A simple vector concurrency lemma
and its applications to celestial mechanics
and to the concurrency of the altitudes, medians, perpendicular bisectors and angle bisectors of a triangle

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Martin Celli
Departamento de Matemáticas
Universidad Autónoma Metropolitana-Iztapalapa
Av. San Rafael Atlixco, 186. Col. Vicentina. Alc. Iztapalapa. CP 09340. Mexico City.
E-mail: cell@xanum.uam.mx

Many references contain vector proofs of the concurrency of the altitudes, medians, perpendicular bisectors and angle bisectors of a triangle. However, at least most of them are not based on a common vector criterion, such as the criterion given in this note, which will also provide us an elementary property of the accelerations of three bodies in gravitational interaction.

1 Our lemma

We will prove the following lemma:

Lemma. *In the plane, let A, B, C be three points and $\vec{U}$, $\vec{V}$, $\vec{W}$ three pairwise non-collinear vectors, whose sum is $\vec{0}$. Then:*
- The scalar $\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}$ (where $\wedge$ denotes the determinant of two vectors) does not depend on the point O.
- The lines $(A,\vec{U})$ (passing through A, with direction $\vec{U}$), $(B,\vec{V})$, $(C,\vec{W})$ are concurrent if, and only if:

$$\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}=0\cdot$$

Proof. For every two points O_1, O_2, we have:

$$(\overrightarrow{O_2A}\wedge\vec{U}+\overrightarrow{O_2B}\wedge\vec{V}+\overrightarrow{O_2C}\wedge\vec{W})-(\overrightarrow{O_1A}\wedge\vec{U}+\overrightarrow{O_1B}\wedge\vec{V}+\overrightarrow{O_1C}\wedge\vec{W})$$

$$=\overrightarrow{O_2O_1}\wedge(\vec{U}+\vec{V}+\vec{W})=0\cdot$$

Therefore, the scalar $\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}$ does not depend on O.
If $(A,\vec{U})$, $(B,\vec{V})$, $(C,\vec{W})$ are concurrent at a point I, then:

$$\overrightarrow{IA}\wedge\vec{U}=\overrightarrow{IB}\wedge\vec{V}=\overrightarrow{IC}\wedge\vec{W}=0\cdot$$

Thus:
$$\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}=\overrightarrow{IA}\wedge\vec{U}+\overrightarrow{IB}\wedge\vec{V}+\overrightarrow{IC}\wedge\vec{W}=0\cdot$$
Conversely, assuming that:
$$\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}=0,$$
and denoting I the intersection point of $(A,\vec{U})$ and $(B,\vec{V})$ (these lines are not parallel as $\vec{U}$ and $\vec{V}$ are not collinear), we have:
$$\overrightarrow{IC}\wedge\vec{W}=-(\overrightarrow{IA}\wedge\vec{U}+\overrightarrow{IB}\wedge\vec{V})=0\cdot$$
So the point I also belongs to $(C,\vec{W})$, and the three lines are concurrent.

Some remarks.

1. Using this lemma, we can study the concurrency of any three lines. As a matter of fact, even when $\vec{U}+\vec{V}+\vec{W}\neq\vec{0}$, by denoting α and β the coordinates of $\vec{W}$ in the basis $(\vec{U},\vec{V})$, we just have to replace $\vec{U}$ with $-\alpha\vec{U}$ and $\vec{V}$ with $-\beta\vec{V}$ (as $\vec{W}$ is not collinear with $\vec{U}$ and $\vec{V}$, we have α, $\beta\neq 0$).

2. We assumed that $\vec{U}$ and $\vec{V}$, $\vec{V}$ and $\vec{W}$, $\vec{W}$ and $\vec{U}$ were not collinear. In fact, these three propositions are equivalent here, since:
$$\vec{U}\wedge\vec{V}=\vec{V}\wedge\vec{W}=\vec{W}\wedge\vec{U}\cdot$$
We can obtain the first equality, for example, seeing that:
$$\vec{U}\wedge\vec{V}-\vec{V}\wedge\vec{W}=-\vec{V}\wedge(\vec{U}+\vec{W})=\vec{V}\wedge\vec{V}=\vec{0}\cdot$$

3. If we choose $O=A$, B or C, one of the three determinants of the expression $\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}$ vanishes and this concurrency criterion involves comparing the areas of two triangles.

