pi}{4}\right)+2(\pi)\left(\sqrt{\sqrt{2}}\right)\left[\frac{\sqrt{2-\sqrt{2}}}{2}\right]+\pi$$

$$A=\frac{1}{2}-\frac{\pi\sqrt{1+\sqrt{2}}}{2}+\pi\sqrt{2\sqrt{2}-2}-\left(\frac{\pi^2+\pi}{4}\right)+\pi$$

$$A=\frac{1}{2}+\frac{\pi}{2}\left[2\sqrt{2\sqrt{2}-2}-\sqrt{1+\sqrt{2}}\right]-\left(\frac{\pi^2+\pi}{4}\right)+\pi$$

$$I=\frac{1}{2}+\frac{\pi}{2}\left[\sqrt{5\sqrt{2}-7}\right]-\left(\frac{\pi^2+3\pi}{4}\right)$$

$$I=\frac{1}{4}\left(2+2\pi\left[\sqrt{5\sqrt{2}-7}\right]-(\pi^2+3\pi)\right)$$