4. Let us denote I the intersection point of the three lines, and α, β, γ the factors defined by:
$$\overrightarrow{AI}=\alpha\vec{U},\overrightarrow{BI}=\beta\vec{V},\overrightarrow{CI}=\gamma\vec{W}\cdot$$
We can notice that the factors α, $-\beta$ (for instance) are also the coordinates of $\overrightarrow{AB}$ in the basis $(\vec{U},\vec{V})$. Moreover, if α, β, $\gamma\neq 0$ and $1/\alpha+1/\beta+1/\gamma\neq 0$, then, by the equality:
$$-(\vec{U}+\vec{V}+\vec{W})=\frac{1}{\alpha}\overrightarrow{IA}+\frac{1}{\beta}\overrightarrow{IB}+\frac{1}{\gamma}\overrightarrow{IC}=\vec{0},$$
the intersection point I is the barycenter of $\{(A,1/\alpha),(B,1/\beta),(C,1/\gamma)\}$. (Using the same equality, we can prove that $1/\alpha+1/\beta+1/\gamma$ only vanishes when A, B, C are collinear.)

2 The concurrency of the altitudes, medians, perpendicular bisectors and angle bisectors of a triangle

Our lemma will give us simple proofs of the concurrency of classical lines of a triangle. First, we can obtain the concurrency of the **altitudes** of a triangle ABC defining $\vec{U}$, $\vec{V}$, $\vec{W}$ as the images of $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ under the vector rotation φ of angle $\pi/2$. The pairwise non-collinearity of $\vec{U}$, $\vec{V}$, $\vec{W}$ comes from the pairwise non-collinearity of $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$, and the sum of $\vec{U}$, $\vec{V}$, $\vec{W}$ is trivially $\vec{0}$. Thus, we just have to use the identity:

$$\vec{r} \wedge \varphi(\vec{s}) = ||\vec{r}||||\vec{s}|| \sin\left((\vec{r}, \vec{s}) + \frac{\pi}{2}\right) = ||\vec{r}||||\vec{s}|| \cos(\vec{r}, \vec{s}) = \vec{r}.\vec{s}$$

(where . denotes the dot product), in order to check that:

$$\overrightarrow{OA} \wedge \vec{U} + \overrightarrow{OB} \wedge \vec{V} + \overrightarrow{OC} \wedge \vec{W} = \overrightarrow{OA}.(\overrightarrow{BO} + \overrightarrow{OC}) + \overrightarrow{OB}.(\overrightarrow{CO} + \overrightarrow{OA}) + \overrightarrow{OC}.(\overrightarrow{AO} + \overrightarrow{OB})$$

$$= -\overrightarrow{OA}.\overrightarrow{OB} + \overrightarrow{OA}.\overrightarrow{OC} - \overrightarrow{OB}.\overrightarrow{OC} + \overrightarrow{OB}.\overrightarrow{OA} - \overrightarrow{OC}.\overrightarrow{OA} + \overrightarrow{OC}.\overrightarrow{OB} = 0\cdot$$

Similarly, we can demonstrate the concurrency of the **medians** by applying the lemma to:

$$\vec{U} = \overrightarrow{AB} + \overrightarrow{AC}, \vec{V} = \overrightarrow{BC} + \overrightarrow{BA}, \vec{W} = \overrightarrow{CA} + \overrightarrow{CB}$$

(it is easy to prove that any of these vectors is twice the vector defined by a vertex and the midpoint of the opposite side). The following equality is trivial:

$$\vec{U} + \vec{V} + \vec{W} = \vec{0}\cdot$$

We obtain the non-collinearity of $\vec{U}$ and $\vec{V}$, for example, seeing that:

$$\vec{U} \wedge \vec{V} = (\overrightarrow{AB} + \overrightarrow{AC}) \wedge (\overrightarrow{BC} + \overrightarrow{BA}) = \overrightarrow{AB} \wedge \overrightarrow{BC} + \overrightarrow{AC} \wedge \overrightarrow{BC} + \overrightarrow{AC} \wedge \overrightarrow{BA} = 3\overrightarrow{AB} \wedge \overrightarrow{AC} \neq 0\cdot$$

At last, we can prove the last hypothesis of the lemma in the following way:

$$\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W} = \overrightarrow{OA}\wedge\overrightarrow{AB}+\overrightarrow{OA}\wedge\overrightarrow{AC}+\overrightarrow{OB}\wedge\overrightarrow{BC}+\overrightarrow{OB}\wedge\overrightarrow{BA}+\overrightarrow{OC}\wedge\overrightarrow{CA}+\overrightarrow{OC}\wedge\overrightarrow{CB}$$

$$= (\overrightarrow{OA} - \overrightarrow{OB}) \wedge \overrightarrow{AB} + (\overrightarrow{OB} - \overrightarrow{OC}) \wedge \overrightarrow{BC} + (\overrightarrow{OC} - \overrightarrow{OA}) \wedge \overrightarrow{CA}$$

$$= \overrightarrow{BA} \wedge \overrightarrow{AB} + \overrightarrow{CB} \wedge \overrightarrow{BC} + \overrightarrow{AC} \wedge \overrightarrow{CA} = 0\cdot$$

In order to prove that the **perpendicular bisectors** of a triangle PQR are concurrent, we must take:

$$A = \frac{Q+R}{2}, B = \frac{R+P}{2}, C = \frac{P+Q}{2}, \vec{U} = \varphi(\overrightarrow{QR}), \vec{V} = \varphi(\overrightarrow{RP}), \vec{W} = \varphi(\overrightarrow{PQ})\cdot$$

Consequently, and as in the case of the altitudes, the only non-trivial calculation is the following:

$$2(\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W})=(\overrightarrow{OQ}+\overrightarrow{OR}).\overrightarrow{QR}+(\overrightarrow{OR}+\overrightarrow{OP}).\overrightarrow{RP}+(\overrightarrow{OP}+\overrightarrow{OQ}).\overrightarrow{PQ}$$

$$=(\overrightarrow{OQ}+\overrightarrow{OR}).(\overrightarrow{OR}-\overrightarrow{OQ})+(\overrightarrow{OR}+\overrightarrow{OP}).(\overrightarrow{OP}-\overrightarrow{OR})+(\overrightarrow{OP}+\overrightarrow{OQ}).(\overrightarrow{OQ}-\overrightarrow{OP})$$

$$=OR^2-OQ^2+OP^2-OR^2+OQ^2-OP^2=0\cdot$$

At last, in order to prove that the **angle bisectors** of a triangle ABC are concurrent, we have to take:

$$\vec{U}=\frac{\overrightarrow{AB}}{AB}+\frac{\overrightarrow{AC}}{AC},\vec{V}=\frac{\overrightarrow{BC}}{BC}+\frac{\overrightarrow{BA}}{BA},\vec{W}=\frac{\overrightarrow{CA}}{CA}+\frac{\overrightarrow{CB}}{CB}\cdot$$

Again, the equality $\vec{U}+\vec{V}+\vec{W}=\vec{0}$ is trivial. Therefore, we just need to notice that:

$$\vec{U}\wedge\vec{V}=\frac{\overrightarrow{AB}}{AB}\wedge\frac{\overrightarrow{BC}}{BC}+\frac{\overrightarrow{AC}}{AC}\wedge\frac{\overrightarrow{BC}}{BC}+\frac{\overrightarrow{AC}}{AC}\wedge\frac{\overrightarrow{BA}}{BA}$$

$$=\frac{1}{AB.BC}\overrightarrow{AB}\wedge\overrightarrow{BC}+\frac{1}{AC.BC}\overrightarrow{AC}\wedge\overrightarrow{BC}+\frac{1}{AC.BA}\overrightarrow{AC}\wedge\overrightarrow{BA}$$

$$=\left(\frac{1}{AB.BC}+\frac{1}{AC.BC}+\frac{1}{AC.BA}\right)\overrightarrow{AB}\wedge\overrightarrow{AC}\neq 0$$

and that:

$$\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}=\overrightarrow{OA}\wedge\frac{\overrightarrow{AB}}{AB}+\overrightarrow{OA}\wedge\frac{\overrightarrow{AC}}{AC}+\overrightarrow{OB}\wedge\frac{\overrightarrow{BC}}{BC}+\overrightarrow{OB}\wedge\frac{\overrightarrow{BA}}{BA}+\overrightarrow{OC}\wedge\frac{\overrightarrow{CA}}{CA}+\overrightarrow{OC}\wedge\frac{\overrightarrow{CB}}{CB}$$

$$=(\overrightarrow{OA}-\overrightarrow{OB})\wedge\frac{\overrightarrow{AB}}{AB}+(\overrightarrow{OB}-\overrightarrow{OC})\wedge\frac{\overrightarrow{BC}}{BC}+(\overrightarrow{OC}-\overrightarrow{OA})\wedge\frac{\overrightarrow{CA}}{CA}$$

$$=\frac{\overrightarrow{BA}\wedge\overrightarrow{AB}}{AB}+\frac{\overrightarrow{CB}\wedge\overrightarrow{BC}}{BC}+\frac{\overrightarrow{AC}\wedge\overrightarrow{CA}}{CA}=0\cdot$$

Some remarks.

1. We can slightly simplify the proofs corresponding to the altitudes, medians and angle bisectors, taking $O=A$ in the computation of $\overrightarrow{OA}\wedge\vec{U}+\overrightarrow{OB}\wedge\vec{V}+\overrightarrow{OC}\wedge\vec{W}$. However, the proofs obtained are no more symmetric and they appear to be more difficult to convert into vectorless proofs, such as the proofs of the next section.

2. It seems that, unfortunately, our lemma does not provide us an interesting proof of the Ceva theorem:
Let ABC be a triangle, α, β, γ three real coefficients, I the barycenter of $\{(B,\alpha),(C,1)\}$, J

the barycenter of $\{(C,\beta),(A,1)\}$, K *the barycenter of* $\{(A,\gamma),(B,1)\}$. *Then the lines* AI, BJ, CK *are concurrent if, and only if:* $\alpha\beta\gamma = 1$.
As a matter of fact, in order to apply the lemma, we need three vectors:

$$\vec{U} = \lambda\overrightarrow{AI}, \vec{V} = \mu\overrightarrow{BJ}, \vec{W} = \nu\overrightarrow{CK},$$

with sum $\vec{0}$. In order to compute them, we would have to solve a linear system in three unknowns λ, μ, ν, parametrized by A, B, C, α, β and γ (as I, J, K are functions of these parameters). Solving this system is not difficult. However, the classical proof of the Ceva theorem, seen as a linear algebra theorem, also involves a linear system in three unknowns: the barycentric coordinates of the intersection point in the system $\{A,B,C\}$. This computation has exactly the same degree of complexity as the implementation of our criterion. Thus, obtaining the Ceva theorem as a consequence of our lemma does not turn out to be easier than proving it directly with linear algebra.

3 Equivalent vectorless proofs

We can rewrite the proof of the concurrency of the **altitudes** (for instance) **without mentioning the general concurrency lemma,** but including the argument of its proof:
The altitudes passing through A and B are not parallel, as their directions $\varphi(\overrightarrow{BC})$ and $\varphi(\overrightarrow{CA})$ are not collinear (since $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are not). Let I be their intersection point. We have:

$$\begin{cases} \overrightarrow{IA}.\overrightarrow{BC} = \overrightarrow{IA}.\overrightarrow{IC} - \overrightarrow{IA}.\overrightarrow{IB} = 0 \\ \overrightarrow{IB}.\overrightarrow{CA} = \overrightarrow{IB}.\overrightarrow{IA} - \overrightarrow{IB}.\overrightarrow{IC} = 0 \end{cases}$$

Summing these equalities we obtain:

$$\overrightarrow{IA}.\overrightarrow{IC} - \overrightarrow{IB}.\overrightarrow{IC} = 0,$$

$$\overrightarrow{IC}.\overrightarrow{AB} = \overrightarrow{IC}.\overrightarrow{IB} - \overrightarrow{IC}.\overrightarrow{IA} = 0\cdot$$

So the point I also belongs to the altitude passing through C: the altitudes are concurrent.

This proof can also be written as a **vectorless proof.** We just have to see that, for example:

$$\overrightarrow{IA}.\overrightarrow{IC} - \overrightarrow{IA}.\overrightarrow{IB} = \frac{AB^2 - IA^2 - IB^2}{2} - \frac{CA^2 - IC^2 - IA^2}{2} = \frac{AB^2 + IC^2 - CA^2 - IB^2}{2}.$$

In the following vectorless proof, we do not consider the trivial cases where a same vertex belongs to two altitudes:
The altitudes passing through A and B are not parallel, as they are perpendicular to non-parallel lines. Let I be their intersection point. By the Pythagorean theorem, we can show that:

$$\begin{cases} AB^2 + IC^2 = CA^2 + IB^2 \text{ as } IA \text{ and } BC \text{ are perpendicular} \\ AB^2 + IC^2 = BC^2 + IA^2 \text{ as } IB \text{ and } CA \text{ are perpendicular} \end{cases}$$

By transitivity:

$$BC^2 + IA^2 = CA^2 + IB^2 \cdot$$

Using this equality and the Pythagorean theorem, we can show that the lines IC and AB are not parallel. Let Ω be their intersection point. By the law of cosines, we have, for instance in the case where Ω is located between A and B and I is located between Ω and C:

$$\Omega B^2 + \Omega C^2 - 2\Omega B.\Omega C \cos(B\Omega I) + \Omega I^2 + \Omega A^2 + 2\Omega I.\Omega A \cos(B\Omega I) = BC^2 + IA^2$$

$$= CA^2 + IB^2 = \Omega C^2 + \Omega A^2 + 2\Omega C.\Omega A \cos(B\Omega I) + \Omega I^2 + \Omega B^2 - 2\Omega I.\Omega B \cos(B\Omega I),$$

$$2(\Omega I - \Omega C)(\Omega A + \Omega B) \cos(B\Omega I) = 0,$$

$$\cos(B\Omega I) = 0 \cdot$$

This implies that IC and AB are perpendicular: I belongs to the three altitudes.

Similarly, we can write the proofs of the concurrency of the **medians, perpendicular bisectors and angle bisectors** as vectorless proofs based on the following ideas.
- As for the medians, a point I belongs to the median passing through A (for example) if, and only if:

$$\overrightarrow{IA} \wedge \overrightarrow{U} = \overrightarrow{IA} \wedge \overrightarrow{IB} + \overrightarrow{IA} \wedge \overrightarrow{IC} = 0,$$

$$\overrightarrow{IA} \wedge \overrightarrow{IB} = \overrightarrow{IC} \wedge \overrightarrow{IA} \cdot$$

This is equivalent to say that the triangles IAB and ICA have the same oriented area. Therefore, if I is the intersection point of the medians passing through A and B, the triangles IAB and ICA, IBC and IAB have the same area. By transitivity, the triangles ICA and IBC also have the same area: the point I also belongs to the median passing through C.
- As for the perpendicular bisectors, we must notice that a point I belongs to the perpendicular bisector of the line segment $\overline{QR}$ (for instance) if, and only if:

$$2\overrightarrow{IA} \wedge \overrightarrow{U} = (\overrightarrow{IQ} + \overrightarrow{IR}).(\overrightarrow{IR} - \overrightarrow{IQ}) = IR^2 - IQ^2 = 0,$$

$$IQ = IR \cdot$$

This is equivalent to the very classical characterization of the perpendicular bisector as the locus of points equidistant from the endpoints of the line segment.
- As for the angle bisectors, a point I belongs to the angle bisector passing through A (for example) if, and only if:

$$\overrightarrow{IA} \wedge \overrightarrow{U} = \overrightarrow{IA} \wedge \frac{\overrightarrow{AB}}{AB} + \overrightarrow{IA} \wedge \frac{\overrightarrow{AC}}{AC} = 0,$$

$$\frac{\overrightarrow{AB}}{AB} \wedge \vec{AI} = \overrightarrow{AI} \wedge \frac{\overrightarrow{AC}}{AC} \cdot$$

This is equivalent to say that the distances from the point I to the lines AB and AC are equal.

4 A link with celestial mechanics

Let us consider a system of three bodies in gravitational interaction, with masses a, b, $c > 0$, making a triangle ABC. Their accelerations $\vec{\lambda}$, $\vec{\mu}$, $\vec{\nu}$ have the following expressions:

$$\begin{cases} \vec{\lambda} = b\frac{\overrightarrow{AB}}{AB^3} + c\frac{\overrightarrow{AC}}{AC^3} \\ \vec{\mu} = c\frac{\overrightarrow{BC}}{BC^3} + a\frac{\overrightarrow{BA}}{BA^3} \\ \vec{\nu} = a\frac{\overrightarrow{CA}}{CA^3} + b\frac{\overrightarrow{CB}}{CB^3} \end{cases}$$

Let us denote:

$$\vec{U} = a\vec{\lambda}, \vec{V} = b\vec{\mu}, \vec{W} = c\vec{\nu}$$

the forces applied to these bodies. Replacing $1/AB$ with $ab/AB^3 \ldots$ in our proof of the concurrency of the angle bisectors, we can easily check that these vectors are pairwise non-collinear and that:

$$\begin{cases} \vec{U} + \vec{V} + \vec{W} = \vec{0} \\ \overrightarrow{OA} \wedge \vec{U} + \overrightarrow{OB} \wedge \vec{V} + \overrightarrow{OC} \wedge \vec{W} = 0 \text{ for every point } O \end{cases}$$

From a physical point of view, the first equality expresses that the resultant force applied to the three-body system is $\vec{0}$, as there is no external force. This implies that the center of mass has a rectilinear motion with constant velocity. The second equality expresses that the torque applied to the three-body system is 0. This implies that its angular momentum is constant.

Applying our lemma, we obtain the following statement:
The lines passing through three non-collinear bodies in gravitational interaction, whose directions are the accelerations of these bodies, are always concurrent.

Martin Celli.
Depto. de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa. Mexico City.
E-mail: cell@xanum.uam.mx